

On the Split Structure of Lifted groups

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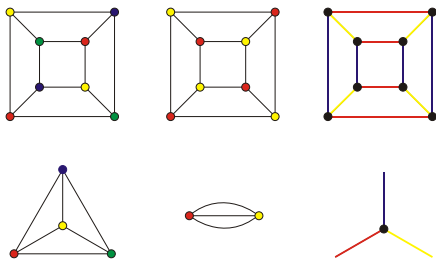
Joint work with Rok Požar

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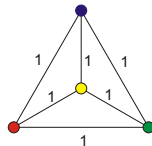
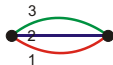
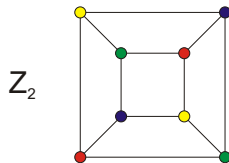
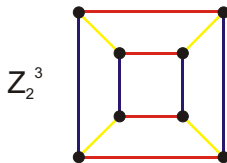
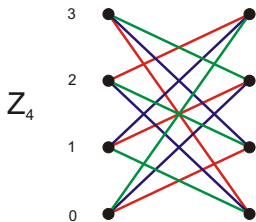


A surjective mapping $p: \tilde{X} \rightarrow X$ arising as quotienting by the action of a semiregular subgroup $\text{CT}_p \leq \text{Aut } \tilde{X}$

$$p^{-1}(v) \text{ and } p^{-1}(e) = \text{orbits of } \text{CT}_p$$

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- **Applications.** Construction of infinite families, Compiling lists, Classification of graphs with interesting symmetry properties.

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Algorithmic and complexity aspects

Split extensions $1 \rightarrow \mathrm{CT}_p \rightarrow \tilde{G} \rightarrow G \rightarrow 1$

Let $\tilde{X} \rightarrow X$ be a G -admissible regular cover given by $\zeta: X \rightarrow \Gamma$. Denote

$$\tilde{g}_{t_g}: \mathrm{fib}_b \rightarrow \mathrm{fib}_{gb}, \quad 1 \mapsto t_g$$

$$\bar{G} = \{\tilde{g}_{t_g} \mid g \in G\} \text{ algebraic transversal to } \mathrm{CT}_p$$

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- $\mathrm{CT}_p \rightarrow \tilde{G} \rightarrow G$ is split \Leftrightarrow there **exists** $t: G \rightarrow \Gamma$, $t_{id} = 1$

$$t_{gh} = t_g g^{\#b}(t_h) \cdot g^{\#b}(\zeta_Q) \zeta_Q^{-1}$$

where $g^{\#b}(\zeta_W) = \zeta_{gW}$, with $W: b \rightarrow b$ and $Q: hb \rightarrow b$ arbitrary.

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- There exists a canonical representation of \tilde{G} as $\Gamma \rtimes_{\theta} G$.

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Theorem 2.

The problem whether a given group lifts along a given abelian regular cover as a split extension of CT_p can be solved in polynomial time (in terms of $r = \text{Betti}(X)$ and $|G|$).

Split extensions – special cases wrt the action of \bar{G}

Some \bar{G} acts transitively.

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Split covers – Split extensions with an invariant section

Trivial consequence of Theorem 1

Theorem 3 (recognition). (M, Nedela, Škovič, 2000) G lifts with an invariant section over $\Omega \Leftrightarrow \tilde{X} \rightarrow X$ **can be reconstructed** by Cayley voltages $\zeta: X \rightarrow \Gamma$ that are $(1, G)$ -**invariant** on Ω :

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- **Special case:** $\Omega = V(X)$. Biggs, Algebraic Graph Theory, 1972

$$g^{\#}: \zeta_x \mapsto \zeta_{gx}, \quad x = \text{arc}.$$

$(1, G)$ -invariance: How difficult?

Define $\text{Cone}_X(\Omega)$, and extend a given ζ on X to ζ^* on $\text{Cone}_X(\Omega)$

s.t. ζ^* trivial on arcs at $*$

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For abelian covers one can use Theorem 2 to construct all complements and check their orbits ...

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Find all connected regular elementary abelian covers of K_4 such that the cyclic group \mathbb{Z}_4 lifts as a split extension with an invariant section.

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In view of Theorem 4, and using results about elementary abelian covers (M, Marušič, Potočnik, 2003) we obtain (up to isomorphism of covering projections)

Line	Condition	Dim	Voltage array
1.	$p \equiv -1 (4)$	1	$[1], [1], [1], [1], [0], [0]$
2.		2	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3.		3	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
4.	$p \equiv 1 (4), \lambda_0^2 = -1$	1	$[1], [1], [1], [1], [0], [0]$
5.		1	$[1], [\lambda_0], [-1], [-\lambda_0], [0], [0]$
6.		2	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\lambda_0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda_0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
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8.		3	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda_0 \\ -\lambda_0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\lambda_0 \\ \lambda_0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
9.	$p = 2$	1	$[1], [1], [1], [1], [1], [1]$
10.		2	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thank you!