Classical groups acting on polytopes

Peter Brooksbank

Bucknell University

Symmetry in Graphs, Maps and Polytopes

Fields Institute

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Motivation

The principal motivation for this work was the paper:

D. Leemans and E. Schulte, *Groups of type $L_2(q)$ acting on polytopes*, Adv. Geometry (2007)

The paper determines the prime powers q such that $\mathrm{PSL}(2, q)$ is the automorphism group of an abstract regular polytope of rank at least 4.

The method uses the very detailed description of the subgroup structure of $\mathrm{PSL}(2, q)$ given originally by Dickson in 1901.

Which members of other families of finite simple groups of Lie type arise as the automorphism group of an abstract regular polytope?

Existing Results

- $\mathrm{PSL}(2, q) \cong \mathrm{Aut}(\mathcal{P})$, where $\mathcal{P}$ has...

  ...**rank 3** if and only if $q \notin \{2, 3, 7, 9\}$ [Sjerve-Cherkassoff]

  ...**rank 4** if and only if $q \in \{11, 19\}$ [Leemans-Schulte]

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  ...**rank 4** if and only if $q \in \{11, 19\}$ [Leemans-Schulte]
- $Sz(q) \cong \mathrm{Aut}(\mathcal{P})$, where $\mathcal{P}$ has **rank 3**. [Leemans]
- All simple groups of Lie type that arise as the automorphism group of a polytope of **rank 3** are known. [Nuzhin]

The Objective

Let $G(q)$ be a quasi-simple **classical group** defined naturally on a vector space V of dimension d over the field $k = \mathrm{GF}(q)$.

To what extent can the geometric properties of $\mathrm{G}(q)$ *be exploited to say something useful about the polytopes upon which it can act?*

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To what extent can the geometric properties of $G(q)$ be exploited to say something useful about the polytopes upon which it can act?

In this talk, I will discuss the case when $G(q) \leqslant \mathrm{GL}(3, q)$.

This is joint work with a former student, Deborah Vicinsky.

String C-groups

A string C-group is a group H together with a generating sequence $\rho_0, \rho_1, \ldots, \rho_{c-1}$ of involutions satisfying the following conditions:

- For $0 \leqslant i < j \leqslant c-1$, $[\rho_i, \rho_j] = 1$ if and only if $|i-j| > 1$.
- **Intersection Property:** For $I, J \subseteq \{0, \ldots, c-1\}$,

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_j : j \in J \rangle = \langle \rho_k : k \in I \cap J \rangle.$$

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Key fact: $H = \langle \rho_0, \ldots, \rho_{c-1} \rangle$ is a string C-group if and only if $H \cong \mathrm{Aut}(\mathcal{P})$ for some abstract regular polytope $\mathcal{P}$ of rank c.

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- $\mathrm{Isom}(\varphi) = \{g \in \mathrm{GL}(\varphi) : g \text{ is an isometry of } \varphi\}$.
- $H \leqslant \mathrm{GL}(V)$ is a classical group if $\mathrm{Isom}(\varphi)' \leqslant H \leqslant \mathrm{Isom}(\varphi)$.

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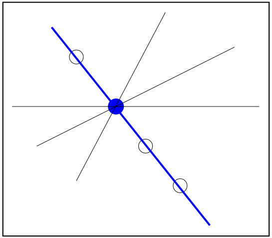
- $H \leqslant \mathrm{GL}(V)$ is a classical group if $\mathrm{Isom}(\varphi)' \leqslant H \leqslant \mathrm{Isom}(\varphi)$.
- Fixing a basis $v_1, \ldots, v_d$ of V one associates to φ the matrix $\Phi = [[\varphi(v_i, v_j)]] \in \mathbb{M}_d(k)$. Then g is an isometry if $g\Phi = \Phi g^{-\mathrm{tr}}$.

Involutions in $\mathrm{SL}(3, q)$, q even



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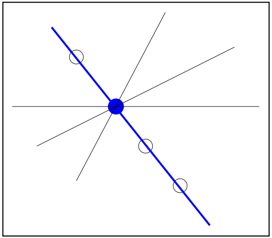

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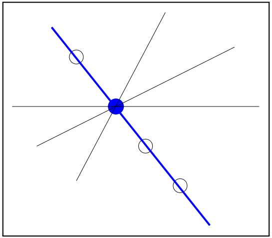

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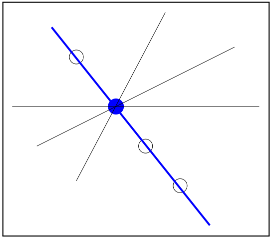

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$$\rho \text{ is conjugate to } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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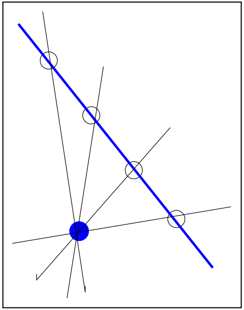

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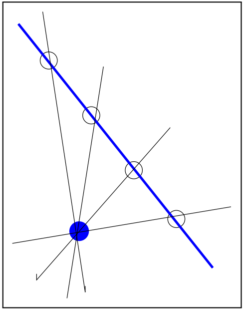

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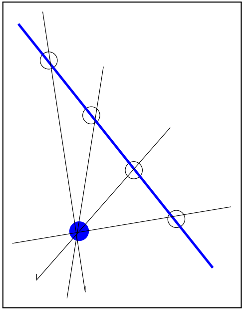

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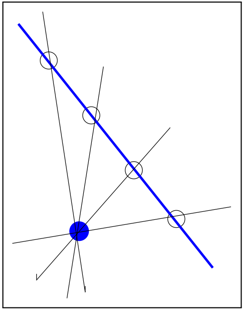

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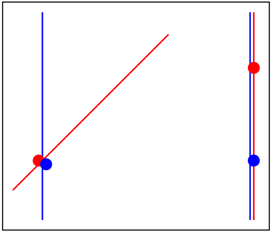

If σ is an involution with center γ and axis Γ, then:

$[\rho, \sigma] = 1 \iff \gamma = \delta$ or $\Gamma = \Delta$.

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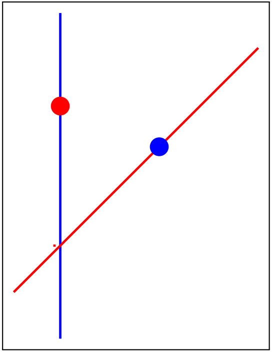

If σ is an involution with center γ and axis Γ, then:

$[\rho, \sigma] = 1 \Leftrightarrow \gamma \in \Delta$ and $\delta \in \Gamma$.

Bounding the rank of a string C-group

Lemma:
If $G = \langle \rho_0, \ldots, \rho_{c-1} \rangle \leqslant \mathrm{SL}(3, q)$ *is a string* C*-group, then* $c \leqslant 4$.

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We just consider the (more interesting) case where q is odd.
For $i = 0, \dots, c-1$, let ρ_i have center δ_i and axis Δ_i.

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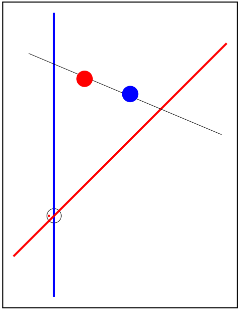
There are two cases to deal with:

[generic case] Here $\delta_0 \neq \delta_1$ and $\Delta_0 \neq \Delta_1$.

[special case] Either $\delta_0 = \delta_1$ or $\Delta_0 = \Delta_1$. *Note: by duality, we need only look at one of these possibilities.*

Proof of Lemma (cont'd)

The generic case: $\boldsymbol{\delta_0 \neq \delta_1}$ and $\boldsymbol{\Delta_0 \neq \Delta_1}$.



Suppose that ρ is any involution commuting with both ρ_0 and ρ_1. Let ρ have center δ and axis Δ.

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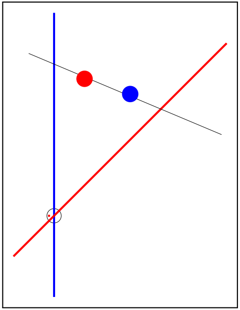

Suppose that ρ is any involution commuting with both ρ_0 and ρ_1. Let ρ have center δ and axis Δ.

Since both δ_0 and δ_1 lie on Δ, and δ lies on both Δ_0 and Δ_1, it follows that $\Delta = \langle \delta_0, \delta_1 \rangle$ and $\delta = \Delta_0 \cap \Delta_1$.

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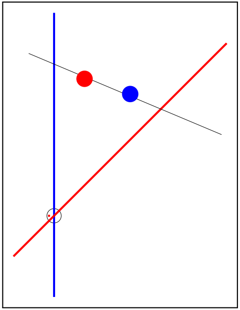

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This uniquely determines ρ, so $c \leqslant 4$.

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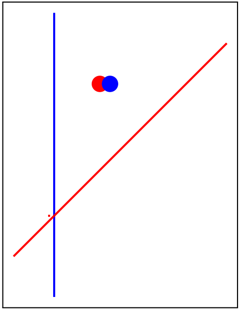

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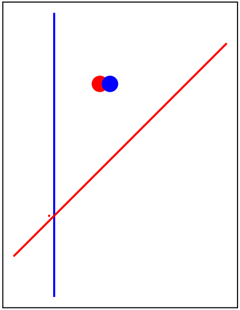

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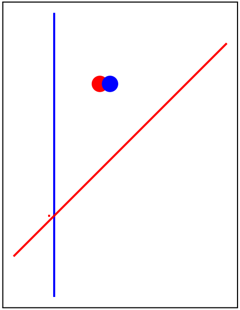

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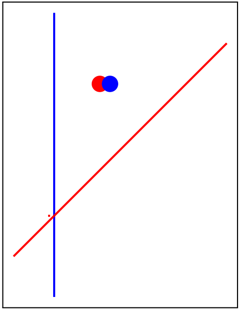

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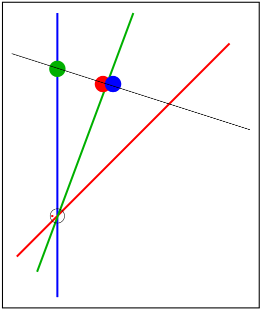

Choose basis e_1, e_2, e_3 for V:
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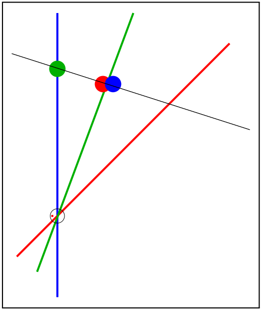

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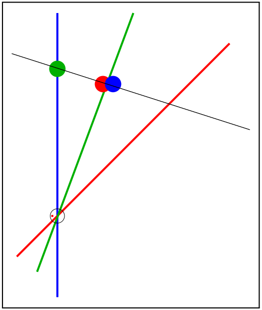

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$(\rho_0\rho_1)^2 = (\rho_2\rho_1)^2$ is a nontrivial transvection. Hence $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$ contains an element of order $p > 2$.

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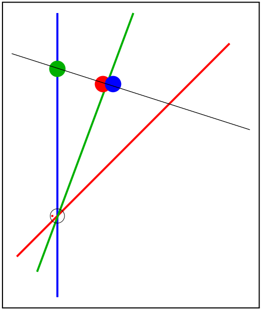

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This contradicts the intersection property, so $c < 5$.

Irreducible string C-subgroups of $\mathrm{GL}(3, q)$

Theorem [B & Vicinsky]

Let $H = \langle \rho_0, \ldots, \rho_{c-1} \rangle \leqslant \mathrm{GL}(3, q)$ be a string C-group acting irreducibly on $V = \mathbb{F}_q^3$. Then q is odd, and H preserves a nondegenerate, symmetric bilinear form φ on V.

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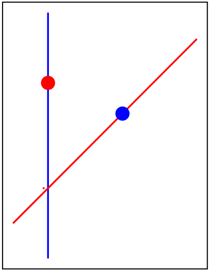

$\mathbb{M}$ = matrix algebra of degree 3 over $k = \mathbb{F}_q$
$\mathbb{S}$ = subspace of symmetric matrices (dim 6)
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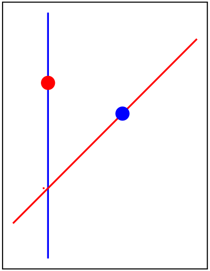

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Choose a basis of V such that

$$\rho_0 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \text{ and } \rho_2 = \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}.$$

Observe $C_{\mathbb{M}}(\rho_0) \cap C_{\mathbb{M}}(\rho_2) = \mathbb{D}$

Proof of Theorem (cont'd)

Let $\Phi \in \mathbb{S}$ represent some symmetric form φ.

$$\begin{aligned}
\langle \rho_0, \rho_2 \rangle \text{ preserves } \Phi \quad & \Longleftrightarrow \quad \rho_i \Phi = \Phi \rho_i^{-\mathrm{tr}} \text{ for } i = 0, 2 \\
& \Longleftrightarrow \quad \rho_i \Phi = \Phi \rho_i \text{ for } i = 0, 2 \\
& \Longleftrightarrow \quad \Phi \in C_{\mathbb{M}}(\rho_0) \cap C_{\mathbb{M}}(\rho_2) \cap \mathbb{S} \\
& \Longleftrightarrow \quad \Phi \in \mathbb{D}
\end{aligned}$$

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Consider the rank 3 case. (The result for rank 4 follows easily.)

Let ρ be any involution of $\mathrm{SL}(3, q)$. Define

$$\mathbb{X}_\rho = \{\Phi \in \mathbb{S} \colon \rho\Phi = \Phi\rho^{-\mathrm{tr}} = \Phi\rho^{\mathrm{tr}}\},$$

the matrices representing symmetric bilinear forms preserved by ρ.

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- $\dim(\mathbb{X}_\rho \cap \mathbb{D}) \geqslant 1$.

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the matrices representing symmetric bilinear forms preserved by ρ.

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H preserves a nondegenerate, symmetric bilinear form φ on V. $\square$

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Thank You!