# The ubiquity of alternating groups

# (as automorphism groups of symmetric structures)

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## **Context**:

Discrete structures with maximum possible symmetry (under certain constraints), such as the following:

- compact Riemann surfaces of genus $g > 1$ with $84(g-1)$ conformal automorphisms [meeting the Hurwitz bound]
- equivalently, regular maps of type $\{3, 7\}$
- 5-arc-transitive cubic graphs
- 7-arc-transitive 4-valent graphs
- hyperbolic 3-manifolds of largest possible symmetry-to-volume ratio
- regular and chiral polytopes.

In these cases, the alternating groups $A_n$ occur frequently as automorphism groups. ['Ubiquitous' $\equiv$ 'found everywhere']

**Now the context is set, a small digression ...**

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## Compact Riemann surfaces and regular maps

Let $G$ be a group of orientation-preserving automorphisms of a compact Riemann surface (or equivalently, a complex algebraic curve) of genus $g > 1$. Then by a theorem of Hurwitz (1893), $|G| \leq 84(g-1)$. Moreover, this bound is attained if and only if $G$ is a quotient of the ordinary $(2,3,7)$ triangle group $\langle\, x, y \mid x^2 = y^3 = (xy)^7 = 1 \,\rangle$.

In particular, if $G$ is the group $\mathrm{Aut}^{\mathrm{o}} M$ of all orientation-preserving automorphisms of a regular map $M$ on an orientable surface of genus $g > 1$, then $G$ is a quotient of the ordinary $(2,k,m)$ triangle group, where $\{k,m\}$ is the type, and the maximum value that $|G|$ can take is $84(g-1)$, which happens when $(k,m) = (3,7)$ or $(7,3)$.

Quotients of the $(2,3,7)$ triangle group are Hurwitz groups.

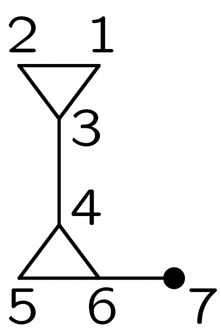
Examples include the following:

- $\mathrm{PSL}(2,7)$, the group of Klein's quartic (of genus 3)
- $\mathrm{PSL}(2,p)$ for prime $p \equiv \pm 1 \bmod 7$
- $\mathrm{PSL}(2,p^3)$ for prime $p \equiv \pm 2, \pm 3 \bmod 7$
- $\mathrm{PSL}(n,q)$ for all $n \geq 287$ and every prime-power $q$
- $\mathrm{Sp}(2n,q)$ for all $n \geq 371$ and every prime-power $q$
- the Ree groups ${}^2G_2(3^{2m+1})$ for all $m \geq 1$
- 12 of the 26 sporadic simple groups, incl. the Monster
- all but finitely many of the alternating groups $A_n$
- extensions by these groups of various other groups.

**Theorem** [MC, 1980] The group $A_n$ is a quotient of the $(2, 3, 7)$ triangle group for all $n > 167$ ... and for many smaller values of $n$ as well.

How to prove this? Ans: Use coset diagrams — which depict permutation representations of finitely-generated groups.

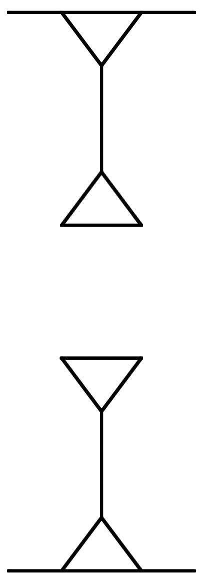
e.g. below is a coset diagram for an action of the $(2, 3, 7)$ triangle group $\langle\, x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \,\rangle$ on 7 points:



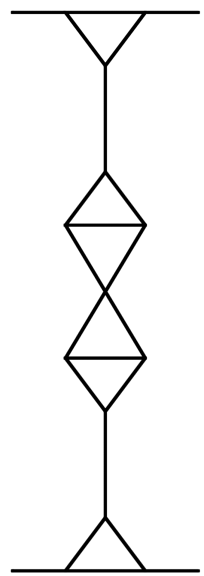

$$x \mapsto (3, 4)(6, 7)$$
$$y \mapsto (1, 2, 3)(4, 5, 6)$$
$$z \mapsto (1, 4, 7, 6, 5, 3, 2)$$

## Composition of coset diagrams

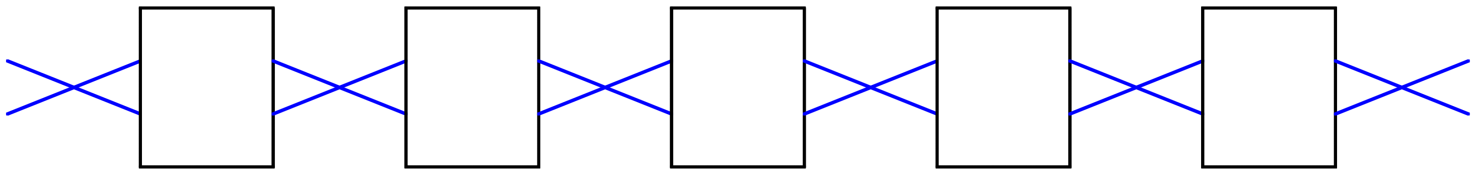
Often two coset diagrams for the same group $G$ on (say) $m$ and $n$ points can be composed to produce a transitive permutation representation of larger degree $m + n$, e.g.



$\Rightarrow$



We can now string together copies of coset diagrams:



This can be used to do all sorts of things, such as prove that certain finitely-presented groups are infinite.

If diagrams $P$ and $Q$ have $m$ points and $n$ points, then we can string together $p$ copies of $P$ and $q$ copies of $Q$ and get a diagram on $m = ap + bq$ points, and if $\gcd(p, q) = 1$, then $m = ap + bq$ can be any sufficiently large positive integer.

Then add a single copy of an extra diagram $R$ (with $r$ points) to disturb the cycle structure of particular elements, and make the permutations from the new diagram generate the alternating group $A_{m+r}$ or the symmetric group $S_{m+r}$.

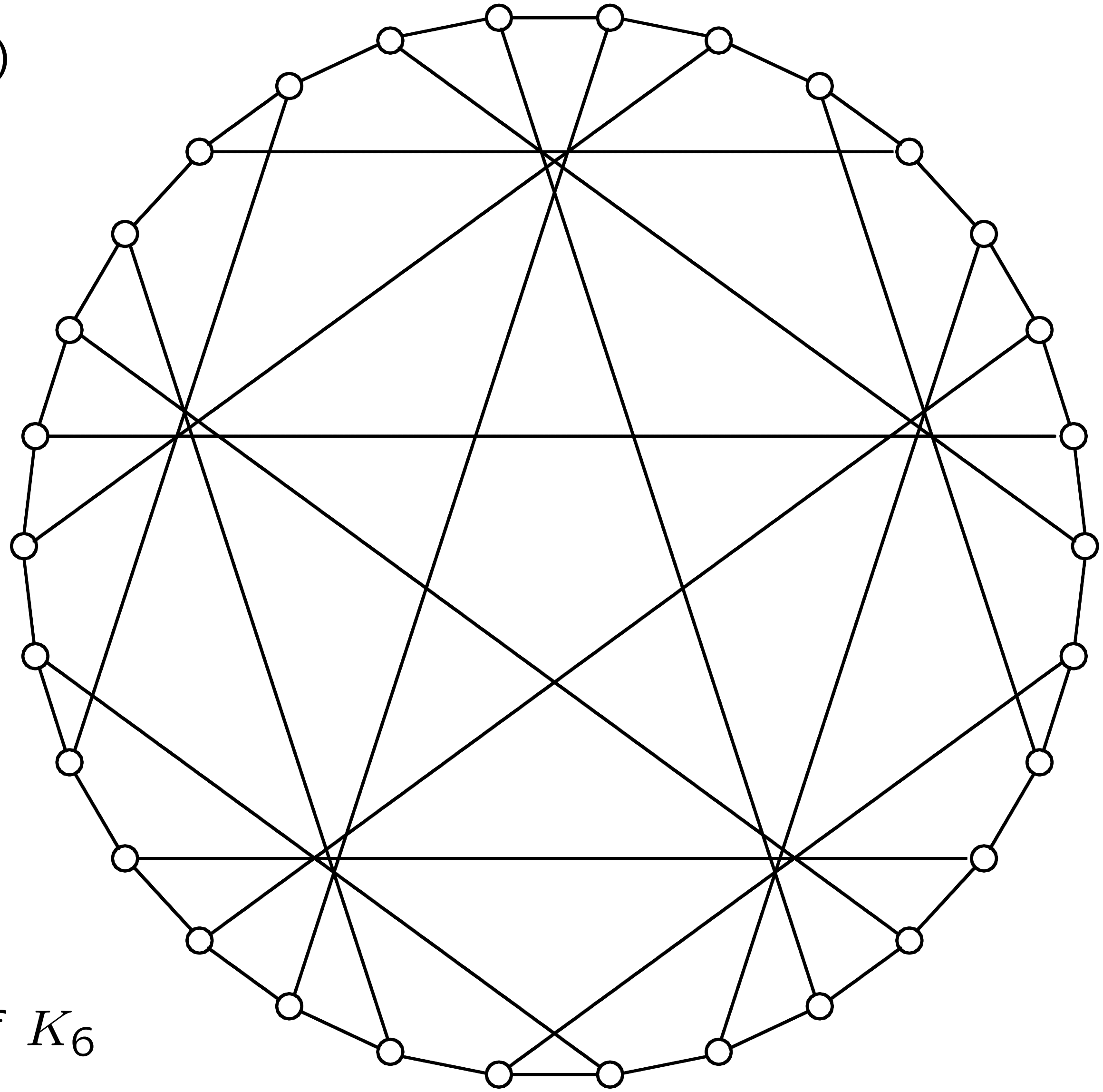
## Symmetric graphs

A graph is called symmetric if its automorphism group has a single orbit on arcs (ordered pairs of adjacent vertices).

An $s$-arc in a graph is a sequence $(v_0, v_1, v_2, \ldots, v_s)$ of $s+1$ vertices such that any two consecutive $v_i$ are adjacent and any three consecutive $v_i$ are distinct. A graph is $s$-arc-transitive if its automorphism group is transitive on $s$-arcs.

Tutte's Theorem (1947): If $X$ is a connected finite symmetric 3-valent graph then $|\mathrm{Aut}\, X| \leq 48|V(X)|$, and this bound is attained if and only if $X$ is 5-arc-transitive.

**Tutte's 8-cage**
**(5-arc-transitive)**



Associated with
1-factorisations of $K_6$

Theorem [Djoković & Miller (1980), adapted slightly] The group $G$ is the automorphism group of a connected finite 5-arc-transitive 3-valent graph if and only if $G$ is a smooth quotient of the finitely-presented group

$$\begin{aligned} G_5 = \langle\, h, p, q, r, s, a \mid {} & h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1, \\ & pq = qp,\ pr = rp,\ ps = sp,\ qr = rq,\ qs = sq, \\ & (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r,\ h^{-1}rh = pqr, \\ & shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle \end{aligned}$$

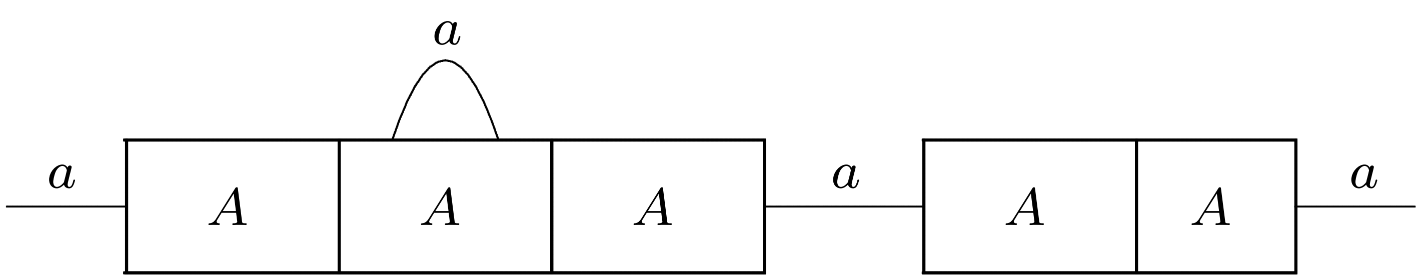
Correspondence between graph and group:

| | | | |
|---|---|---|---|
| Vertex-stabilizer: | $S_4 \times C_2$ | $\cong$ | $\langle h, p, q, r, s\rangle$ |
| Edge-stabilizer: | $(D_4 \times C_2){:}C_2$ | $\cong$ | $\langle a, p, q, r, s\rangle$ |
| 1-arc-stabilizer: | $D_4 \times C_2$ | $\cong$ | $\langle p, q, r, s\rangle$ |
| 2-arc-stabilizer: | $C_2 \times C_2 \times C_2$ | $\cong$ | $\langle p, q, r\rangle$ |
| 3-arc-stabilizer: | $C_2 \times C_2$ | $\cong$ | $\langle p, q\rangle$ |
| 4-arc-stabilizer: | $C_2$ | $\cong$ | $\langle p\rangle$ |

**Theorem** [MC, 1988] For all but finitely many $n$, both the alternating group $A_n$ and the symmetric group $S_n$ are the automorphism groups of 5-arc-transitive 3-valent graphs.

[So in particular, 5-arc-transitive cubic graphs are plentiful.]

How to prove this? Use coset diagrams for the group $G_5$ to construct permutation representations, with composition achieved by using 2-cycles of the edge-reversing generator $a$ to link together sub-orbits of the arc-stabilizer $A = \langle p, q, s, r \rangle$ between orbits of the vertex-stabilizer $V = \langle h, p, q, s, r \rangle$:



Theorems [Richard Weiss (1981)] There are no finite 8-arc-transitive graphs of valency $k > 2$, and if $X$ is a finite 7-arc-transitive graph, then its valency is $1 + 3^t$ for some $t$.

[Proof relies on the classification of doubly-transitive groups.]

Moreover, if $X$ is a finite connected 7-arc-transitive 4-valent graph, its automorphism group is a quotient of the group

$$R_{4,7} = \langle\, h, p, q, r, s, t, u, v, b \mid h^4 = p^3 = q^3 = r^3 = s^3 = t^3 = u^3 = v^2 = b^2 = (hu)^3 = (uv)^2 = (huv)^2 = [h^2, u] = [h^2, v] = [q, r] = [q, s] = [q, t] = [r, s] = [r, t] = [p, q] = [p, r] = [p, s] = [p, t] = 1,\ [s, t] = p,\ h^{-1}ph = p,\ h^{-1}qh = q^{-1}r,\ h^{-1}rh = qr,\ h^{-1}sh = pq^{-1}r^{-1}s^{-1}t^{-1},\ h^{-1}th = p^{-1}qr^{-1}s^{-1}t,\ u^{-1}pu = p,\ u^{-1}qu = q,\ u^{-1}ru = q^{-1}r,\ u^{-1}su = s,\ u^{-1}tu = pqrst,\ vpv = p^{-1},\ vqv = q^{-1},\ vrv = r,\ vsv = s,\ vtv = t^{-1},\ bpb = q^{-1},\ bqb = p^{-1},\ brb = s^{-1},\ bsb = r^{-1},\ btb = u^{-1},\ bub = t^{-1},\ bvb = v,\ bh^2b = h^2v \,\rangle.$$

**Theorem** [MC & Cameron Walker, 1998] For all but finitely many $n$, both the alternating group $A_n$ and the symmetric group $S_n$ are the automorphism groups of 7-arc-transitive 4-valent graphs.

[So in particular, 7-arc-transitive cubic graphs are plentiful.]

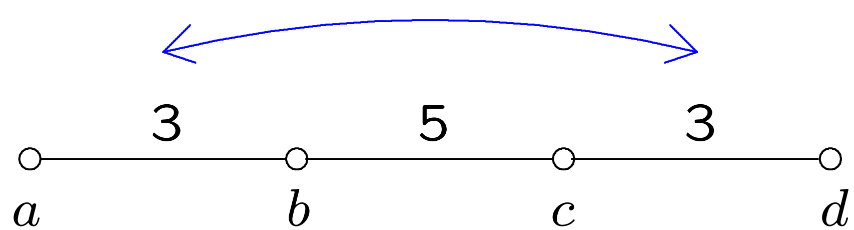
How to prove this? Use coset diagrams for the group $R_{4,7}$ to construct permutation representations, with composition achieved by using 2-cycles of the edge-reversing generator $b$ to link sub-orbits of the arc-stabilizer $A = \langle h^2, p, q, r, s, t, u, v \rangle$ between orbits of the vertex-stabilizer $V = \langle h, p, q, r, s, t, u, v \rangle$.

# Hyperbolic 3-manifolds

A hyperbolic $n$-manifold $M$ is a quotient space $H^n/K$, where $K$ is a torsion-free discrete subgroup of the group $\mathrm{Iso}^+(H^n)$ of orientation-preserving isometries of hyperbolic space $H^n$.

For each $n \geq 3$, there exists an upper bound on the quotient $|\mathrm{Iso}^+(M)|/\mathrm{vol}(M)$, and this bound is attained for some $M$.

By a recent theorem of Gehring, Marshall & Martin (2009), the largest value of this 'symmetry-to-volume' ratio occurs for the case $n = 3$ when the group $\mathrm{Iso}^+(M)$ is a finite smooth quotient of the orientation-preserving subgroup of the normaliser in $\mathrm{Iso}(H^3)$ of the $[3, 5, 3]$-Coxeter group:



The Coxeter group $[3,5,3]$ has four involutory generators $a,b,c,d$ subject to defining relations $a^2 = b^2 = c^2 = d^2 = (ab)^3 = (bc)^5 = (cd)^3 = (ac)^2 = (ad)^2 = (bd)^2 = 1$, and its normalizer in $\mathrm{Iso}(H^3)$ is obtained by adding a new involutory generator $t$ that conjugates $(a,b,c,d)$ to $(d,c,b,a)$, thereby reversing the Dynkin diagram. Call this group $[3,5,3]\!:\!2$.

For largest symmetry-to-volume ratio, the group $\mathrm{Iso}^+(M)$ must be a finite smooth quotient of the subgroup generated by $ab, bc, da$ and $t$. Call this group $[3,5,3]^{\circ}\!:\!2$.

This group $[3,5,3]^{\circ}\!:\!2$ is the 3-dimensional analogue of the ordinary (2,3,7)-triangle group (from the 2-manifold case).

**Theorem** [MC & Anna Torstensson, 2003] For all but finitely many $n$, both the alternating group $A_n$ and the symmetric group $S_n$ are the symmetry groups of compact hyperbolic 3-manifolds with largest possible symmetry to volume ratio. In fact, all but finitely many $A_n$ and $S_n$ are smooth quotients of both $[3,5,3]^{\mathrm{o}}\!:\!2$ and $[3,5,3]\!:\!2$.

The proof uses coset diagrams for the group $[3,5,3]\!:\!2$, but with a slightly different method of composition, involving sets of 2-cycles of the 'reflecting' generator $t$.

## Locally $s$-arc-transitive edge-transitive graphs

A graph $X$ is said to be locally $s$-arc-transitive if the stabilizer in Aut $X$ of every vertex $v$ is transitive on all the $s$-arcs emanating from $v$.

If $X$ is also vertex-transitive, then $X$ is $s$-arc-transitive. But if $X$ is edge-transitive but not vertex-transitive, then Aut $X$ has two orbits on vertices, say $U$ and $V$, and hence $X$ is bipartite, with parts $U$ and $V$.

**Theorem** [Stellmacher, 1996] If the finite connected edge-transitive graph $X$ is locally $s$-arc-transitive, then $s \leq 9$.

Until July this year, the only known graphs meeting this bound were generalised octagons associated with the Ree groups ${}^2F_4(2^{2n+1})$ and certain covers of these.

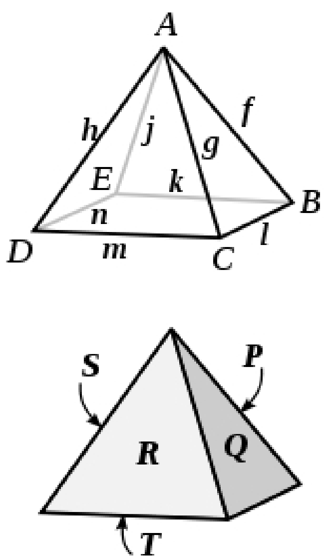
**Theorem** [MC, July 2011] For all but finitely many $n$, the alternating group $A_n$ is the automorphism group of an edge-transitive, locally 9-arc-transitive bipartite graph (with vertices of valency 3 in one part and 5 in the other).

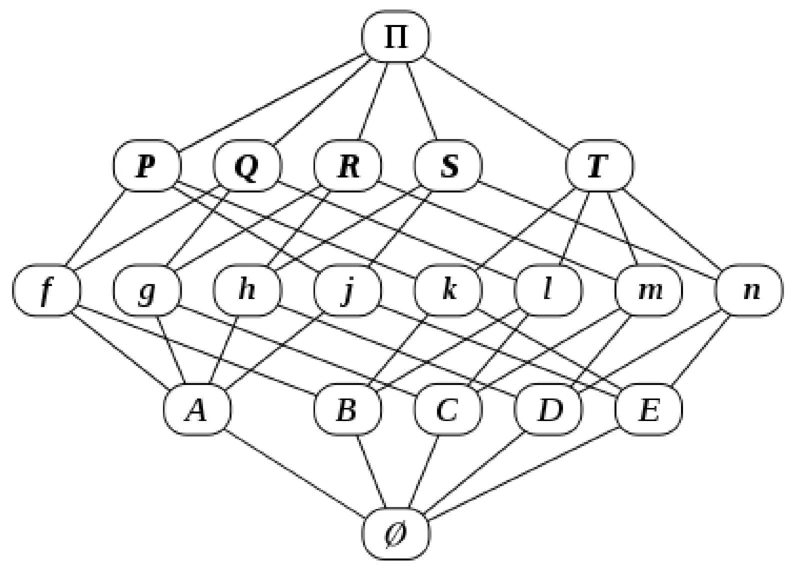
The proof uses the fact that the Ree group ${}^2F_4(2)$ (of order 35,942,400) is a product of subgroups $A$ and $B$ of orders 12288 and 20480 with intersection $C = A \cap B$ of order 4096 having index 3 in $A$ and 5 in $B$. These three subgroups act as the stabilizers of two (adjacent) vertices $u$ and $v$ and the edge $\{u, v\}$, respectively.

The rest of the proof involves constructing transitive permutation representations of the free product $A *_C B$, using coset diagrams for each of $A$ and $B$, linked together using sub-orbits of the intersection $C = A \cap B$.

# Regular and chiral polytopes

Abstract polytopes are generalisations of geometric structures that can be viewed as a partially ordered set:




For the most regular of these, the symmetry group is always an image of a Coxeter group.

**Theorem** [Fernandes & Leemans, 2011] For every $n > 3$, the symmetric group $S_n$ is the automorphism group of some regular abstract $r$-polytope, for each $r$ such that $3 \leq r \leq n-1$.

**Corollary**: For any given $r \geq 3$, all but finitely many $S_n$ are the automorphism group of a regular polytope of rank $r$.

## Chiral polytopes

In an abstract polytope, a flag is a maximal chain (of mutually incident elements), and two flags are said to be adjacent if they differ in just one element. The polytope $\mathcal{P}$ said to be chiral if any two adjacent flags lie in different orbits of the automorphism group of $\mathcal{P}$. If $\mathcal{P}$ is maximally chiral (so that $\mathrm{Aut}\,\mathcal{P}$ has just two orbits on flags, with adjacent flags in different orbits), then $\mathrm{Aut}\,\mathcal{P}$ is a smooth quotient of the orientation-preserving subgroup of some Coxeter group.

The first known finite (maximally) chiral polytopes of rank greater than 4 were discovered only recently: rank 5 by Conder/Hubard/Pisanski, and ranks 6,7,8 by Conder/Devillers).

**Theorem** [Daniel Pellicer, 2010] There exist (maximally) chiral polytopes of rank $r$ for all $r \geq 3$.

**Conjecture** [MC, 2011] For any given $r \geq 3$, all but finitely many of the alternating groups $A_n$ and symmetric groups $S_n$ are the automorphism group of a (maximally) chiral polytope of rank $r$.

It's likely this is provable by constructing transitive permutation representations of the group $[3, 3, \ldots, 3, k]^{o}$ from coset diagrams for the subgroup $[3, 3, \ldots, 3]^{o}$ (isomorphic to $A_r$).

**Thank You!**