

The Schur Indices of Irreducible Characters of the Groups of Abstract Regular Polytopes

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Workshop on Graphs, Maps, and Polytopes

Fields Institute

Toronto, Ontario

October 24 - 27, 2011

Schur Indices?

Let F be a subfield of \mathbb{C} .

Let A be a finite-dimensional semisimple F -algebra.

Then

$$A \simeq \bigoplus_{\chi} M_{r_{\chi}}(D_{\chi}),$$

where

- χ runs over the Galois conjugacy classes of irreducible characters of $A \otimes \mathbb{C}$,
- $D_{\chi} \simeq \text{End}_A(V)$ is a division algebra that centralizes the image of A in $\text{End}_F(V)$, for any irreducible A -module V whose complex character involves χ .

The center of D_{χ} is isomorphic to $F(\chi)$, the extension of F obtained by adjoining the field of character values of χ . The *Schur index* of χ over F is $\sqrt{[D_{\chi} : F(\chi)]}$.

Real Schur Indices and the Frobenius-Schur indicator

If A is a finite-dimensional semisimple \mathbb{R} -algebra, the division algebra centralizers D_χ involved in simple components of A come in one of three types: \mathbb{R} , \mathbb{C} , or \mathbb{H} .

If $A = \mathbb{R}G$ is the group algebra of a finite group, the type that occurs for a given $\chi \in \text{Irr}(A \otimes \mathbb{C})$ is conveniently determined by a character-theoretic function called the *Frobenius-Schur indicator*: if

$$FS(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2),$$

then

- $FS(\chi) = 1 \iff D_\chi = \mathbb{R}$ (orthogonal rep, real Schur index 1),
- $FS(\chi) = 0 \iff D_\chi = \mathbb{C}$ (imaginary rep, real Schur index 1), and
- $FS(\chi) = -1 \iff D_\chi = \mathbb{H}$ (symplectic rep, real Schur index 2).

Abstract Polytopes

An *abstract n -polytope* is a poset \mathcal{P} , consisting of *faces*, satisfying the following:

\mathcal{P} has a unique minimal face F_{-1} and a unique maximal face F_n . Every maximal chain (*flag*) in \mathcal{P} contains $n + 2$ faces, including F_{-1} and F_n .

\mathcal{P} is strongly connected with respect to incidence of faces in flags, and has the “diamond property”: whenever $F < (?) < G$ appears in a flag, there are exactly 2 distinct faces that are solutions to $(?)$.

Faces incident to F_{-1} are called *vertices*. The set of vertices of \mathcal{P} will be denoted \mathcal{P}_0 . A pair of distinct vertices of \mathcal{P} is a *diagonal* of \mathcal{P} .

Automorphism Groups of Abstract Regular Polytopes

The automorphism group of an abstract polytope \mathcal{P} is the group of order-preserving permutations of \mathcal{P} . The abstract polytope is *regular* if the automorphism group of \mathcal{P} acts transitively on flags.

If we fix a base flag $F_{-1} < F_0 < F_1 < \cdots < F_n$, then the diamond property implies that $\text{Aut}(\mathcal{P})$ has $n - 1$ *involutory* generators $\rho_0, \rho_1, \dots, \rho_{n-1}$, with ρ_j defined so that it fixes the base flag except at position F_j .

These involutory generators of $\text{Aut}(\mathcal{P})$ satisfy the relations of a possibly infinite string Coxeter group $W = [m_1, m_2, \dots, m_{n-1}]$, where m_i is the order of $\rho_{i-1}\rho_i$ for $i = 1, \dots, n - 1$.

Automorphism Groups of Abstract Regular Polytopes, II

The automorphism groups of abstract regular polytopes are precisely the finite homomorphic images of string Coxeter groups $W = \langle S \rangle$ modulo their normal subgroups N of finite index that satisfy the *intersection property* with respect to the set S of distinguished involutory generators:

$$\text{if } I, J \subseteq S, \text{ then } \langle I \rangle \cap \langle J \rangle = \langle I \cap J \rangle.$$

Such a finite homomorphic image of a string Coxeter group is called a *string C-group*. Conversely, given a finite homomorphic image $\Gamma = W/N$ whose set of involutory generators satisfy the intersection property, Γ is the automorphism group of an abstract regular polytope, whose structure can be recovered from Γ and the particular set of generators \bar{S} .

Remark: McMullen calls arbitrary finite homomorphic images of string Coxeter groups *pre-polytopal*.

Let \mathcal{P} be an abstract regular polytope with automorphism group Γ , with distinguished involutory generating set

$$\overline{S} = \{\rho_0, \rho_1, \dots, \rho_{n-1}\}.$$

Let $\Gamma_0 = \langle \rho_1, \dots, \rho_{n-1} \rangle$ be the stabilizer of F_0 in the base flag.

A *realization* of a polytope \mathcal{P} is a map $\mathcal{P}_0 \rightarrow V$, V a finite-dimensional Euclidean space, for which elements of Γ induce isometries of V . McMullen (1989) showed the congruence classes of realizations of \mathcal{P} form a convex cone whose dimension r is the number of equivalence classes of diagonals of \mathcal{P} under the action of Γ . McMullen showed that

$$r = \{\Gamma_0 g \Gamma_0 \cup \Gamma_0 g^{-1} \Gamma_0 : g \in \Gamma - \Gamma_0\},$$

the dimension of the realization cone of the simplex realization is $\{\Gamma_0 g \Gamma_0 : g \in \Gamma - \Gamma_0\}$, and

every realization decomposes as a sum of pure realizations (associated to a single irreducible character of Γ).

Monson's question

Character-theoretic formulas for r in terms of real representations of Γ given by McMullen were later revised in Monson and McMullen (2001), when it was taken into account that non-scalar elements in the division algebra centralizer of an irreducible real representation act as congruences.

Monson found the example of $\Gamma = [5, 5]_5$ that has a complex conjugate pair of characters for which the centralizer is \mathbb{C} . (Now we know of quite a few more such examples.)

Monson's Question: If Γ is a string C -group, and V is an irreducible $\mathbb{R}\Gamma$ -module, can $\text{End}_{\mathbb{R}\Gamma}(V) = \mathbb{H}$?

In other words, is there an irreducible complex character χ of a string C -group with $FS(\chi) = -1$? Or is it the case that every real Schur index of these groups is 1?

What was known about finite string Coxeter groups

- Every irreducible character of a Weyl group has rational Schur index 1. (And hence real Schur index 1).
- Every irreducible character of a finite irreducible Coxeter group has real Schur index 1. (They all have rational Schur index 1 except for the rational-valued non-parabolic irreducible character of degree 48 of the Coxeter group of type H_4 , whose 2- and 3-local indices are 2.)
- There are 23 examples of characters of finite irreducible unitary reflection groups with real Schur index 2 (24 with rational Schur index 2), many of these with string diagrams. For example, $3[3]3, 3[4]3, 4[3]4, 4[4]3, 5[3]5, 5[4]3, 3[5]3, 3[3]3[3]3$ and $3[3]3[4]2$ all have irreducible characters with real Schur index 2.

Looking for an example?

- Rational Schur indices of $SL(n, q)$ were calculated by Turull (2001). Some of these have real Schur index 2, even for $PSL(n, q)$.
- Using GAP's character functions, at BIRS in 2004 we calculated $FS(\chi)$ for all irreducible characters for all string C -groups in the atlas of regular polytopes ($|\Gamma| \leq 2000$).
- For nonabelian simple groups: Schur indices are known for all but the infinite family of type $PSU(2n, q^2)$ and most families of simple groups of Lie type. The only known examples with real Schur index 2 occur for 2 characters of the sporadic group McL , some cases of $PSL(n, q)$ with n odd, and for cuspidal unipotent characters of groups of type ${}^2A_{n+1}(q)$ ($\forall q, \exists n$), ${}^2F_4(2^{2k+1})$, and ${}^2E_6(q)$ ($\exists q$).

By [Hartley and Hulpke, 2010] McL is not the automorphism group of a regular polytope, so it is not a string C -group. I know of no string- C characterization for (small?) simple groups of Lie type.

- Tiep (2005) asked: If every element of the finite simple group G is *strongly real* (conjugate to its inverse via an involution), is G *totally orthogonal* ($FS(\chi) = 1, \forall \chi \in Irr(G)$)?

The answer is no if the finite group is not simple, but $FS(\chi) \geq 0$ ($\forall \chi$) for the only published counterexample (which has order 32)...

Totally orthogonal groups are always generated by involutions [Wang & Grove, 1988]. New results of Guralnick and Montgomery (2009) concerning Frobenius-Schur indicators of the Drinfel'd double $D(G)$ point to a relationship between these issues.

- Fact: Any element of finite order in a Coxeter group is strongly real. So if $\chi(g) \neq \chi(g^{-1})$ in a string C -group Γ , the preimage of g in the overlying string Coxeter group W must have infinite order. This is the case whenever $FS(\chi) = 0$. This will also be true in the pre-polytopal case.

- Herman and Monson (2004) compared McMullen's revised formula for r with Frame's formula for counting double cosets of a subgroup in a finite group, and found that whenever $FS(\chi) = 0$ or -1 , the dimension of the essential Wythoff space for the pure realizations related to χ is always 0 or 1.

Thus the existence of such characters would not complicate the structure of the realization cone of the polytope, and there is no geometric reason (so far) not to expect examples with $FS(\chi) = -1$.