The symmetries of McCullough-Miller space

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Universal Coxeter groups

For a positive integer n, the universal Coxeter group of rank n is the group W_n presented by

$$\langle a_1, \ldots, a_n \mid a_1^2 = \cdots = a_n^2 = e \rangle.$$

That is, W_n is the free product of n copies of the group of order two.

In W_n there are exactly n conjugacy classes of involutions, each represented by a generator a_i.

$\mathrm{Aut}(W_n) = \mathrm{Aut}^0(W_n) \rtimes \Sigma_n$ where

- $\mathrm{Aut}^0(W_n)$ consists of those automorphisms which map each generator to a conjugate of itself;
- Σ_n consists of those automorphisms which permute the generators.

$\mathrm{Aut}^0(W_n)$ is generated by partial conjugations. For $j \in \{1, \ldots, n\}$ and $D \subset \{1, \ldots, n\} \setminus \{j\}$, the partial conjugation x_{jD} is the automorphism determined by the rule:

$$a_k \mapsto \begin{cases} a_j a_k a_j & \text{if } k \in D; \\ a_k & \text{if } k \notin D. \end{cases}$$

$\mathrm{Out}(W_n) = \mathrm{Aut}(W_n)/\mathrm{Inn}(F_n)$ is the group of outer automorphisms of W_n.

It follows that

$$\mathrm{Out}(W_n) = \mathrm{Out}^0(W_n) \rtimes \Sigma_n.$$

For $n \geq 3$, $\mathrm{Out}^0(W_n)$ is an infinite group.

Why take an interest in $\mathrm{Out}(W_n)$?

The following is taken from [Farb and Margalit, p.76]:

> *"... we have*
>
> $$\mathrm{GL}(2,\mathbb{Z}) \cong \mathrm{Mod}^{\pm}(S_{1,1}) \cong \mathrm{Out}(F_2). \qquad (1)$$
>
> *Therefore, we can think of* $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{Mod}^{\pm}(S)$ *and* $\mathrm{Out}(F_n)$ *as three generalizations of the same group."*

This perspective is the source of an analogy that has driven the development of much of the theory of $\mathrm{Out}(F_n)$.

We have the following:

$$\mathrm{PGL}(2,\mathbb{Z}) \cong \mathrm{Mod}^{\pm}(\mathcal{O}_{0;2,2,2,\infty}) \cong \mathrm{Out}(W_3). \qquad (2)$$

Therefore, we can think of $\mathrm{PGL}(n,\mathbb{Z})$, $\mathrm{Mod}^{\pm}(\mathcal{O})$ and $\mathrm{Out}(W_{n+1})$ as three generalizations of the same group.

Geometric models of groups

A simplicial complex K is a geometric model for a group G if there exists a homomorphism $m\colon G \to \mathrm{Aut}(\mathrm{K})$ (that is, if G acts on K via m).

The smaller the kernel of m, the less the model simplifies G.

The larger $m(G)$ in $\mathrm{Aut}(\mathrm{K})$, the greater the expectation that $\mathrm{Aut}(\mathrm{K})$ in its entirety, rather than the subgroup $m(G)$, can offer insights into G.

Following Bridson-Vogtmann, we say that K is an accurate geometric model of G if there exists an *isomorphism* $m\colon G \to \mathrm{Aut}(\mathrm{K})$.

Some groups with accurate geometric models

Group	Accurate geometric model	Credits
Algebraic group (satisfying certain hypothesis)	Spherical building	Tits
Mapping class group associated to a surface of genus at least two	Complex of curves	Royden, Ivanov
Outer automorphisms of F_n for $n \geq 3$	Spine of outer space	Bridson-Vogtmann

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Outer automorphisms of W_n for $n \geq 4$	McCullough-Miller space	

McCullough-Miller space is a general construction

Given an arbitrary group G, and a fixed decomposition of G as a free product of groups, we write $\Sigma\,\mathrm{Out}(G)$ for the group of "symmetric outer automorphisms" of G—these are outer automorphisms which first permute the free factors, and then act by conjugation on each free factor.

McCullough-Miller space (MM-space) is a contractible simplicial complex equipped with a $\Sigma\,\mathrm{Out}(G)$-action; that is, MM-space is a geometric model for $\Sigma\,\mathrm{Out}(G)$.

MM-space is constructed by gluing together copies of the hypertree complex (to be described below) in a manner which encodes the structure of $\Sigma\,\mathrm{Out}(G)$.

McCullough-Miller in a particular case

We write K_n for the MM-space corresponding to W_n with its canonical decomposition. Since $\Sigma\,\mathrm{Out}(W_n) = \mathrm{Out}(W_n)$, K_n is a contractible simplicial complex equipped with an $\mathrm{Out}(W_n)$-action. Our main result is the following.

Theorem

For $n \geq 4$, $\mathrm{Out}(W_n) \cong \mathrm{Aut}(\mathrm{K}_n)$; that is, K_n is an accurate geometric model for $\mathrm{Out}(W_n)$.

Hypergraphs

A hypergraph Γ is an ordered pair (V_Γ, E_Γ) consisting of a set of distinguishable vertices V_Γ, and a collection (often a set) E_Γ of hyperedges, each of which is a subset of V_Γ containing at least two elements.

A graph (without loops) is a hypergraph in which each hyperedge contains exactly two vertices.

A hypergraph Θ is a hypertree if the corresponding labeled bipartite graph is a tree.

We write $\mathcal{HT}_n$ for the hypertrees with vertex set $\{1, \ldots, n\}$.

n	1	2	3	4	5	6	7	8	...
$\#\mathcal{HT}_n$	1	1	4	29	311	4447	79745	1722681	...

(See sequence A030019 in the OEIS)

Hypergraphs and hypertrees

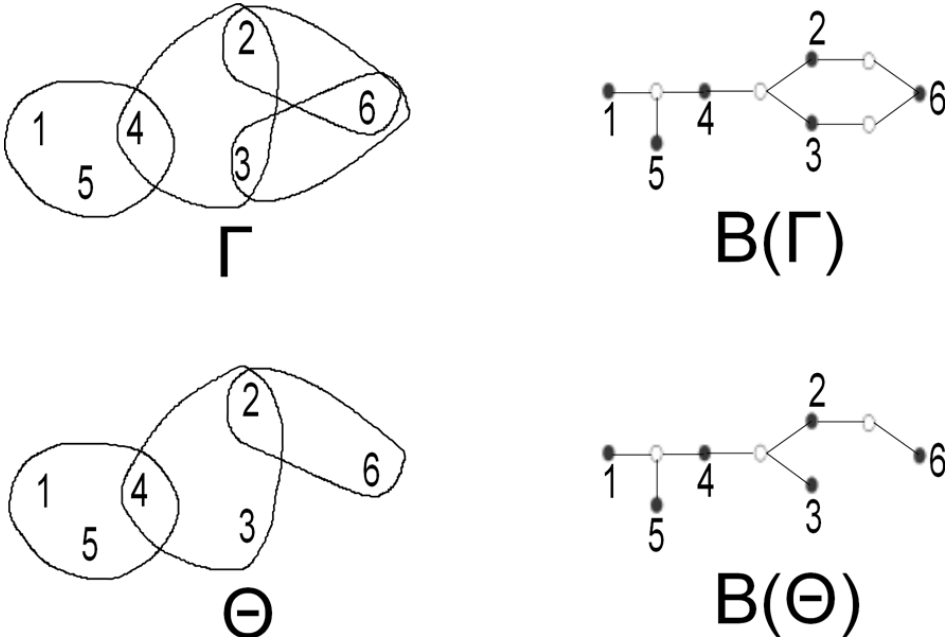
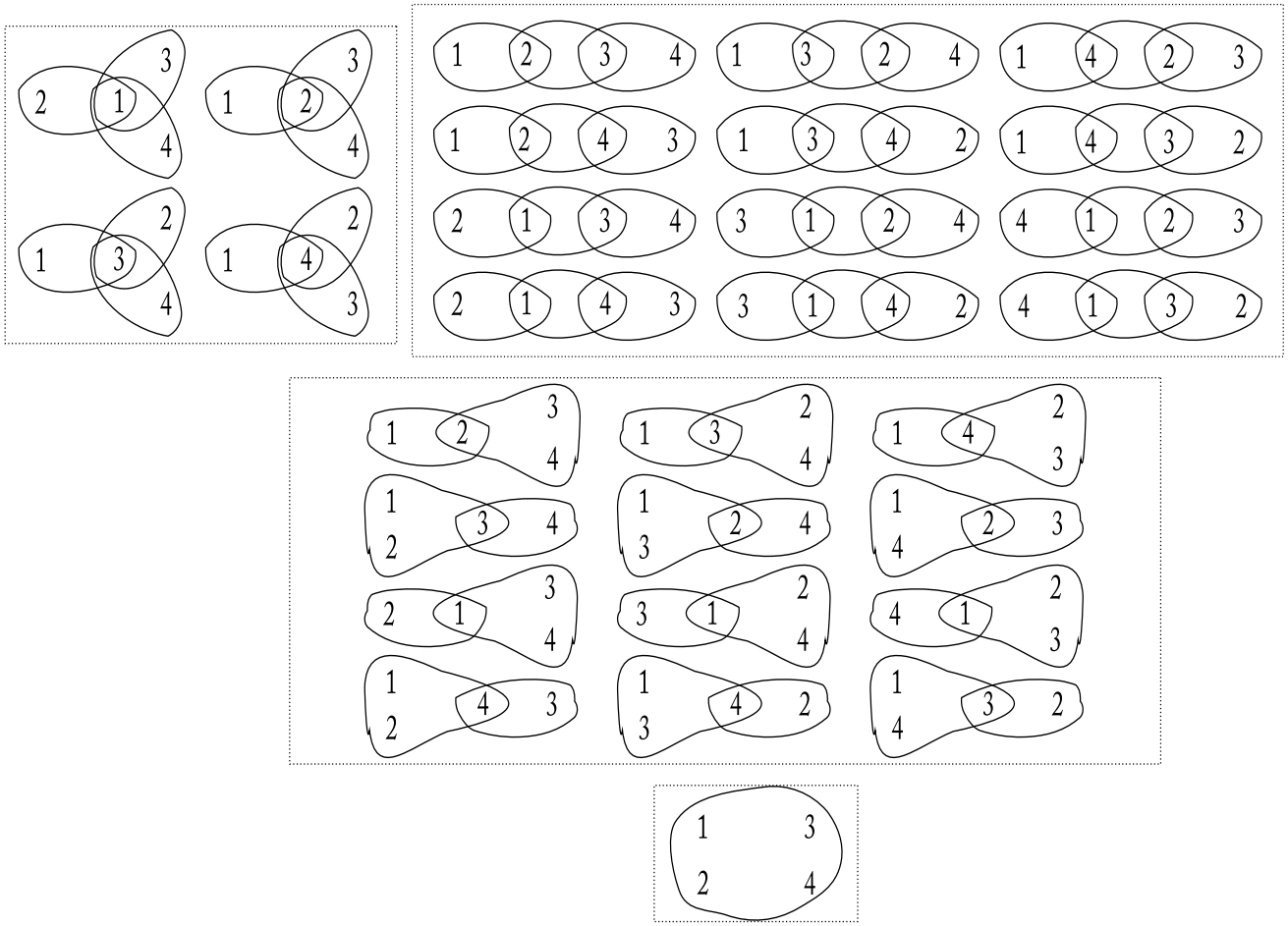


Figure: Γ is a hypergraph but not a hypertree, Θ is a hypertree.

Example: The elements of $\mathcal{HT}_4$



If two distinct hyperedges intersect nontrivially, they can be replaced by their union to give a hypertree with one less hyperedge; this is called folding.

Folding determines a partial order $\leq$ on $\mathcal{HT}_n$. The poset has a unique minimal element Θ_n^0.

We write HT_n for the simplicial realization of $\mathcal{HT}_n$; it is called the hypertree complex (of rank n).

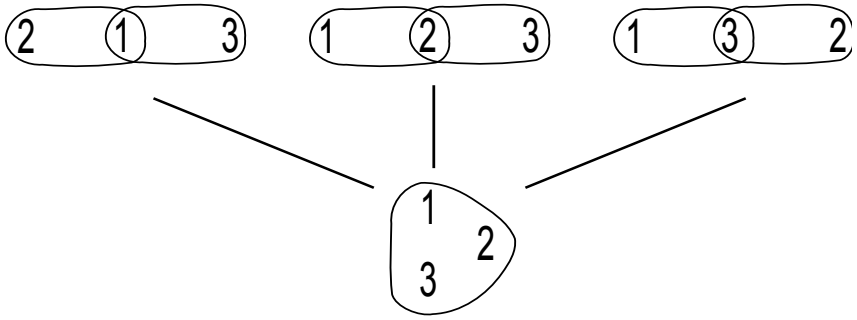


Figure: $(\mathcal{HT}_3, \leq)$.

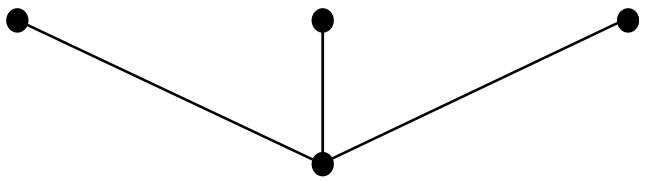


Figure: HT_3.

Example: The link in HT_4 of Θ_4^0

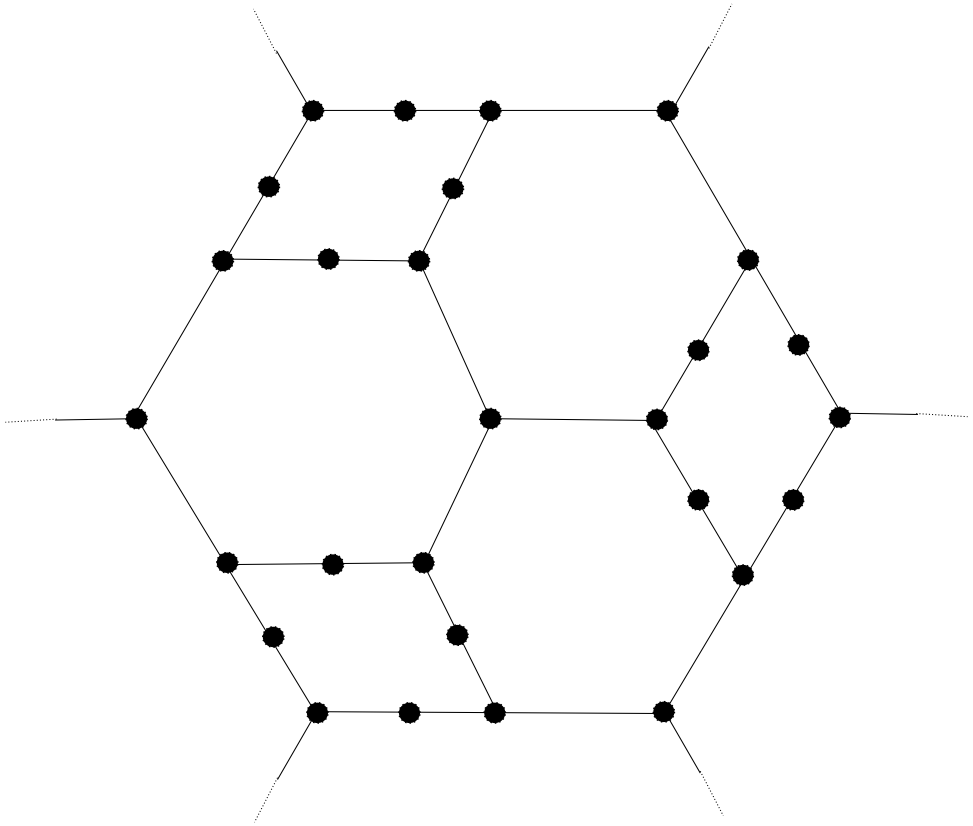


Figure: The endpoints of antipodal dashed edges should be identified to create HT_4^+, the link in HT_4 of Θ_4^0. This figure copied from MM.

Hypertrees and partial conjugations

Given a partial conjugation x_{iD}, and a hypertree $\Theta \in \mathcal{HT}_n$, we say x_{iD} is carried by Θ if: for all $d \in D$ and for all $j \in [n] \setminus D$, the simple walk in Θ from j to d visits i.

We say $\alpha \in \mathrm{Out}^0(W)$ is carried by Θ if α can be written as a product of partial conjugations, each of which is carried by Θ.

Lemma

If x_{iD} and x_{jK} are partial conjugations carried by Θ, then $x_{iD}x_{jK} = x_{jK}x_{iD}$. Thus if α is carried by Θ, then α can be written as a commuting product of partial conjugations, each of which is carried by Θ.

To construct the McCullough-Miller space K_n corresponding to $\mathrm{Out}(W_n)$ with its canonical decomposition:

- begin with one copy of HT_n for each element of $\mathrm{Out}^0(W_n)$; vertices corresponding to Θ_n^0 are called nuclear vertices.
- vertices in different copies of HT_n are identified if and only if they correspond to the same hypertree Θ, and the difference between the corresponding elements of $\mathrm{Out}^0(W_n)$ is a product of partial conjugations carried by Θ;
- the identification of vertices induces identifications of higher-dimensional simplices.

The elements of $\mathrm{Out}^0(W_n)$ act on K_n by permuting the copies of HT_n; that is, by permuting the stars of nuclear vertices.

Outline of proof

To prove the theorem we show that, for an arbitrary automorphism $f \in \mathrm{Aut}(\mathrm{K}_n)$:

(1) *The nuclear vertices are the vertices of maximal valence in* K_n. Thus f maps the star of the nuclear vertex (one of the copies of HT_n) to the star of another nuclear vertex. By construction, $\mathrm{Out}^0(W_n)$ acts transitively on the stars of nuclear vertices. Thus there exists $\phi \in \mathrm{Out}^0(W_n)$ such that $\phi^{-1}f$ fixes setwise the star of the nuclear vertex corresponding to the identity automorphism.

Outline of proof (cont)

(2) *The only automorphisms of* HT_n *are those induced by permuting the set* $\{1, \ldots, n\}$. Thus there exists $\sigma \in \Sigma_n$ such that $\sigma^{-1}\phi^{-1}f$ fixes pointwise the star of the nuclear vertex corresponding to the identity automorphism.

(3) Finally, we show that adjacent copies of HT_n share sufficiently many vertices that if the copy corresponding to $\alpha \in \mathrm{Out}^0(W)$ is fixed pointwise, then the copy corresponding to αx_{iD} is fixed pointwise too. Since the partial conjugations generate $\mathrm{Out}^0(W)$, this suffices to prove that $\sigma^{-1}\phi^{-1}f$ fixes pointwise K_n.