Polytopes of high rank arising from almost simple groups

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Workshop on Symmetry in Graphs, Maps and Polytopes
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1. Introduction

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- Finite almost simple groups
- i.e. G such that $S \leq G \leq Aut(S)$ for some simple group S.

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More specific questions: given a family of (almost) simple groups

- What is the highest rank of a polytope having a group of that family as regular automorphism group ?
- How many polytopes are there up to isomorphism ?

2. String C-groups and CPR graphs

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$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

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A C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a **string C-group** if its generators satisfy the following relations.

$$(\rho_j \rho_k)^2 = 1_G \forall j, k \in \{0, \ldots n-1\} with \mid j-k \mid \geq 2$$

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The **Schläfli symbol** of a string C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is the ordered sequence $\{o(\rho_0\rho_1), \ldots, o(\rho_{n-1}\rho_n)\}$ where $o(g)$ denotes the order of the element $g \in G$.

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A **CPR graph** of a string C-group $\mathcal{P} := (G, \{\rho_0, \ldots, \rho_{d-1}\})$ of rank d is a permutation representation of $\Gamma := \langle \rho_0, \ldots, \rho_{d-1} \rangle$ represented on a graph as follows.

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Let ϕ be an embedding of Γ into the symmetric group S_n for some n. The **CPR graph** $\mathcal{G}$ of $\mathcal{P}$ determined by ϕ is the multigraph with n vertices, and with edge labels in the set $\{0, \ldots, d-1\}$, such that any two vertices v, w are joined by an edge of label j if and only if $(v)(\phi(\rho_j)) = w$.

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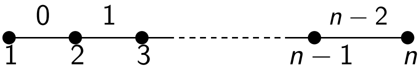

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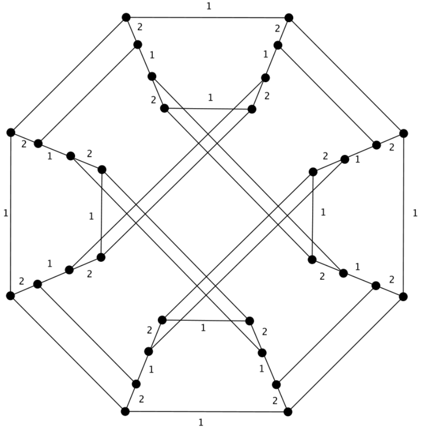

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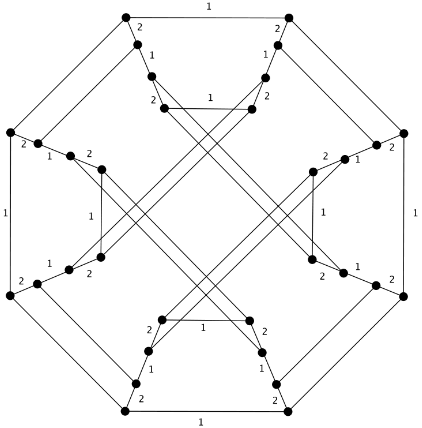

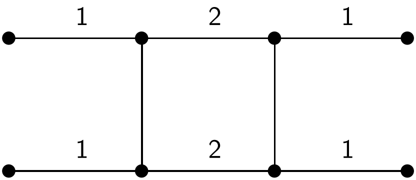

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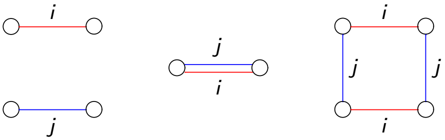
Since we deal with string C-groups, the connected components of the graphs induced by edges with labels i and j for $|i-j|>1$ must either be single vertices, single edges, double edges, or alternating squares.

This is due to the fact that if ρ_i and ρ_j commute, we have $\rho_i^{\rho_j} = \rho_i$ and therefore, when conjugating ρ_i by ρ_j, the set of edges corresponding to ρ_i in the CPR graph is stabilized. In other words, ρ_j must permute the edges corresponding to ρ_i.



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- Hartley-Hulpke : all polytopes with automorphism group a sporadic simple group up to the Held group (CDM 2010).

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4. The Leemans-Vauthier atlas

D. Leemans and L. Vauthier, *An atlas of abstract regular polytopes for small groups*, Aequationes Math. **72** (2006), 313-320
The groups analysed are subdivided into six families, namely

- Sporadic groups and their automorphism groups;
- Alternating groups and their automorphism groups;
- $PSL(2, q)$ groups and their automorphism groups;
- Other linear groups and their automorphism groups;
- Unitary groups and their automorphism groups;
- Suzuki groups and their automorphism groups.

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- Suzuki groups : 3.

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Those that don't have : $A_6 = L_2(9)$, A_7, A_8, $L_2(7)$, $L_3(q)$, $L_4(2^n)$, $U_4(2^n)$, $U_3(q)$, $PSp_4(3)$, M_{11}, M_{22}, M_{23}, McL.

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Theorem (Proc. AMS, 2006)

Let $Sz(q) \leq G \leq Aut(Sz(q))$ *with* $q = 2^{2e+1}$ *and* $e > 0$ *a positive integer. Then* G *is a C-group if and only if* $G = Sz(q)$*. Moreover, if* $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ *is a string C-group, then* $n = 3$*.*

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- if $Sz(q) < G \leq Aut(Sz(q))$, then G is not the automorphism group of an abstract regular polytope;
- if $G = Sz(q)$, there exists an abstract regular polytope $\mathcal{P}$ such that $G = Aut(\mathcal{P})$. Moreover, if $\mathcal{P}$ is an abstract regular polytope such that $G = Aut(\mathcal{P})$, then $\mathcal{P}$ must be an abstract polyhedron, i.e. a rank three polytope.

5. Almost simple groups of Suzuki type

Theorem (with Kiefer, JCTA, 2010)

A given Suzuki group $Sz(q)$ with $q = 2^{2e+1}$ and $e \geq 0$, acts on

$$\frac{1}{2}\sum_{2f+1|2e+1}\mu\left(\frac{2e+1}{2f+1}\right)\sum_{\substack{n|2f+1\\ n\neq 1}}\lambda(n)\psi(n,2f+1)$$

polyhedra up to isomorphism and duality, where

$$\lambda(n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\cdot 2^d$$

$$\psi(n,2f+1) = \sum_{m|\frac{2f+1}{n}}\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)(2^{nd}-1)}{m}$$

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e	q	Number of polytopes
1	8	7
2	32	93
3	128	1143
4	512	14476
5	2048	190371
6	8192	2580165
7	32768	35788085
8	131072	505278705
9	524288	7233587739
10	2097152	104715242943
11	8388608	1529754761127
12	33554432	22517996123568
13	134217728	333599964936448
14	536870912	4969489216225845

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Theorem (Sjerve-Cherkassoff, CMR Proc. Lect. Notes, 1993)

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Earlier work by Conder in the early 1980's covers all but a few cases of the results of S-C for the alternating and symmetric groups. These days, it takes a few seconds to handle the missing cases for S_n with MAGMA.

8. Symmetric groups S_n

Moreover, Conder observed that one may just take $x = (1,2)$ and $y = (1,2,3,\dots,n-1,n)$ as standard generators of S_n, which have been known for at least 100 years. This generating pair is "reflexible" within S_n, e.g. by conjugation by $t = (1,2)(3,n)(4,n-1)\dots(n/2, 3+n/2)(1+n/2, 2+n/2)$ if n is even or $(1,2)(3,n)(4,n-1)\dots((n+1)/2,(n+5)/2)$ if n is odd. In particular, S_n is generated by the three involutions xt, yt and t, for any $n > 2$.

8. Symmetric groups S_n

In the LV atlas, symmetric groups up to $n = 9$ are available.

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
S_5	4	1	0	0	0	0
S_6	2	4	1	0	0	0
S_7	35	7	1	1	0	0
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1. the $(n-1)$-simplex is, up to isomorphism, the unique regular $(n-1)$-polytope having S_n as automorphism group;
2. for $n \geq 7$, there exists, up to isomorphism and duality, a unique regular $(n-2)$-polytope whose automorphism group is S_n.

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Looking at these results, three observations are easily made.

1. the $(n-1)$-simplex is, up to isomorphism, the unique regular $(n-1)$-polytope having S_n as automorphism group;
2. for $n \geq 7$, there exists, up to isomorphism and duality, a unique regular $(n-2)$-polytope whose automorphism group is S_n.
3. there are polytopes of rank r with $3 \leq r \leq n-1$; moreover, there is more than one r-polytope provided $r \leq n-3$.

8. Symmetric groups S_n

Let S be a family of elements of a group G. We say that S is an **independent set** if $s \notin \langle S \backslash \{s\} \rangle$, for all $s \in S$.

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Proposition (Whiston, J. Algebra 2000)

The size of an independent set in S_n is at most $n-1$, with equality only if the set generates the whole group S_n.

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$\Rightarrow$ maximal rank is $n-1$!

8. Symmetric groups S_n

Theorem (with Fernandes, Adv. Math 2011)

For $n \geq 5$, the $(n-1)$-simplex is, up to isomorphism, the unique polytope of rank $n-1$ having a group S_n as automorphism group. For $n = 4$, there are, up to isomorphism and duality, two abstract regular polyhedra whose automorphism group is S_4, namely the hemicube and the tetrahedron. Finally, for $n = 3$, there is, up to isomorphism, a unique abstract regular polygon whose automorphism group is S_3, namely the triangle.

8. Symmetric groups S_n

Sketch proof :

Proposition (Cameron, Cara, J. Algebra 2002)

If S is an independent generating set for S_n of size $n-1$, with $n \geq 7$, then one of the following holds:

1. *S is a set of transpositions;*
2. *There exist a transposition $s \in S$ and a set of transpositions T such that*

$$S = \{s\} \cup \{(st)^{\varepsilon(t)} \mid t \in T\}, \quad \textit{where } \varepsilon(t) = \pm 1.$$

8. Symmetric groups S_n

Theorem (with Fernandes, Adv. Math. 2011)

For $n \geq 7$, there exists, up to isomorphism and duality, a unique $(n-2)$-polytope $\mathcal{P}$ having a group S_n as automorphism group. The Schläfli symbol of $\mathcal{P}$ is $\{4, 6, 3^{n-5}\}$.

Remarks :

- $n = 6$: four polytopes of rank 4, of respective Schläfli symbols $\{3, 4, 4\}$, $\{3, 6, 4\}$, $\{4, 4, 4\}$ and $\{4, 6, 4\}$.
- $n = 5$: four polytopes of rank 3, of respective Schläfli symbols $\{4, 5\}$, $\{4, 6\}$, $\{5, 6\}$ and $\{6, 6\}$.
- $n \geq 8$: at least two non-isomorphic $(n-3)$-polytopes with automorphism group S_n, for instance the ones with respective Schläfli symbols $\{4, 6, 3^{n-6}\}$ and$\{4, 6, 3^{n-8}, 6, 4\}$.
- The facet of the $(n-2)$-polytope with automorphism group S_n is a $(n-3)$-polytope with automorphism group S_{n-1}.

8. Symmetric groups S_n

Sketch proof:
O'Nan-Scott theorem : list of the maximal subgroups of S_n, putting them in three categories, namely

- the primitive groups,
- the imprimitive but transitive groups and
- the intransitive groups.

Naturally, any proper subgroup of S_n belongs to one of these three categories as well.

8. Symmetric groups S_n

Exhaustive lists of primitive subgroups of S_n exist up to a certain degree (see for instance Buekenhout, Leemans, J. Symbolic Comput. 1996, or Roney-Dougal, J. Algebra 2005).

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Every non trivial normal subgroup of a primitive group is transitive.

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Proposition (Wielandt, 1964)

Every non trivial normal subgroup of a primitive group is transitive.

Lemma

If $r \geq 3$, then Γ_1 and Γ_{r-2} are not primitive.

8. Symmetric groups S_n

Lemma (***)

Let $i \in \{0, \ldots, n-3\}$ *with* $n \geq 7$*. If* Γ_i *is imprimitive, then it is intransitive.*

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Proposition (Maroti, J. Algebra 2002)

If G is a primitive subgroup of S_n not containing A_n, then $\mid G \mid < 3^n$. Moreover, if $n > 24$, then $\mid G \mid < 2^n$.

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Γ_1 *and* Γ_{n-4} *are intransitive.*

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Γ_0 *and* Γ_{n-3} *are not primitive.*

Lemma

Γ_i *is not primitive for* $i \in \{2, \ldots, n-5\}$.

8. Symmetric groups S_n

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Use this to determine the possible shape of the CPR graph associated to a rank $(n-2)$ regular polytope of S_n and show that, up to isomorphism and duality, this graph is unique.

8. Symmetric groups S_n

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Theorem

Let Γ be a subgroup of S_n with $n \geq 7$. If Γ is the automorphism group of an abstract regular polytope of rank $n-2$, with a connected Coxeter diagram, then Γ is isomorphic to S_n.

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Theorem

Let Γ be a subgroup of S_n with $n \geq 7$. If Γ is the automorphism group of an abstract regular polytope of rank $n-2$, with a connected Coxeter diagram, then Γ is isomorphic to S_n.

Corollary

The group A_n with $n \geq 7$ has no abstract regular polytope of rank $n-2$.

8. Symmetric groups S_n

Theorem (with Fernandes, Adv. Math 2011)

Let $n \geq 4$. For every $r \in \{3, \ldots, n-1\}$, there exists at least one r-polytope $\mathcal{P}$ having a group S_n as automorphism group. Its Schläfli symbol is $\{n-r+2, 6, 3^{r-3}\}$.

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Theorem (rephrased)

Given any integer $r > 2$, the symmetric group S_n has at least one polytope of rank r whenever $n > r$.

8. Symmetric groups S_n

Can we count how many polytopes there are ?

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Can we count how many polytopes there are ?
One step in that direction : number of pairs of commuting involutions.

8. Symmetric groups S_n

Theorem (with Kiefer, submitted)

Let $n > 3$ be a positive integer. Set $\lambda(k)$ and $\psi(k,n)$ as follows.

$$\lambda(k) = \lfloor (\frac{k+1}{2})^2 \rfloor$$

$$\psi(k,n) = \begin{cases} \left[\frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor\right)\right]^2 + \frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor\right) & \text{if } n \equiv 0,1 \text{ mod } 4, \\ \left[\frac{1}{2}\left(k - \lfloor\frac{n-2k}{2}\rfloor - 1\right)\right]^2 + k - \lfloor\frac{n-2k}{2}\rfloor & \text{if } n \equiv 2,3 \text{ mod } 4. \end{cases}$$

If $n \neq 6$, there are, up to isomorphism,

$$-\frac{3}{2}\cdot\lfloor\frac{n}{2}\rfloor + \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\lambda(k)\cdot(\frac{1}{2}\lfloor\frac{n-2k}{2}\rfloor + 1) - \frac{1}{2}\cdot\sum_{k=\lfloor\frac{n}{4}\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}\psi(k,n)$$

pairs of commuting involutions in $Sym(n)$.
If $n = 6$, there are, up to isomorphism, five pairs of commuting involutions in $Sym(n)$.

8. Symmetric groups S_n

Counting is still an OPEN QUESTION !!!
Probably hopeless.

Theorem (with Kiefer, submitted)

Let $n > 3$ be a positive integer. Set $\lambda''(k)$ and $\mu(n)$ as follows.

$$\lambda''(k) = \begin{cases} \lambda(k) - 1 & \textit{if } k \leq \lfloor \frac{n}{4} \rfloor + 1, \\ \lambda(k) - \psi(k,n) - 1 & \textit{if } k > \lfloor \frac{n}{4} \rfloor + 1, \end{cases}$$

$$\mu(n) = -2 \cdot \lfloor \frac{n}{4} \rfloor + \sum_{\substack{k=1 \\ k \text{ even}}}^{\lfloor \frac{n}{2} \rfloor} \left[\lambda_e(k) \cdot \lceil \frac{1}{2} \cdot (\lfloor \frac{n-2k}{2} \rfloor + 1) \rceil + \lambda_o(k) \cdot \lfloor \frac{1}{2} \cdot (\lfloor \frac{n-2k}{2} \rfloor + 1) \rfloor \right]$$

where $\lambda_e(k) = \frac{k^2}{8} + \frac{3k}{4} + 1$ and $\lambda_o(k) = \frac{k^2}{8} + \frac{k}{4}$.
If $n \neq 6$, there are, up to isomorphism,

$$\frac{1}{2}\left(\mu(n) + \sum_{\substack{k=1 \\ k \text{ even}}}^{\lfloor \frac{n}{2} \rfloor} \lambda''(k) \right)$$

pairs of commuting involutions in $Alt(n)$.
If $n = 6$, there is, up to isomorphism, a unique pair of commuting involutions in $Alt(n)$.

9. Alternating groups A_n

Question (Hartley, 2006) :
Find regular, chiral, or other polytopes whose automorphism groups are alternating groups A_n. In particular, given a rank r, for which n does A_n occur as the automorphism group of a regular or chiral polytope of rank r?

9. Alternating groups A_n

- First main results on regular 3-polytopes with automorphism group A_n : Conder in his doctoral thesis, published in JLMS, 1980 and Quart. J. Oxford 1981.

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- Pellicer, PhD thesis and EJC 2008.

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A_5 : Up to isomorphism and duality, the only two polytopes with group A_5 are the hemi-icosahedron and the hemi-great dodecahedron.

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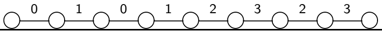
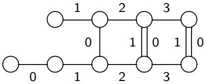
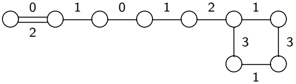
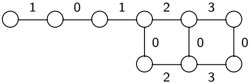
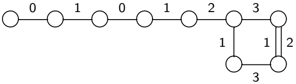
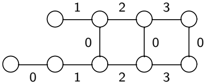
A_6 : no polytope

A_7 : no polytope

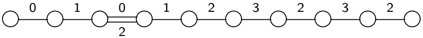
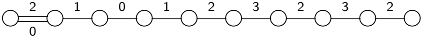
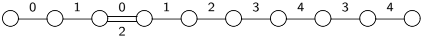
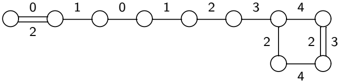
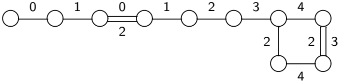
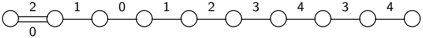
A_8 : no polytope

A_9 : 41 rank three and 6 rank four string C-groups.

{5,6,5}	
{5,6,3}	
{10,12,6}	
{10,6,3}	
{10,4,5}	
{10,3,3}	

9. Alternating groups A_n

A_{10} : 94 of rank three, 2 of rank four and 4 of rank five.

{5,10,6}	
{5,6,6}	
{5,5,6,5}	
{5,6,4,5}	
{5,10,4,5}	
{5,3,6,5}	

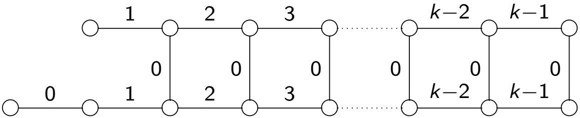
9. Alternating groups A_n

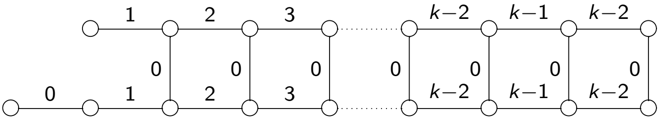
A_{11} : rank three and six.
NO rank four or five !!!

9. Alternating groups A_n

Theorem (with Fernandes and Mixer, JCTA 2012)

For each rank $k \geq 3$, there is a regular k-polytope $\mathcal{P}$ with automorphism group isomorphic to an alternating group A_n for some n. In particular, for each even rank $r \geq 4$, there is a regular polytope with Schläfli type $\{10, 3^{r-2}\}$ and group isomorphic to A_{2r+1}, and for each odd rank $q \geq 5$, there is a regular polytope with Schläfli type $\{10, 3^{q-4}, 6, 4\}$ and group isomorphic to A_{2q+3}.

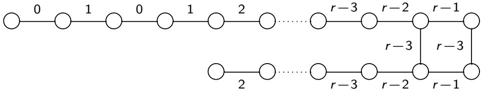
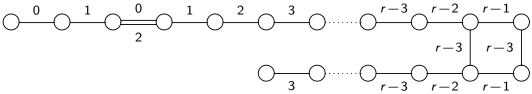




9. Alternating groups A_n

Theorem (with Fernandes and Mixer, submitted)

For each $n \notin \{3,4,5,6,7,8,11\}$, there is a rank $\lfloor \frac{n-1}{2} \rfloor$ string C-group representation of the alternating group A_n.

Group	Schläfli Type	CPR Graph
A_{2r+2} ($r \geq 6$)	$\{5, 6, 3^{r-6}, 6, 6, 3\}$	
A_{2r+1} ($r \geq 7$)	$\{5, 5, 6, 3^{r-7}, 6, 6, 3\}$	

9. Alternating groups A_n

Conjecture (Fernandes, Leemans, Mixer)

Let $n \geq 12$. The highest rank of a string C-group having A_n as automorphism group is $\lfloor \frac{n-1}{2} \rfloor$.

10. Future Work

- Prove latter conjecture for the maximum rank of a string C-group with group A_n.

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- Either find examples of rank four for the Ree groups or prove that they do not exist.
- Try to count string C-groups of rank 3, ... for S_n and A_n.
- Prove similar results for chiral polytopes.

THANK YOU !