Stability of polyhedra

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Definition:
Stable face: projection of the mass center onto the plane of a particular face is not outside of the face.

Thm: Every tetrahedron is stable on at least two faces.

Proof:

Let the faces be indexed by 1, 2, 3, 4 so that

\[ d_1 \leq d_2 \leq d_3 \leq d_4 \]

where \( d_i \) is the distance between M and the plane of the ith face.
Two-tip Tetrahedron

A. Heppes (1967)
Gömböc (2006) : a mono monostatic body by G. Domokos and P. Várkonyi

Photo:2010
Skeletal densities:

- body density: $\delta_V$
- face density: $\delta_F$
- edge density: $\delta_E$
Comments on M. Goldberg (1967):
‘ Every tetrahedron has at least 2 stable faces paper: R. Dawson mentions incompleteness in 84, refers to J. Conway.

A.B. (2011):
Every tetrahedron with:

- uniform body density $\delta V$
- uniform face density $\delta F$
- uniform edge density $\delta E$

has at least two stable faces.
Volume: $V$  
Surface area: $F$  
Total edge length: $E$  

body density: $\delta_V$  
face density: $\delta_F$  
edge density: $\delta_E$  

faces: $1, \ldots, 4$  
face and its area: $a_i$  
face perimeter: $p_i$  

$$d(M_V, a_i) = \frac{3}{4} \frac{V}{a_i}$$

$$d(M_F, a_i) = \frac{V}{a_i} \frac{F-a_i}{F}$$

$$d(M_E, a_i) = \frac{3}{2} \frac{V}{a_i} \frac{E-p_i}{E}$$

$$x = \frac{F-a}{F} L$$

$$d(M, a_i) = \frac{\delta_V V \frac{3}{4} \frac{V}{a_i} + \delta_F F \frac{V}{a_i} \frac{F-a_i}{F} + \delta_E E \frac{3}{2} \frac{V}{a_i} \frac{E-p_i}{E}}{\delta_V V + \delta_F F + \delta_E E}$$
Questions from 1967:

• Are there polyhedra with exactly one stable face?

• If yes what is the smallest possible face number of such polyhedra?

• Is it 4?
A 19 faceted polyhedron which has exactly one stable face.
Gömböc (2006) : a mono monostatic body by G. Domokos and P. Várkonyi

Through the media people were told that:

A 3D shape made out of homogeneous material, which rolls back to the same position, just like the loaded toy called ‘stand up kid’.
G. Domokos and P. Varkonyi
Is the sliced tube just as good as the Gömböc?
Types of stabilities:

Stable equilibrium
Unstable equilibrium
Additional balance points

The distance function measured from the mass center has:

Local minima
Local maxima
Saddle point
The number of stable equilibria: \( s \)

The number of unstable equilibria: \( u \)

The number of other balanced equilibria: \( t \)

Euler type formula holds:
\[
s + u - t = 2
\]
\[
1 + 2 - 1 = 2
\]

Arnolds question: Is there a shape for which one satisfies the Euler type formula with
\[
1 + 1 - 0 = 2 ?
\]
Gömböc: a mono monostatic body by G. Domokos & P. Várkonyi

# of stable equilibria : \( s = 1 \)
# of unstable equilibria : \( u = 1 \)
# of other balanced equilibria: \( t = 0 \)
A 19 faceted polyhedron which has exactly one stable face.

Spiral of \(N\) segments \(N = 6\)
It was believed that:
19 is the smallest face number of uni stable polyhedra.

A.B. (2011)
There is a uni stable polyhedron with 18 faces. One can modify this polyhedron by adding 3 faces so that the stable face has arbitrary small diameter.