Polyhedral Realizations and Non-Realizability for Vertex-Minimal Triangulations of Closed Surfaces in $\mathbb{R}^3$

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Triangulated Surfaces

Definition

A 2-MANIFOLD is a topological space, in which every point has an open neighborhood homeomorphic to $\mathbb{R}^2$. Connected, compact 2-manifolds are called CLOSED SURFACES .

Genus:

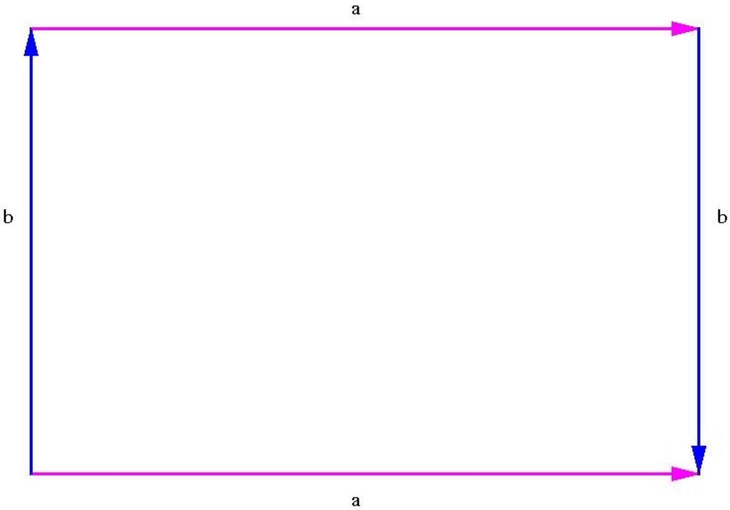
- M_g: orientable of genus g, i.e. connected sum of g Tori ($g = 0$ sphere)
- N_h: non-orientable of genus h, i.e. connected sum of h Projective Planes

Definition

A TRIANGULATION Δ of a closed surface M^2 is a simplicial complex, such that $|\Delta| \cong M^2$.

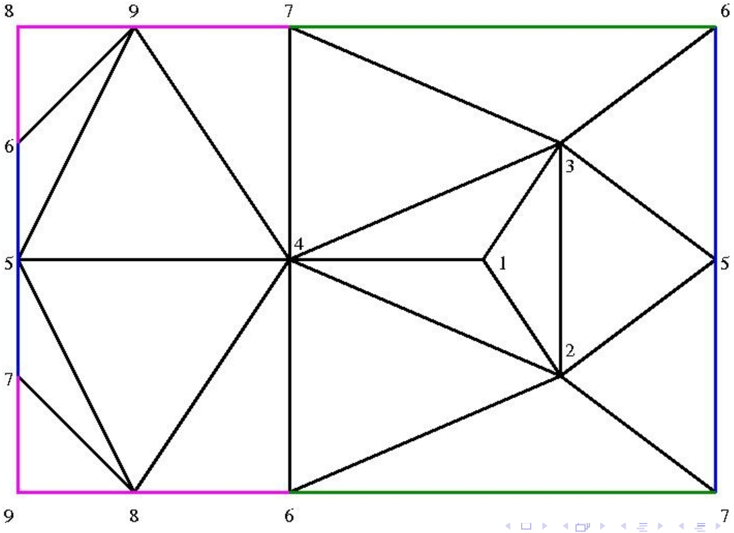
Example

A polygon representing a Klein Bottle:



Example

A triangulation of a Klein Bottle:



Triangulations and Polyhedral Realizations

Embedding An EMBEDDING of a closed surface M^2 into $\mathbb{R}^3$ is an injective map $\phi : M^2 \to \mathbb{R}^3$.

Immersion An IMMERSION of a closed surface M^2 into $\mathbb{R}^3$ is a locally injective map $\phi : M^2 \to \mathbb{R}^3$.

Polyhedral Realization A POLYHEDRAL REALIZATION of a triangulation Δ is a map $\phi : |\Delta| \cong M^2 \to \mathbb{R}^3$ such that:

- ϕ is a simplex-wise linear embedding w.r.t. Δ if M^2 is orientable, a simplex-wise linear immersion if M^2 is non-orientable
- edges of Δ are mapped to straight line segments
- triangles of Δ are mapped to planar, non-degenerate triangles

Differences Between the Smooth and Polyhedral Case

The existence of a triangulation does not guarantee its realizability in $\mathbb{R}^3$.

- there may obstructions if the number of vertices is small or minimal
- f-vector for our triangulations: $(f_0, f_1, f_2) = (n, 3n - 3\chi, 2n - 2\chi)$
- to date: Tetrahedron and Császár's torus are the only known examples of realizations of minimal triangulations with *complete* edge graph

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Consider

n_t	the number of vertices needed to triangulate a surface
n_p	the number of vertices needed to find a realizable triangulation

What is the gap between n_t and n_p (if there is one)?

Construction of Realizations

How do you find polyhedral realizations of vertex-minimal (or few-vertex) triangulations of a given surface?

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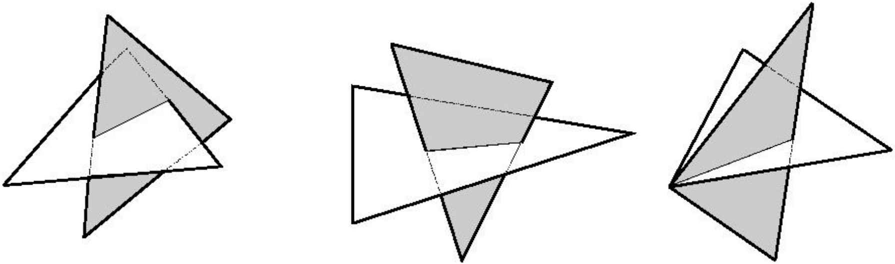
- 'by hand': Császár (1949), Brehm (1981, 1990), Bokowski and Brehm (1987-1989), Cervone (1994)
- algorithmically: Bokowski and Lutz (2006-2008), Hougardy, Lutz, and Zelke (2010)

Treatment of the Orientable Case for Small Genus (Hougardy, Lutz, Zelke, 2010)

- assigning vertex coordinates induces a simplex-wise linear map into $\mathbb{R}^3$ for any triangulation of a closed surface
- key idea: manipulation of vertex coordinates on the lattice of points with integer coordinates
- decrease OBJECTIVE FUNCTION by moving one vertex at a time by a unit step
- result: All vertex-minimal triangulations of orientable surfaces of genus $g \leq 4$ are polyhedrally realizable. Some of genus 5 are also realizable.

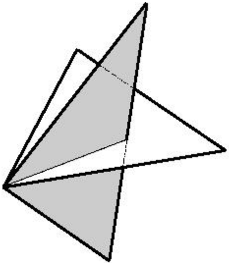
The Objective Function



$$f_{\text{obj}} = \sum_{\text{pairs of triangles}} \text{length of the intersection segment}$$

- requirement of *sufficiently general position of vertices* ensures triangles only intersect as above (segments!)
- absolute minimum 0 indicates embedding

Modification for Immersions and Symmetric Realizations



- only self-intersections in the neighborhood of a vertex are disallowed
- the following proved to be a viable alternative (Brehm, L., in preparation):

$$f_{\text{obj}} = \sum_{\substack{\text{all pairs of non-adjacent triangles} \\ \text{with common vertex}}} \text{length of the intersection segment}$$

In addition, we successfully imposed compatible symmetry conditions on the vertices yielding more beautiful results and speeding up the computation.

Results

Minimal Realizations of Triangulated Orientable Surfaces:

Typ	n_t	n_p	symmetries realized
M_1	7	7	$\mathbb{Z}_2$ (maximal)
M_2	10	10	$\mathbb{Z}_4$ *
M_3	10	10	$\mathbb{Z}_4$ (maximal)
M_4	11	11	$\mathbb{Z}_2$ (maximal)
M_5	12	12	$\mathbb{Z}_2$ *
M_6	12	≥ 13	–

*...Brehm, L.

Results

Minimal Realizations of Triangulated Non-Orientable Surfaces:

Typ	n_t	n_p	symmetries realized
N_1	6	9	$\mathbb{Z}_3$
N_2	8	9	$\mathbb{Z}_2$
N_3	9	9 *	–
N_4	9	≤ 10 *	–
N_5	9	10 *	$\mathbb{Z}_3$ *
N_6	10	10 *	$\mathbb{Z}_2$ *

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Known Gaps Between n_t and n_p

How do you prove that a triangulation is not geometrically realizable?

- few results: Klein Bottle (Cervone, 1994), Möbiusband (Brehm, 1983), not necessarily vertex-minimal examples for M_g with $g \geq 5$ (Schewe, 2010)
- algorithmic treatment possible, but difficult

$\rightarrow$ use geometric, topological, combinatorial methods, focus on the non-orientable case

Polyhedral Non-Immersibility of Triangulated N_h

Assumption: polyhedral immersion $\phi\colon |\Delta| \to \mathbb{R}^3$ exists

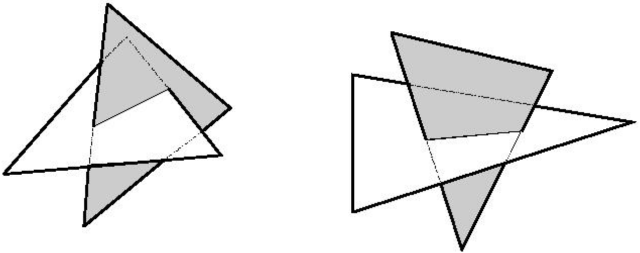
Key idea: consider the necessary self-intersection D_ϕ of the image

- assume certain genericity conditions (always fulfillable) which make D_ϕ into a finite set of closed curves **and**
- enable to show statements about the intersections of $\phi^{-1}(D_\phi)$ with edge cycles (simply closed)
- *edge cut analysis* (Cervone) is helpful

$\to$ Derive a contradiction!

Edge-Cut Analysis

Observation: An edge ab incident to triangles abc and abd cannot pierce a triangle efg if $\{e, f, g\} \cap \{a, b, c, d\} \neq \emptyset$.



In order for two triangles to intersect in space, exactly two of the triangles' six edges must pierce one of the triangles under consideration.

Methods for Proving Polyhedral Non-Immersibility

(Edge-) Cycle conditions:

- cycles in M^2 with orientable tubular neighborhood need to have an even number of intersections with $\phi^{-1}(D_\phi)$
- cycles in M^2 without orientable tubular neighborhood need to have an odd number of intersections with $\phi^{-1}(D_\phi)$ (at least one!)

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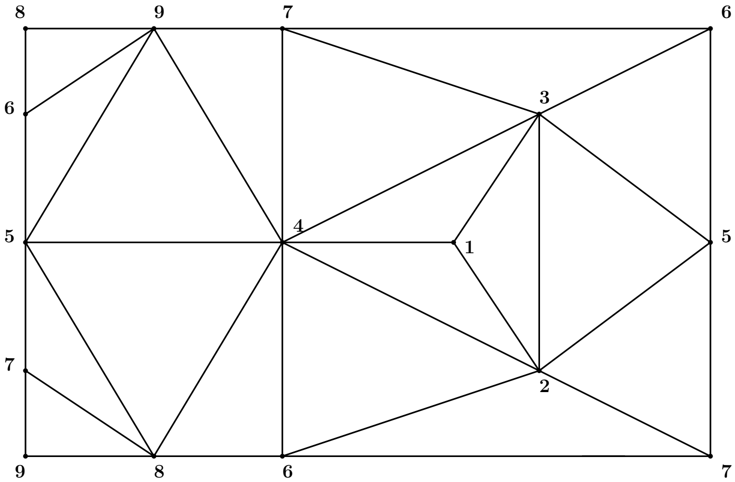
Further considerations:

- triple point conditions (Banchoff, 1974)
- linking numbers (Brehm)
- identification of geometric obstructions (Cervone)
- exploiting automorphisms

Example: A Non-Realizable Triangulation of the Klein Bottle



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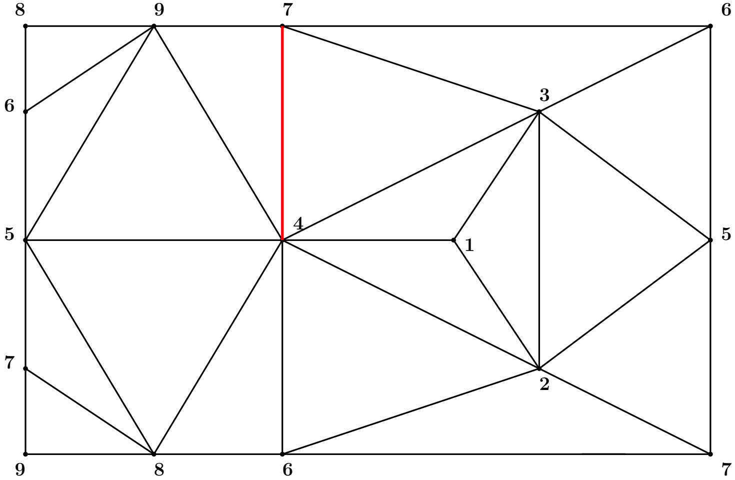

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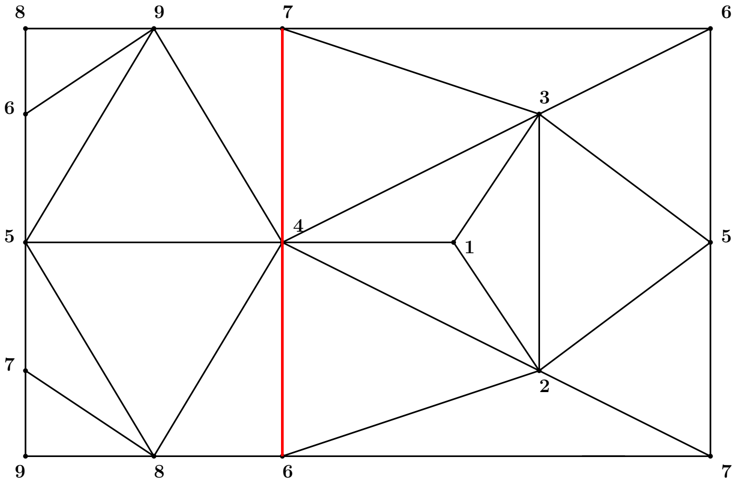

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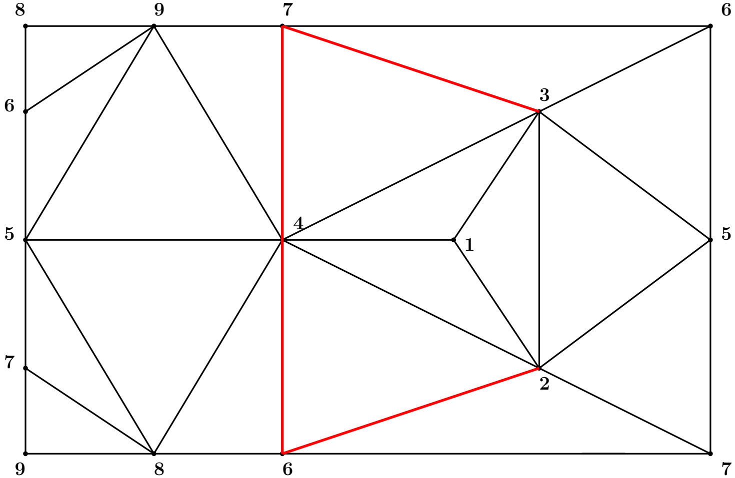

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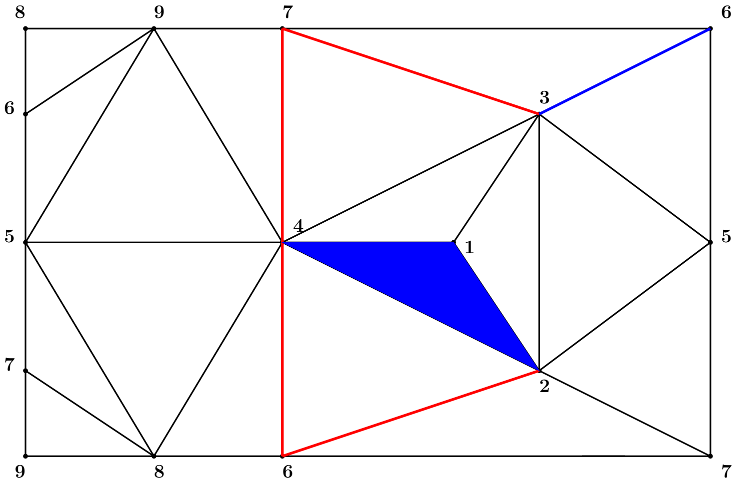

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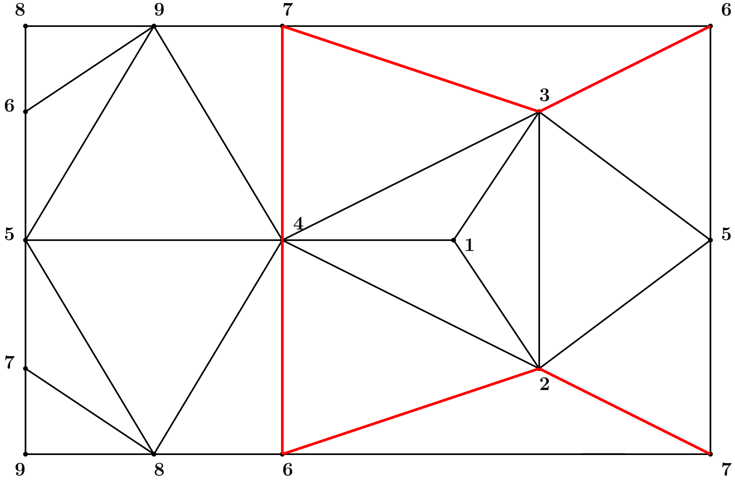

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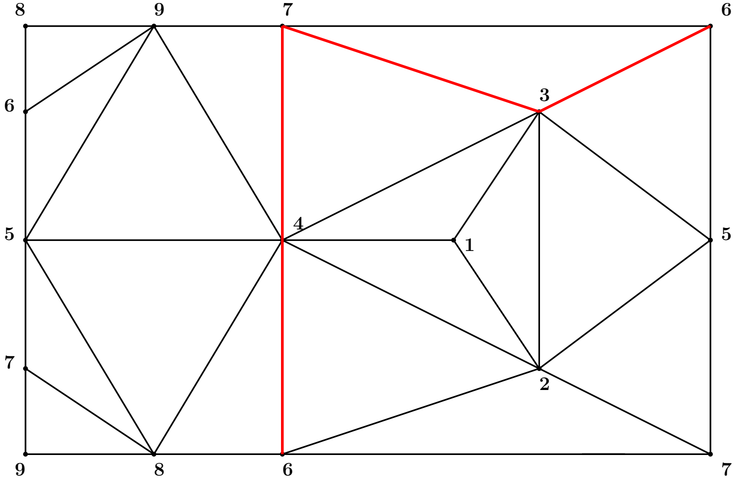

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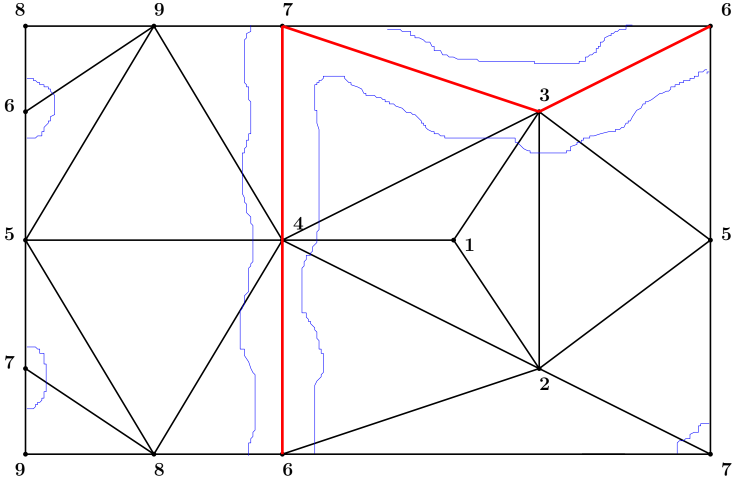

Results

$\rightarrow$ The example on the previous slide is not geometrically realizable.

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Applicability of edge cycle conditions in conjunction with edge-cut analysis and triple point considerations:

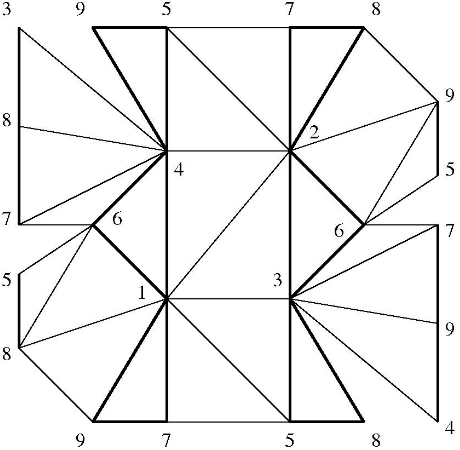
- works for some triangulations of Projective Planes with 9 vertices
- as well as for some triangulations of Klein Bottles with 9 vertices
- with 'a lot more trickery': vertex-minimal triangulations of N_5 with 9 vertices are not geometrically realizable (L., in preparation)
- conjecture: works for other non-orientable surfaces as well

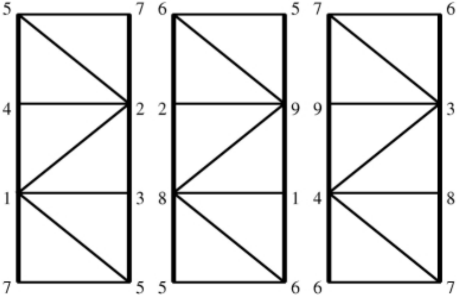
Thank you!

Questions?

Time for Pictures?

Triangulations of N_5 With 9 Vertices





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