Basic Systems of Integers

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ABSTRACT. A subset of one of the algebraic systems $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ is a *basic system* of integers if: (1) the trace and the norm of each element are rational integers; (2) the elements form a discrete subring of $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ with the units forming a finite multiplicative group or loop; and (3) when $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ is taken as a two-, four-, or eight-dimensional vector space over $\mathbb{R}$, the elements are the points of a two-, four-, or eight-dimensional lattice spanned by the units. The ring of rational integers can be regarded as a one-dimensional basic system. The rings of Gaussian and Eisenstein complex integers are well known. A. I. Weiss and I proved that there are exactly three basic systems of integral quaternions. Here I show that there are just four such systems of octonions. As lattice points, the integers of each basic system are the vertices of some regular or uniform Euclidean honeycomb of dimension 1, 2, 4, or 8.

Complex Numbers

The field $\mathbb{C}$ of *complex numbers* is a two-dimensional vector space over $\mathbb{R}$ with a commutative multiplication of vectors, defined by the mapping

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^{2\times 2}, \quad \text{with} \quad (x,\, y) \longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

and the usual matrix multiplication. The transpose of the matrix for a complex number $\mathbf{z} = (x,\, y) = x + y\mathrm{i}$ is the matrix for its *conjugate* $\bar{\mathbf{z}} = (x,\, -y) = x - y\mathrm{i}$.

Each complex number $\mathbf{z}$ has a *trace*

$$\operatorname{tr} \mathbf{z} = \mathbf{z} + \bar{\mathbf{z}} = 2x$$

and a *norm*

$$N(\mathbf{z}) = \mathbf{z}\,\bar{\mathbf{z}} = x^2 + y^2.$$

Quaternions

The division ring $\mathbb{H}$ of *quaternions* is a four-dimensional vector space over $\mathbb{R}$ with a noncommutative multiplication of vectors. Each quaternion

$$\mathbf{x} = x_0 + x_1 \mathrm{i} + x_2 \mathrm{j} + x_3 \mathrm{k}$$

has a *conjugate*

$$\widetilde{\mathbf{x}} = x_0 - x_1 \mathrm{i} - x_2 \mathrm{j} - x_3 \mathrm{k},$$

in terms of which we can define its *trace*

$$\mathrm{tr}\, \mathbf{x} = \mathbf{x} + \widetilde{\mathbf{x}} = 2x_0$$

and its *norm*

$$N(\mathbf{x}) = \mathbf{x}\,\widetilde{\mathbf{x}} = x_0{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2.$$

Octonions

The alternative division ring $\mathbb{O}$ of *octonions* is an eight-dimensional vector space over $\mathbb{R}$ with a nonassociative multiplication of vectors. Each octonion

$$\mathbf{x} = x_0 + x_1 e_1 + \cdots + x_7 e_7$$

has a *conjugate*

$$\widetilde{\mathbf{x}} = x_0 - x_1 e_1 - \cdots - x_7 e_7,$$

in terms of which we can define its *trace*

$$\operatorname{tr} \mathbf{x} = \mathbf{x} + \widetilde{\mathbf{x}} = 2x_0$$

and its *norm*

$$N(\mathbf{x}) = \mathbf{x}\,\widetilde{\mathbf{x}} = {x_0}^2 + {x_1}^2 + \cdots + {x_7}^2.$$

The Rank Equation

The trace is additive and the norm is multiplicative:

$$\begin{aligned} \operatorname{tr}\,(\mathbf{x}+\mathbf{y}) &= \operatorname{tr}\,\mathbf{x} + \operatorname{tr}\,\mathbf{y}, \\ N(\mathbf{x}\,\mathbf{y}) &= N(\mathbf{x})\cdot N(\mathbf{y}). \end{aligned}$$

Every complex number, quaternion, or octonion $\mathbf{a}$ satisfies what Dickson (1923) called its "rank equation":

$$\mathbf{x}^2 - (\mathbf{a} + \widetilde{\mathbf{a}})\mathbf{x} + \mathbf{a}\,\widetilde{\mathbf{a}} = 0.$$

The nonzero complex numbers, quaternions, or octonions form a multiplicative group $GL(\mathbb{C})$ or $GL(\mathbb{H})$ or *Moufang loop* $GM(\mathbb{O})$.

Integers

According to Dickson (1923), a set of complex, quaternionic, or octonionic integers should have the following properties:

(1) for each number in the set, the coefficients of its rank equation are rational integers;
(2) the set is closed under subtraction and multiplication;
(3) the set contains 1;
(4) the set is *maximal*, i.e., not a subset of a larger set meeting the other criteria.

Basic Systems

A *basic system* of integral elements is a subset of $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ such that:

(1) the trace and the norm of each element are rational integers;

(2) the elements form a subring of $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ with a set of invertible *units* (elements of norm 1) closed under multiplication;

(3) when $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ is taken as a vector space over $\mathbb{R}$, the elements are the points of a two-, four-, or eight-dimensional lattice spanned by the units.

A basic system is *maximal* if it is not a subset of a larger set meeting the other criteria.

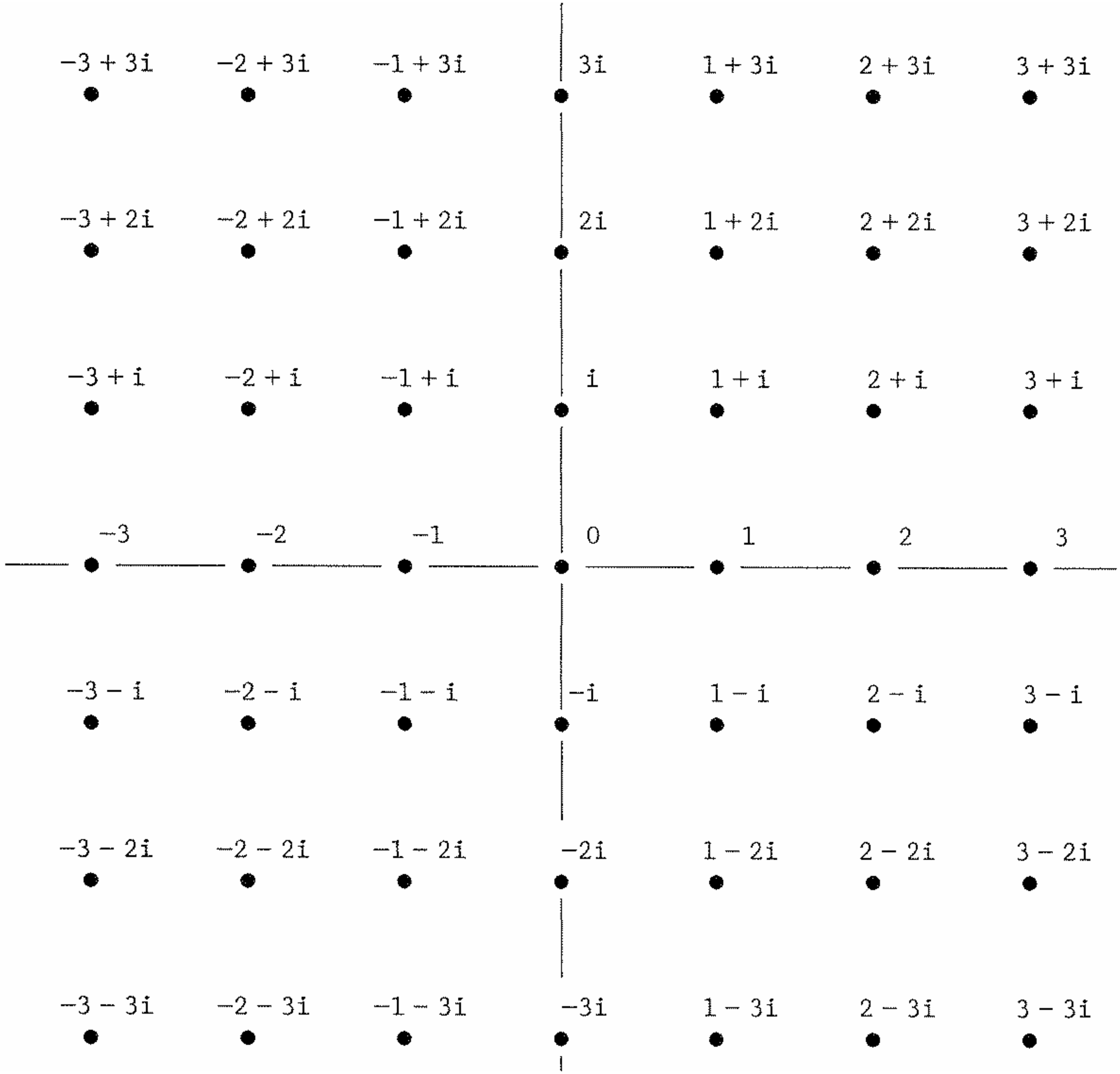
Real and Complex Integers

The only basic system of integral real numbers is the ring of *rational* integers $\mathbb{Z}$, with two units, 1 and -1.

The two basic systems of integral complex numbers are the rings of *Gaussian* and *Eisenstein* integers

$$\mathbb{G} = \mathbb{Z}[i] \quad \text{and} \quad \mathbb{E} = \mathbb{Z}[\omega],$$

where $i = \sqrt{-1}$ and $\omega = -\frac{1}{2}+\frac{1}{2}\sqrt{-3}$. The units 1 and i of $\mathbb{G}$ span the square lattice C_2, and the units 1 and ω of $\mathbb{E}$ span the hexagonal lattice A_2.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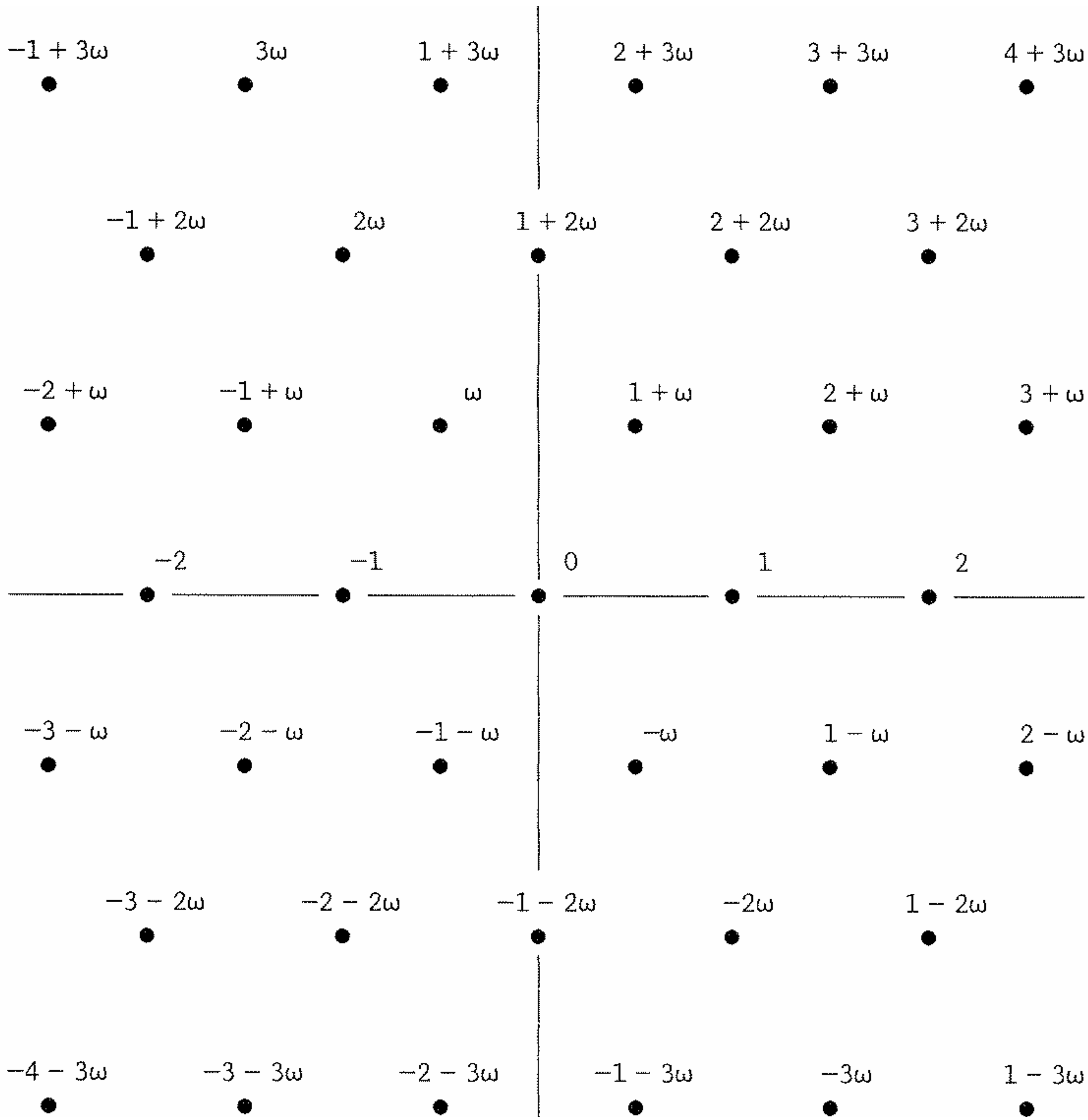
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−1 − 2ω
−2ω
1 − 2ω
−4 − 3ω
−3 − 3ω
−2 − 3ω
−1 − 3ω
−3ω
1 − 3ω

Quaternionic Integers

There are three basic systems of integral quaternions (Johnson & Weiss 1999), viz., the rings of *Hamilton*, *hybrid*, and *Hurwitz* integers

$$\mathbb{H}\mathrm{am} = \mathbb{Z}[\mathrm{i}, \mathrm{j}], \quad \mathbb{H}\mathrm{yb} = \mathbb{Z}[\omega, \mathrm{j}], \quad \mathbb{H}\mathrm{ur} = \mathbb{Z}[u, v],$$

where $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$ and where

$$u = \tfrac{1}{2} - \tfrac{1}{2}\mathrm{i} - \tfrac{1}{2}\mathrm{j} + \tfrac{1}{2}\mathrm{k} \quad \text{and} \quad v = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{i} - \tfrac{1}{2}\mathrm{j} - \tfrac{1}{2}\mathrm{k}.$$

The systems have 8, 12, and 24 units, respectively, spanning four-dimensional lattices C_4, $A_2 \oplus A_2$, and D_4.

Quaternionic Systems

System	Units	Lattice	Honeycomb
$\mathbb{H}$am	8	C_4	$\{4, 3, 3, 4\}$
$\mathbb{H}$yb	12	$A_2 \oplus A_2$	$\{3, 6\} \times \{3, 6\}$
$\mathbb{H}$ur	24	D_4	$\{3, 3, 4, 3\}$

Cayley–Graves Integers

The simplest basic system of octonionic integers consists of all octonions

$$\mathbf{g} = g_0 + g_1 e_1 + \cdot \cdot \cdot + g_7 e_7$$

whose eight components are all rational integers. We denote this system by $\mathbb{O}cg$ and call its elements the *Cayley–Graves* integers or, following Conway & Smith (2003), the *Gravesian octaves*. There are 16 units, namely,

$$\pm 1, \ \pm e_1, \ \pm e_2, \ \pm e_3, \ \pm e_4, \ \pm e_5, \ \pm e_6, \ \pm e_7,$$

spanning an eight-dimensional lattice C_8.

Gravesian Multiplication Table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Coxeter–Dickson Integers

Dickson (1923) showed that certain sets of octonions having all eight coordinates in $\mathbb{Z}$, four in $\mathbb{Z}$ and four in $\mathbb{Z}+\frac{1}{2}$, or all eight in $\mathbb{Z}+\frac{1}{2}$ form a system of octonionic integers. Coxeter (1946) found that there are actually seven of these systems. Each system has 240 units, consisting of the 16 Gravesian units and 224 others having coordinates of the type

$$(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, 0, 0, 0, 0).$$

We shall denote any of these seven systems by $\mathbb{O}$cd and call its elements the *Coxeter–Dickson* integers or the *Dicksonian octaves*. The elements of any one system are the points of a lattice E_8.

Dickson's Notation

To bring out the connection with the complex numbers and the quaternions, we may follow Dickson and denote the Gravesian units by

$$1, \quad i, \quad j, \quad k, \quad e, \quad ie, \quad je, \quad ke$$

and their negatives. This notation can be related to the one we have been using by the mapping

$$\begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ i & j & e & k & je & -ke & ie \end{array}$$

Coxeter also defines $h = \frac{1}{2}(i + j + k + e)$. Then

$$\mathbb{O}cg = \mathbb{Z}[i, j, e] \quad \text{and} \quad \mathbb{O}cd = \mathbb{Z}[i, j, h].$$

Coupled Hurwitz Integers

The ring $\mathbb{H}$ur of Hurwitz integers is a four-dimensional lattice D_4, spanned by 1, u, v, and w, where

$$u = \tfrac{1}{2} - \tfrac{1}{2}i - \tfrac{1}{2}j + \tfrac{1}{2}k \quad \text{and} \quad v = \tfrac{1}{2} + \tfrac{1}{2}i - \tfrac{1}{2}j - \tfrac{1}{2}k$$

and $w = (uv)^{-1}$. The quaternionic ring $\mathbb{H}$ur has an octonionic analogue, obtained by adjoining the unit e. This is the system $\mathbb{O}$ch of *coupled Hurwitz* integers or *Hurwitzian octaves*, which can be realized as an eight-dimensional lattice $D_4 \oplus D_4$. There are 48 units, consisting of the 16 Gravesian units and 32 others having four coordinates equal to $\pm\frac{1}{2}$ and the other four equal to 0. In Dickson's notation

$$\mathbb{O}\text{ch} = \mathbb{Z}[u, v, e].$$

Compound Eisenstein Integers

The ring $\mathbb{H}\mathrm{yb}$ of hybrid integers is a four-dimensional lattice $A_2 \oplus A_2$, spanned by 1, ω, j, and ωj, where

$$\omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i \quad \text{and} \quad \omega j = -\tfrac{1}{2}j + \tfrac{1}{2}\sqrt{3}\,k.$$

The quaternionic ring $\mathbb{H}\mathrm{yb}$ has an octonionic analogue, the system $\mathbb{O}\mathrm{ce}$ of *compound Eisenstein* integers or *Eisensteinian octaves*, which can be realized as an eight-dimensional lattice $4A_2 = A_2 \oplus A_2 \oplus A_2 \oplus A_2$. There are 24 units. In Dickson's notation

$$\mathbb{O}\mathrm{ce} = \mathbb{Z}[\omega, j, e].$$

Alternatively, if we let $g = \tfrac{1}{2}(i - j + ie - je)$, then $\mathbb{O}\mathrm{ce} = \mathbb{Z}[u, g, e]$.

Finite Moufang Loops

Boddington & Rumynin (2007) and Curtis (2007) have shown that a finite loop of octonions is either associative (and hence a group), a nonassociative double of a finite group of quaternions, or the loop of units in the system of Coxeter–Dickson integers. If commutative, such a loop spans a two-dimensional subspace of $\mathbb{O}$ isomorphic to $\mathbb{C}$; if associative but noncommutative, it spans a four-dimensional subspace of $\mathbb{O}$ isomorphic to $\mathbb{H}$; if nonassocative, it spans the eight-dimensional space $\mathbb{O}$.

Since the units of a basic system of integral octonions form a finite loop, such a system is either a double of a basic system of integral quaternions—i.e., $\mathbb{O}$cg, $\mathbb{O}$ce, or $\mathbb{O}$ch—or the system $\mathbb{O}$cd of Coxeter–Dickson integers.

Basic Systems of Integers

System	Units	Lattice	Honeycomb
$\mathbb{Z}$	2	C_1	$\{\infty\}$
$\mathbb{G}$	4	C_2	$\{4, 4\}$
$\mathbb{E}$	6	A_2	$\{3, 6\}$
$\mathbb{H}$am	8	C_4	$\{4, 3, 3, 4\}$
$\mathbb{H}$yb	12	$2A_2$	$\{3, 6\}^2$
$\mathbb{H}$ur	24	D_4	$\{3, 3, 4, 3\}$
$\mathbb{O}$cg	16	C_8	$\{4, 3^6, 4\}$
$\mathbb{O}$ce	24	$4A_2$	$\{3, 6\}^4$
$\mathbb{O}$ch	48	$2D_4$	$\{3, 3, 4, 3\}^2$
$\mathbb{O}$cd	240	E_8	$\{3^5, 3^{2,1}\}$

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