Few-Orbit Polytopes

Egon Schulte
Northeastern University

Symmetry in Graphs, Maps, and Polytopes

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Convex Regular Polytopes — Review

Platonic solids $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$

DIMENSION $n \geq 4$

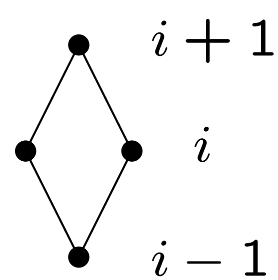
name	symbol	#facets	group	order
simplex	{3,3,3}	5	S_5	120
cross-polytope	{3,3,4}	16	B_4	384
cube	{4,3,3}	8	B_4	384
24-cell	{3,4,3}	24	F_4	1152
600-cell	{3,3,5}	600	H_4	14400
120-cell	{5,3,3}	120	H_4	14400
simplex	{3,. . . ,3}	n+1	S_{n+1}	$(n+1)!$
cross-polytope	{3,. . . ,3,4}	2^n	B_{n+1}	2^nn!
cube	{4,3,. . . ,3}	2n	B_{n+1}	2^nn!

Abstract Polytopes P of rank n

(Grünbaum, Danzer, 70's)

P	ranked partially ordered set
i-faces	elements of rank i (= -1,0,1,...,n)
i=0	vertices
i=1	edges
i=n-1	facets

- Faces F_{-1}, F_n (of ranks -1, n)
- Each flag of P contains exactly n+2 faces
- P is connected
- Intervals of rank 1 are diamonds:



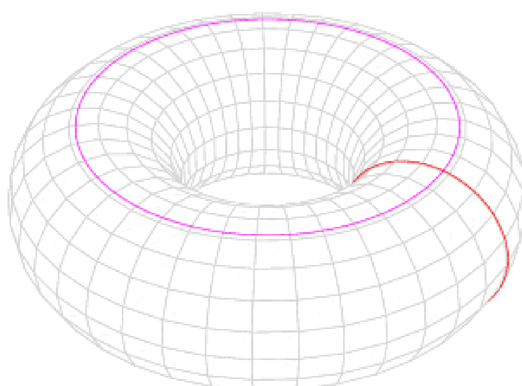
P is *regular* iff $\Gamma(P)$ flag transitive.

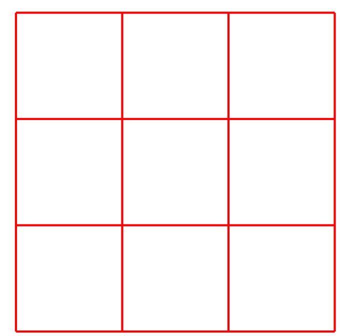
P is *chiral* iff $\Gamma(P)$ has two orbits on flags such that adjacent flags always are in different orbits.

P is a *2-orbit polytope* iff $\Gamma(P)$ has two orbits on flags.

Nothing new in ranks 0, 1, 2 (points, segments, polygons)!

Rank 3: Maps (2-cell tessellations) on closed surfaces.





$\{4,4\}_{(3,0)}$

(0,0) (3,0)

History: Klein, Dyck, Brahana, Coxeter .... and many people in the room.

$$\boxed{\text{regular polytopes} \quad \Longleftrightarrow \quad \text{C-groups}}$$

C-group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$

- $\rho_i^2 = 1$

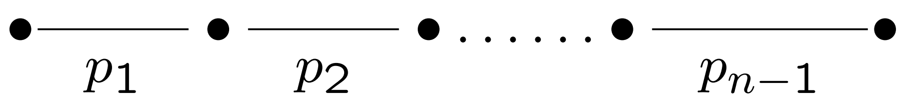
  $(\rho_i \rho_j)^2 = 1 \;\; (|i-j| \geq 2)$

  $(\rho_0 \rho_1)^{p_1} = (\rho_1 \rho_2)^{p_2} = \ldots = (\rho_{n-2} \rho_{n-1})^{p_{n-1}} = 1$

  & in general additional relations!

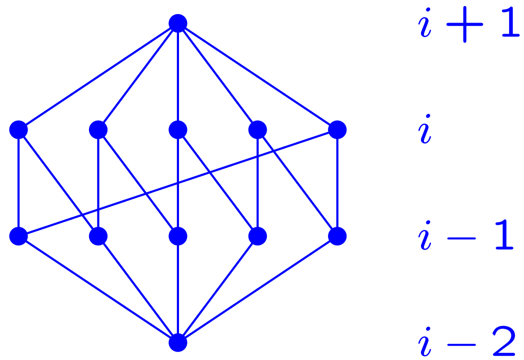
- $\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle$

- Quotient of the Coxeter group



Chirality (Weiss & S., late 80's)

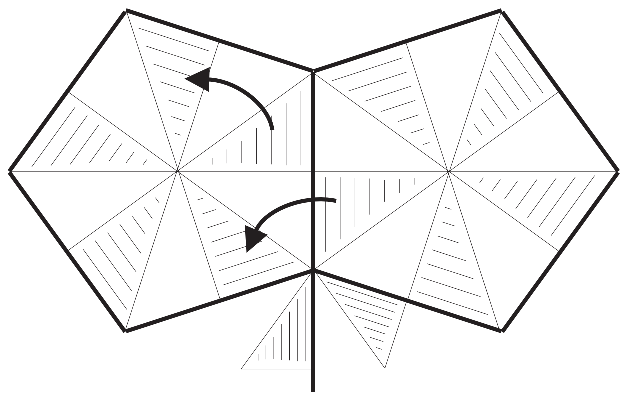
Local definition: P not regular, but for some *base* flag $\Phi := \{F_1, F_0, \ldots, F_n\}$ there exist $\sigma_1, \ldots, \sigma_{n-1} \in \Gamma(\mathsf{P})$ such that σ_i fixes each face in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces in the section F_{i+1}/F_{i-2}.



σ_i cyclically permutes vertices (edges) of the p_i-gon

- Maximal rotational symmetry but no reflexive symmetry!
- No "classical" geometric objects to start from. Convex polytopes cannot be chiral!

- Rank 3: Yes on 2-torus! Occurrence very sporadic, at least for small genus g (next for $g = 7$).



Generators σ_1, σ_2 for type $\{p, q\}$ in rank 3.

$$\sigma_1^p = \sigma_2^q = (\sigma_1 \sigma_2)^2 = 1$$

- Rank 4: Generators $\sigma_1, \sigma_2, \sigma_3$ for type $\{p, q, r\}$ in rank 4.

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1 \sigma_2)^2 = (\sigma_2 \sigma_3)^2 = (\sigma_1 \sigma_2 \sigma_3)^2 = 1$$

For $\{\{4,4\}_{(b,c)}, \{4,3\}\}$, single extra relation $(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = 1$.

• Intersection property in rank 4

$$\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \langle\epsilon\rangle = \langle\sigma_2\rangle \cap \langle\sigma_3\rangle, \quad \langle\sigma_1,\sigma_2\rangle \cap \langle\sigma_2,\sigma_3\rangle = \langle\sigma_2\rangle$$

• Two enantiomorphic forms: Chiral polytopes occur in a "right-hand" and a "left-hand" version (choice of base flag).

• Two sets of generators: $\sigma_1,\ldots,\sigma_{n-1}$ for base flag, and $\sigma_1^{-1},\sigma_1^2\sigma_2,\sigma_3,\ldots,\sigma_{n-1}$ for 0-adjacent flag.

The key: not equivalent under an involutory group automorphism of the group of the polytope!! You cannot think of σ_j as $\rho_{j-1}\rho_j$!

• Automorphism group is a quotient of the rotation subgroup of the Coxeter group $\bullet\underset{p}{\text{———}}\bullet\underset{q}{\text{———}}\bullet\underset{r}{\text{———}}\bullet$ by a normal subgroup which is *not* normal in the full Coxeter group!

Polytopes associated with the groups

Regular polytopes: Γ generated by $\rho_0, \ldots, \rho_{n-1}$

j-faces: right cosets of $\Gamma_j := \langle \rho_i \mid i \neq j \rangle$

Chiral polytopes: Γ generated by $\sigma_1, \ldots, \sigma_{n-1}$

j-faces: right cosets of

$$\Gamma_j := \begin{cases} \langle \sigma_2, \ldots, \sigma_{n-1} \rangle \text{ if } j = 0, \\ \langle \sigma_1,\ldots,\sigma_{j-1}, \sigma_{j+2},\ldots,\sigma_{n-1},\, \sigma_j\sigma_{j+1} \rangle \text{ if } j = 1, \ldots, n-2, \\ \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \text{ if } j = n-1. \end{cases}$$

Partial order in both cases:

$\Gamma_j\varphi \leq \Gamma_k\psi$ iff $j \leq k$ and $\Gamma_j\varphi \cap \Gamma_k\psi \neq \emptyset$.

Plenty of examples in rank 4! Locally toroidal chiral polytopes. (Up to mid 90's — Coxeter, Weiss & S., Monson, Nostrand)

Key idea: The rotation subgroups of the relevant hyperbolic Coxeter groups have nice representations as groups of complex Möbius transformations over $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, ... These groups have generators behaving like $\sigma_1, \sigma_2, \sigma_3$.

Construct polytopes by modular reduction of the corresponding groups of 2×2 matrices.

Example: Rotation group of $\bullet\underset{4}{\overline{\qquad}}\bullet\underset{4}{\overline{\qquad}}\bullet\underset{3}{\overline{\qquad}}\bullet$. Work over $\mathbb{Z}_m$, with -1 a quadratic residue mod m.

Gives chiral 4-polytopes of type $\{\{4,4\}_{(b,c)},\{4,3\}\}$ with group $PSL_2(\mathbb{Z}_m)$ or $PSL_2(\mathbb{Z}_m) \rtimes C_2$, with $m = b^2 + c^2$ and $(b,c) = 1$, $b, c > 0$. (Work modulo the ideal of $\mathbb{Z}[i]$ generated by $b + ic$.)

How about chiral polytopes of higher ranks?

- Mid 90's: Finite examples in rank 5 or higher?
- Hartley, McMullen & S. in 1999: The n-torus is the only compact euclidean space form admitting a regular or chiral tessellation. Chirality can only occur when $n = 2$!
- Conder, Hubard & Pisanski in 2006: Finite rank 5 examples.
- Breda, Jones & S. in 2009: More in ranks 4, 5. Parasite constructions in higher ranks.
- Conder & Devillers in 2009: Examples in ranks $6, 7, 8$.
- Pellicer in 2009: Finite examples in every rank.
- Cunningham in 2010: More of a similar nature as BJS.
- Classification results are needed!!

Parasite construction (joint work with A.Breda and G.Jones)

Input: A chiral n-polytope P and a (directly) regular n-polytope Q, both suitably chosen.

$$\Gamma^+(P) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle = W^+/M = \Gamma(P),$$
$$\Gamma^+(Q) = \langle \sigma'_1, \ldots, \sigma'_{n-1} \rangle = W^+/K,$$

with $W = \langle r_0, \ldots, r_{n-1} \rangle$ given by $\bullet \underset{\infty}{\rule{1.5em}{0.4pt}} \bullet \underset{\infty}{\rule{1.5em}{0.4pt}} \bullet \cdots\cdots \bullet \underset{\infty}{\rule{1.5em}{0.4pt}} \bullet$

$$\begin{tikzcd}[arrows=-]
 & W^+ \arrow[dl] \arrow[d] \arrow[dr] & \\
M \arrow[d] & K{=}K^{r_0} \arrow[dl] \arrow[dr] & M^{r_0} \arrow[d] \\
M \cap K = N & & N^{r_0} = (M \cap K)^{r_0}
\end{tikzcd}$$

Mix of P and Q (denoted $P \diamondsuit Q$)

Subgroup $\Gamma^+(P) \diamondsuit \Gamma^+(Q)$ of $\Gamma^+(P) \times \Gamma^+(Q)$ generated by

$$\tau_1 := (\sigma_1, \sigma_1'), \ldots\ldots, \tau_{n-1} := (\sigma_{n-1}, \sigma_{n-1}').$$

- Often the direct product! Intersection property must hold!

- Local chirality criterion: find a pair of "enantiomorphic words" that have different orders.

- Global chirality criterion: show that the "chirality group" is non-trivial (for example, this is the case when $|\Gamma^+(P)|$ does not divide $|\Gamma^+(Q)|$).

[The chirality group of a polytope measures algebraically how far the polytope is from being regular.]

Chirality groups

P chiral or directly regular, $\Gamma^+(P) = W^+/M$, with W the universal string Coxeter group $\bullet \underset{\infty}{\text{———}} \bullet \underset{\infty}{\text{———}} \bullet \cdots\cdots \bullet \underset{\infty}{\text{———}} \bullet$

$M^W := MM^{r_0}$

$M_W := M \cap M^{r_0}$

$$\begin{tikzcd}[column sep=small] & M^W \arrow[dl, no head] \arrow[dr, no head] & \\ M \arrow[dr, no head] & & M^{r_0} \arrow[dl, no head] \\ & M_W & \end{tikzcd}$$

all subgroups normal in W^+

Informally,

- W^+/M_W — smallest "regular" cover of P.
- W^+/M^W — largest "regular" pre-polytope covered by P.

- M^W/M, M/M_W, M^W/M^{r_0}, M^{r_0}/M_W all isomorphic. Called the chirality group $X(P)$. Normal subgroup of $\Gamma^+(P)$.

- $X(P)$ is trivial iff P is regular ($M = M^{r_0}$).

$X(P)$ measures how far P is from being regular.

- Other extreme case: P totally chiral.

$X(P) = \Gamma^+(P)$ (that is, $M^W = W^+$). Simple groups.

- Notion originated in maps and hypermaps (work by Breda, Jones, Nedela, Skoviera).

Computation of $X(P)$

Suppose $\Gamma^+(P) = \langle s_1, \ldots, s_{n-1} \mid \mathcal{R} \rangle$, with $W^+ = \langle s_1, \ldots, s_{n-1} \rangle$. Then $X(P)$ is the normal closure in $\Gamma^+(P)$ of the set $\mathcal{R}^{r_0}$, viewed in $\Gamma^+(P)$, consisting of those words in $s_1, \ldots, s_{n-1}$ obtained from the relators in $\mathcal{R}$ by replacing s_1 by s_1^{-1} and s_2 by $s_1^2 s_2$, while keeping all other s_j invariant.

Sometimes computationally feasible.

Nice chirality criterion

Let P be chiral and totally chiral, Q directly regular, and suppose $\Gamma^+(P) \diamondsuit \Gamma^+(Q)$ has the intersection property. If $\Gamma^+(P) \times \Gamma^+(Q)$ does not have a subgroup which has a quotient $\Gamma^+(P) \times \Gamma^+(P)$ (in particular if P and Q are finite and $|\Gamma^+(P)|$ does not divide $|\Gamma^+(Q)|$), then $P \diamondsuit Q$ is a chiral polytope.

Applications

Locally spherical case in rank 4

Let p be a prime, let $p \equiv \pm 1 \bmod 5$ or $p = 5$, and let $\mathcal{Q}$ be a finite locally spherical directly regular 4-polytope of type $\{5, 3, 5\}$ such that $p(p^2 - 1)$ does not divide $|\Gamma(Q)|$. Then there exists a locally spherical chiral 4-polytope of type $\{5, 3, 5\}$ with group $PSL_2(p) \times \Gamma^+(\mathcal{Q})$.

Example: $Q = \{\frac{5}{2}, 3, 5\}$ with group H_4, and Jones & Long (or Conder) results for $PSL_2(p)$. Resulting automorphism groups are $PSL_2(p) \times H_4^+$.

Nice tessellated hyperbolic 3-manifolds! Similar results for types $\{3, 5, 3\}$, $\{5, 3, 4\}$!

Any rank

Let P be a finite chiral n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, let Q be a finite directly regular n-polytope of type $\{q_1, \ldots, q_{n-1}\}$, and let p_j, q_j be relatively prime for each j.

(a) Then $P \diamondsuit Q$ is a chiral or directly regular n-polytope of type $\{p_1q_1, \ldots, p_{n-1}q_{n-1}\}$ with group $\Gamma(P) \times \Gamma^+(Q)$.

(b) Moreover, if P is *totally chiral* and $\Gamma(P) \times \Gamma^+(Q)$ does not have a subgroup which has a quotient $\Gamma(P) \times \Gamma(P)$ (*this holds in particular if* $|\Gamma(P)|$ *does not divide* $|\Gamma^+(Q)|$), then $P \diamondsuit Q$ is chiral.

Rank 5

Chiral component: P the universal chiral 5-polytope

$$\{\{\{3,4\},\{4,4\}_{(2,1)}\},\{\{4,4\}_{(2,1)},\{4,3\}\}\}$$

with automorphism group S_6 (Conder, Hubard, Pisanski). Now $X(P) = A_6$, so P not totally chiral.

Regular component: Q the regular cubic 5-toroid $\{4,3,3,4\}_{(s^k,0^{4-k})}$, with $s \geq 2$ and $k = 1,2$ or 4 (McMullen & S.).

Result: an infinite series of chiral 5-polytopes $P \diamondsuit Q$ of type $\{12,12,12,12\}$, with groups $[4,3,3,4]^+_{(s^k,0^{4-k})} \times S_6$, of orders $138\,240 s^4$, $276\,480 s^4$ or $1\,105\,920 s^4$ as $k = 1,2$ or 4.

Use local chirality criterion in this case!

Work in progress on 2-orbit polytopes,

joint with Isabel Hubard!

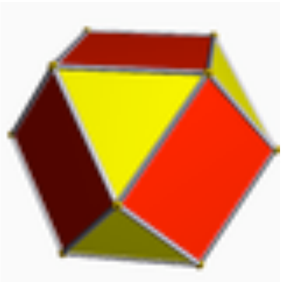
Later ......

Puts chiral polytopes in wider context! Studies groups of 2-orbit polytopes!

Semiregular Polytopes (with Barry Monson)

Convex Polytopes: Facets are *regular* (convex) polytopes. Geometric symmetry group vertex-transitive.

- Plane — regular polygons $\{p\}$



- 3-space — Archimedean solids, and prisms and antiprisms
- Three polytopes for $n = 4$, and one each for $n = 5, 6, 7, 8$.
- $t_1\{3, 3, 3\}$, snub 24-cell, $t_1\{3, 3, 5\}$, and half-5-cube.
- Gosset polytopes 2_{21}, 3_{21}, 4_{21} related to E_6, E_7 and E_8.

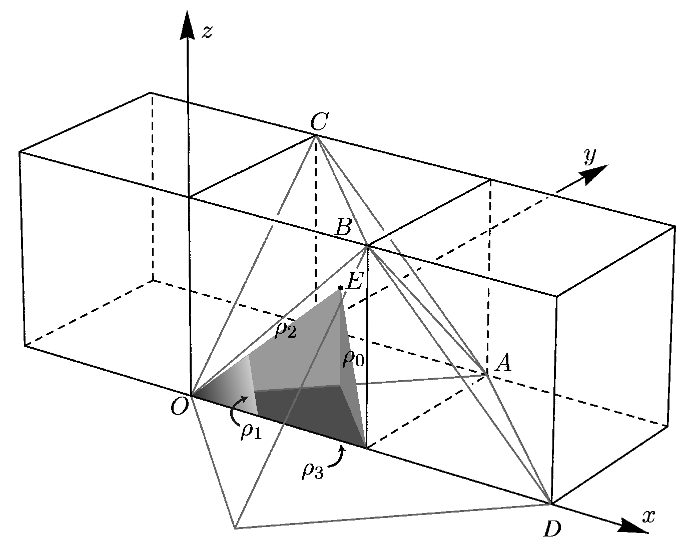
Semiregular polytopes are *uniform* polytopes.

Semiregular abstract polytopes

• Facets *regular* (abstract) polytopes, and automorphism group vertex-transitive.

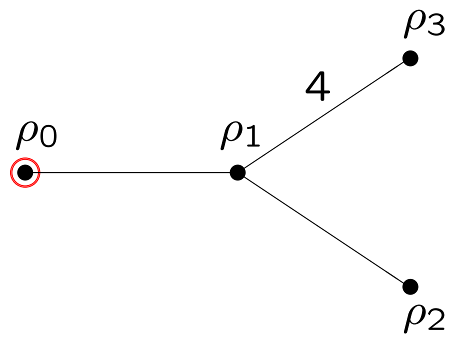
• $n = 3$: Any vertex-transitive (abstract) polyhedron is semiregular. Weak condition!

• $n = 4$: semiregular tessellation T of $\mathbb{E}^3$ by tetrahedra and octahedra. *Alternating!*



cuboctahedral vertex-figures

- Wythoff's construction

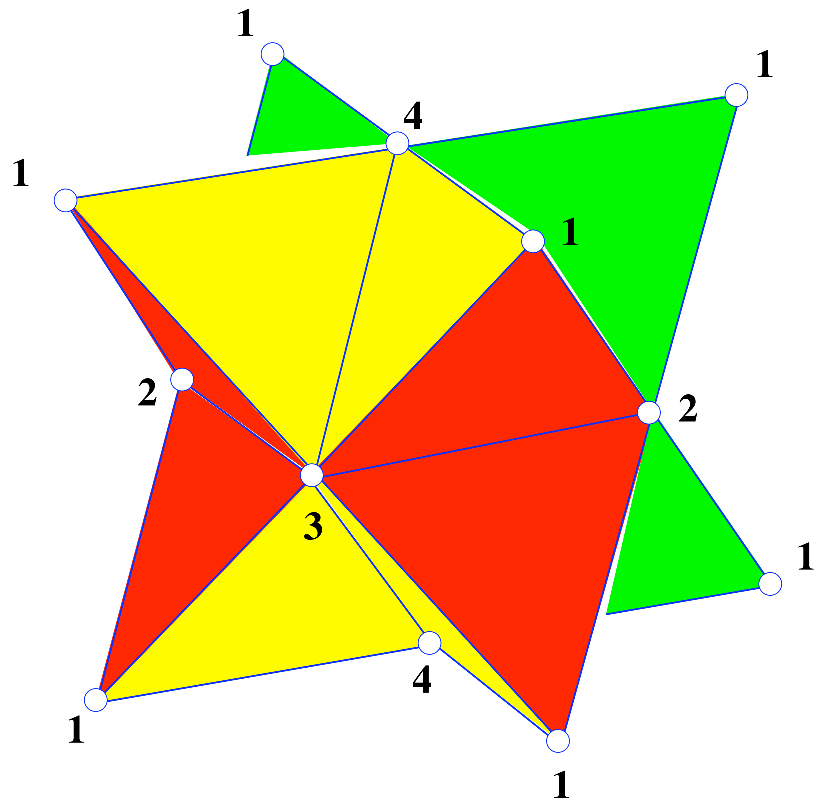


$\Gamma(T) = D_4$

(4 facets around an edge)

- How about alternating semiregular polytopes with hemi-octahedra (non-orientable) and tetrahedra as facets?

Tomotope: 4 vertices, 12 edges, 16 triangles, 4 tetrahedral and 4 hemi-octahedral facets. Vertex-figures are hemi-cuboctahedra. Group order is 96.
(Monson, Pellicer, Williams)



Tomotope

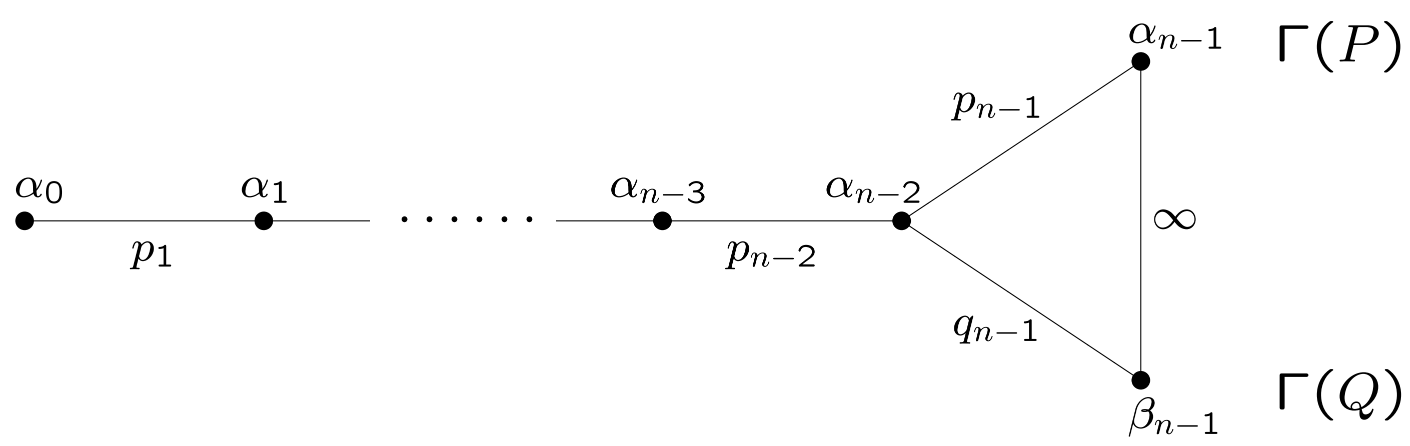
“Stella octangula” inscribed in cube. Make toroidal-type identifications for vertices and edges lying in the boundary of the cube. Finally, further identify antipodal faces of all ranks. One hemi-octahedra is in the core; and three have colors red, yellow and green, and run around the belt

Universal alternating semiregular polytopes

Input: any regular n-polytopes P and Q with isomorphic facets K

Freely assemble alternate copies of P and Q to get the universal $(n+1)$-polytope $U = U(P,Q)$!

$$\Gamma := \Gamma(P) *_{\Gamma(K)} \Gamma(Q) = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$$



- The universal $U(P,Q)$ exists for all compatible P and Q, and is semiregular and alternating (copies of P and Q appear alternately around each ridge).

- $U(P,Q)$ is regular if and only if $P \cong Q$. In this case $\Gamma(U(P,Q)) = \Gamma \rtimes C_2$.

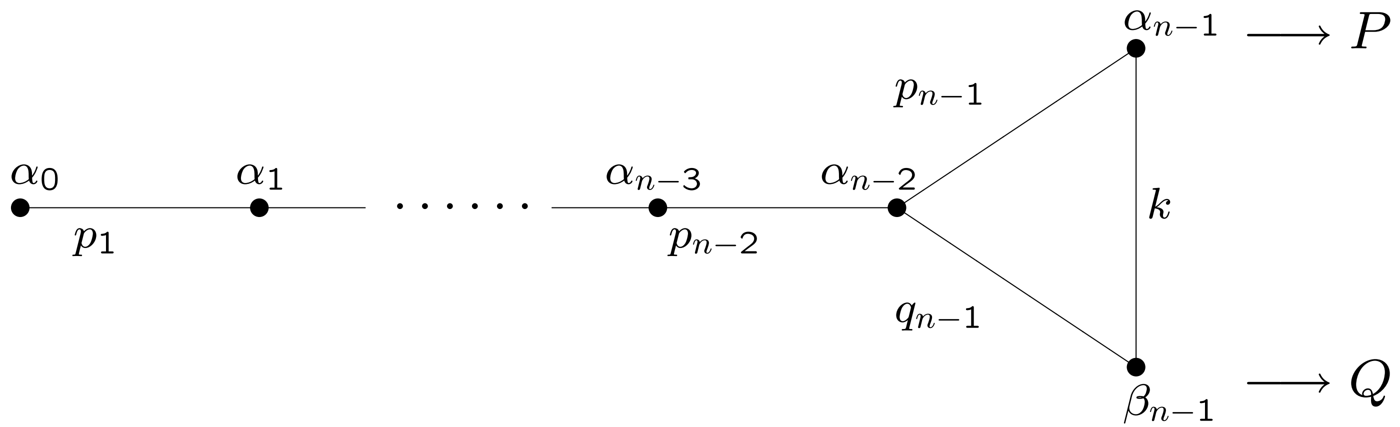
- Otherwise $\Gamma(U(P,Q)) = \Gamma$.

Questions:

- Can we have finite alternating examples?
- Can we preassign the number $2k$ of facets around a ridge?

Tail-triangle group $\Gamma = \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, \beta_{n-1} \rangle$,

with intersection property



Rank $n+1$ polytope S

- *j-faces for $j \leq n-2$*

  (right) cosets of $\Gamma_j := \langle \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{n-1}, \beta_{n-1} \rangle$

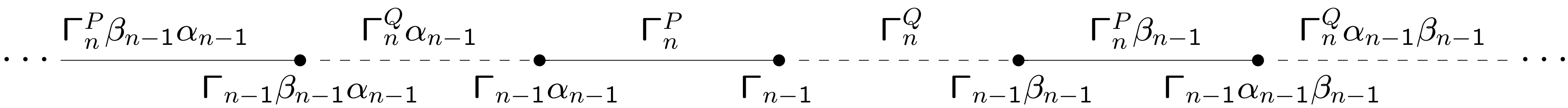
- *$(n-1)$-faces*

  cosets of $\Gamma_{n-1} := \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$

- *n-faces*
  cosets of Γ_n, with Γ_n either given by $\Gamma_n^P := \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ or $\Gamma_n^Q := \langle \alpha_0, \ldots, \alpha_{n-2}, \beta_{n-1} \rangle$.

- partial order: $\Gamma_j \nu < \Gamma_k \mu$ iff $j < k$ and $\Gamma_j \nu \cap \Gamma_k \mu \neq \emptyset$

- top rank 2 section a $2k$-gon



Many examples by modular reduction applied to crystallographic Coxeter groups!

.... The End ....

Thank you!

Abstract Few-Orbit Polytopes

Among abstract or geometric polytopes, the regular polytopes stand out as those with maximal symmetry—their combinatorial automorphism group or geometric symmetry group has just one orbit on the flags. We discuss various classes of polytopes with few flag orbits under the respective group. Important classes include chiral polytopes, and more generally two-orbit polytopes, as well as "alternating" semiregular polytopes. We report about joint work with Antonio Breda and Gareth Jones, as well as with, separately, Isabel Hubard and Barry Monson.

Dim.	Symbol	f_0	f_{d-1}	Group
$d=3$	$\{3,\frac{5}{2}\}$	12	20	H_3
	$\{\frac{5}{2},3\}$	20	12	
	$\{5,\frac{5}{2}\}$	12	12	
	$\{\frac{5}{2},5\}$	12	12	
$d=4$	$\{3,3,\frac{5}{2}\}$	120	600	H_4
	$\{\frac{5}{2},3,3\}$	600	120	
	$\{3,5,\frac{5}{2}\}$	120	120	
	$\{\frac{5}{2},5,3\}$	120	120	
	$\{3,\frac{5}{2},5\}$	120	120	
	$\{5,\frac{5}{2},3\}$	120	120	
	$\{5,3,\frac{5}{2}\}$	120	120	
	$\{\frac{5}{2},3,5\}$	120	120	
	$\{5,\frac{5}{2},5\}$	120	120	
	$\{\frac{5}{2},5,\frac{5}{2}\}$	120	120	

Regular Star-Polytopes in $\mathbb{E}^d$ ($d \geq 3$)

Honeycombs Euclidean space

n=2: with triangles, hexagons, squares
{3,6}, {6,3}, {4,4}

n$\geq$2: with cubes, {4,3,...,3,4}

n=4: with 24-cells, {3,4,3,3}
with cross-polytopes, {3,3,4,3}

Hyperbolic space

n=2: each symbol {p,q} with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$

n=3: # =15 {3,5,3}, {4,3,5}, {6,3,3}, ...

n=4: # =7 {5,3,3,4}, {3,4,3,4}, ...

n=5: # =5 {3,3,4,3,3}, {3,3,3,4,3}, ...

n$\geq$6: none

Locally toroidal case in rank 4

Let p be a prime with $p \equiv 1 \bmod 8$, let $b, c > 0$ be integers with $p = b^2 + c^2$, let $m \geq 2$, and let $\mathcal{Q}$ be a finite directly regular 4-polytope of type $\{\{4,4\}_{(m,0)}, \{4,3\}\}$ such that $p(p^2-1)$ does not divide $|\Gamma(Q)|$.

Then there exists a chiral 4-polytope of type $\{\{4,4\}_{(m,0)}, \{4,3\}\}$ if $p \mid m$ or $\{\{4,4\}_{(mb,mc)}, \{4,3\}\}$ if $p \nmid m$, whose group is $PSL_2(p) \times \Gamma^+(\mathcal{Q})$. The facets themselves are regular or chiral, respectively, in the two cases.

Many examples for $\mathcal{Q}$ via modular reduction (Monson & S.).

Similar results for types $\{4,4,4\}$, $\{6,3,3\}$, $\{6,3,4\}$, $\{6,3,5\}$, $\{6,3,6\}$, $\{3,6,3\}$!