## A Family of Unsatisfying Graphs

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The tiles of the game Tantrix are hexagons with colored paths running from edge to edge. Permuting the edges of such a hexagon permutes the endpoints of the paths; this action gives rise to a natural correspondence between Tantrix tiles and the vertices of a graph. Different ways of defining this graph and different choices of tile size and path colors give rise to a plethora of graphs. In this talk we pose the question "Are these graphs worth studying?"

## Tantrix: The Real Thing

Tantrix is a game like dominoes, played by matching colored lines on hexagonal tiles.

http://www.tantrix.com/
http://www.tantrix.com/english/TantrixTiles.html

## Idealized Tiles

Our tantrix tiles differ from Tantrix tiles as follows.

- Allow a fixed number $m$ of path colors.
- Include tiles with all pairs of opposite sides joined by paths.
- Tiles are $2 n$-gonal ( $n$ fixed).


## Tantrix Graph

Our tiles form the vertices of a graph.

- Each tile corresponds to a vertex.
- Vertices are joined by an edge if the corresponding tiles are "the same" on all sides except two adjacent ones. (Think of swapping the ends of two paths.)

Unless otherwise specified, we:

- Ignore over- and under-crossings of paths, attending only to which sides the paths are incident with.
- Consider tiles that differ only by rotation to be identical.
- Consider tiles that differ only by reflection to be different.
- Consider colored paths and graph edges to be undirected.
- Allow at most one edge between any pair of tiles.


## Monochromatic

If we choose $n=3$ and $m=1$ we get the graph with five vertices shown below. The mirror symmetry of the graph seems coincidental.


## Hexagonal Tiles, Three Colors

When $m=n=3$ and we consider mirror images to be equivalent, we get the graph shown below. Note the 3-fold symmetry permuting the colors. The graph is almost, but not quite, 3-regular.


## Directed Edges, Degree and Loops

In the graph below, $m=n=2$. Vertex $a$ is connected to tile $b$ by a directed edge if tile $b$ can be obtained by swapping two sides of tile $a$. The degree of an edge is the number of possible swaps transforming one tile into the other.


## Introducing the Permutahedron

The $k$-permutahedron is a polytope whose vertices correspond to the permutations of ( $123 \ldots k$ ). Two vertices are joined by an edge if the corresponding permutations differ by an adjacent transposition.


## Relating Tantrix to the Permutahedron

By arranging the numbers ( $12 \ldots 2 n$ ) about the sides of an $2 n$-gon, we get a correspondence between the $2 n$-gonal tantrix tiles and the vertices of the permutahedron. The resulting tantrix graph has the edge graph of the permutahedron as a subgraph. Edges in the tantrix graph corresponding to transposing the first and last side of the $2 n$-gon are not represented in the permutahedron.

## $n=m=2$

We aligned the two-colored, square tantrix tiles with the vertices of the three dimensional permutahedron.
The permutahedron is a zonotope, and in some cases the tiles on opposite sides of a zone look the same. (The adjacent transposition corresponding to the edges along the zone transpose two ends of the same path.) By identifying pairs of vertices along the zones, we obtained a polytope with 24 vertices whose graph was a sort of tantrix graph. There were many duplicated tiles among those vertices.


## Why that Relationship is Unsatisfying

The $2 n$-permutahedron has $(2 n)$ ! vertices.
There are something like $\frac{(2 n)!}{2^{n}} 2 n$-sided tantrix tiles.
In addition, some edges of the tantrix graph are not edges of the permutahedron.

## Remarks on Symmetry

Some tantrix graphs are symmetric.

- If $m$ is the number of path colors, there is a natural action of $S_{m}$ on the tantrix graph.
- If tiles are not equivalent to their reflections, reflecting each tile in the graph sends the graph to itself.
- A similar statement holds for rotations of tiles.

We may define the $2 n$-gonal tantrix tiles to be the orbits of the circular arrangements of $1,1,2,2,3,3, \ldots, n, n$ under the action of the symmetric group.

## Conjectures

- Is it true that for any $m$ there is some zone of the $2 m-1$ permutahedron whose edges all connect identical tiles? Conjecture: No.
- Can we get an interesting tantrix graph by shrinking edges of the $2 m-1$ permutahedron?
Conjecture: No.
- Which tantrix graphs are Hamiltonian? Conjecture: Most of them.


## Open Questions

- Is there a family of tantrix graphs which are the edge graphs of a family of polytopes?
- Which tantrix graphs are planar?
- Do we get an intersting tantrix graph if we consider orientations or over- and under-lappings of paths on tiles?
- What can we say about graphs based only on tiles present in the game Tantrix?
- Which tantrix graphs are Eulerian?
- Is there a nicely symmetric embedding of an $m$ colored tantrix graph in $R^{m-1}$ ?


## Counting Tiles

An ugly case-by-case argument gives the following formula for the number of $2 m$-gonal Tantrix tiles:

$$
\underbrace{(m-1) \frac{(2 m-2)!}{2^{m-1}}}_{\text {1's on same "side" }}+\underbrace{\frac{1}{2} \frac{\overbrace{(2 m-2)!}^{2^{m-1}}}{\text { ignore symmetry }}+\overbrace{\frac{1}{2}(m-1)!}^{\text {symmetric }}}_{\text {1's opposite each other }}
$$

## Contact Info

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