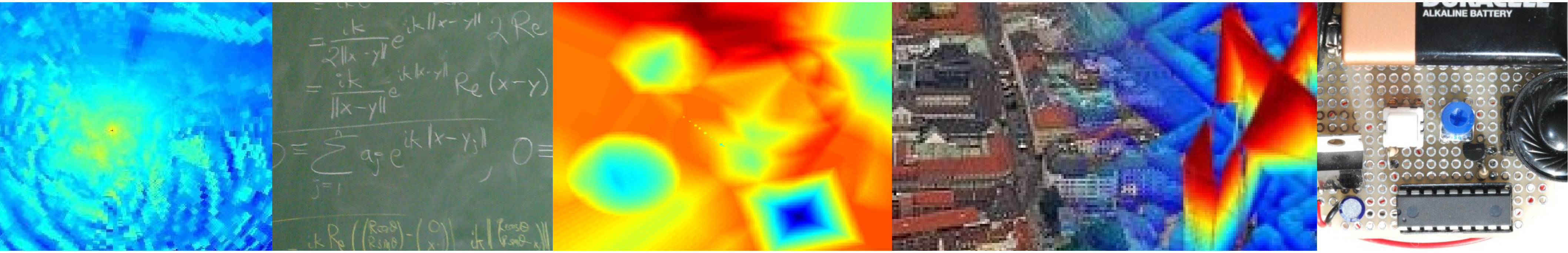
Euler Integral Transforms and Applications



Michael Robinson

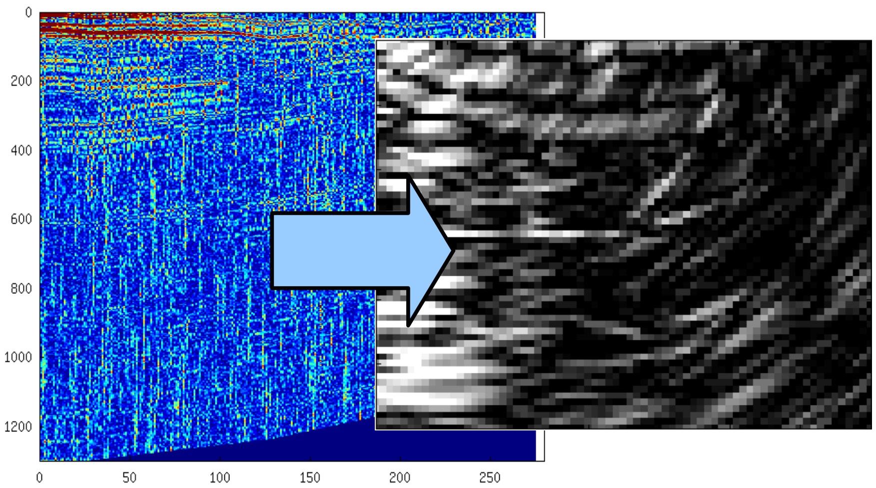
Penn

Acknowledgements

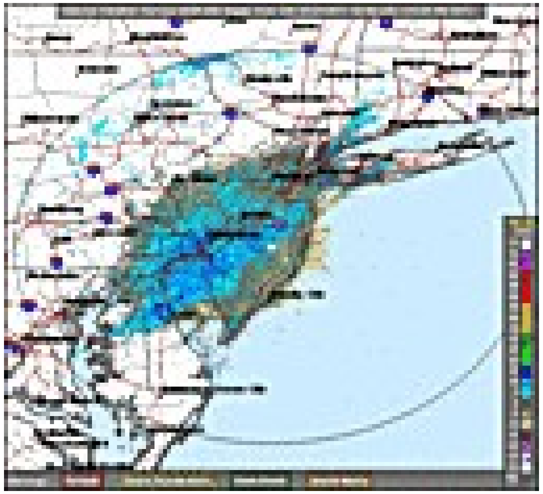
- Collaborators:
  - Robert Ghrist (Penn)
  - Sam Krupa (Penn)
  - Yuliy Baryshnikov (UIUC)
- Funding by:
  - DARPA # HR0011-07-1-0002
  - ONR # N000140810668
- Website reference:

http://www.math.upenn.edu/~robim/

Signal processing



(image courtesy of Fir0002/Flagstaffotos)



(image courtesy of NOAA)

(image courtesy of FAA)

(image courtesy of www.pixeleye.com)

(image courtesy of www.mgef.org)

Topological signal processing

- Topological data analysis
  - Gunnar Carlsson (Stanford)
  - Many others
- Persistent cohomology (especially circular coordinates)
  - Vin daSilva (Pomona)
  - Mikael Vejdemo-Johanssen (Stanford)
- Euler calculus
  - Robert Ghrist (Penn)
  - Yuliy Baryshnikov (UIUC)
- Statistical topology
  - Shmuel Weinberger (Chicago)
  - Robert Adler (Technion)

This is a non-exhaustive list
Mostly, it shows my proclivities toward addressing data from actual systems

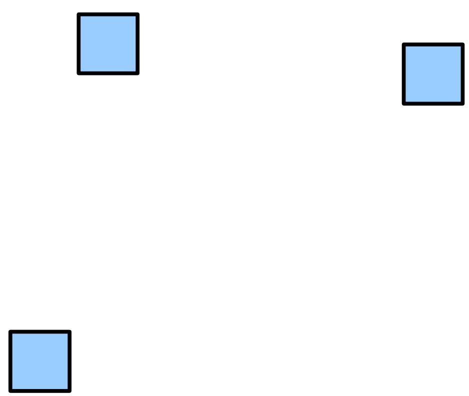
Problem statement

- Task: Develop filters that can **localize centers** and **discriminate shapes** of targets given a dense field of sensors that return **anonymous integer counts** of detected targets in their vicinity
- Assume:
  - Sensors are distributed evenly in the plane
  - Sensors have no knowledge of their absolute position
  - They only know about their position relative to other nearby sensors
  - Therefore, we look for methods based on **topological invariants**

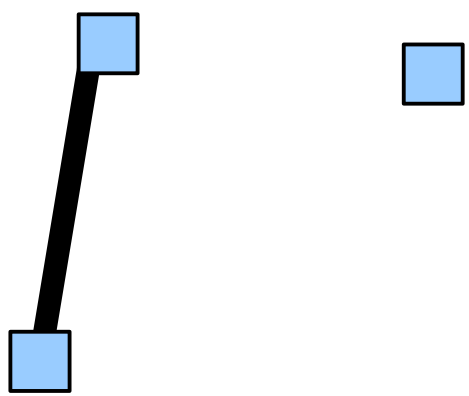
Constructible functions

- We consider exclusively constructible integer-valued functions $CF(\mathbb{R}^n;\mathbb{Z})$
  - Roughly, their graphs have a finite cell decomposition
- Constructible sets have an Euler characteristic
  - For a constructible set A, its Euler characteristic χ is the sum $\sum(-1)^{\dim(C_i)}$ where $\cup C_i = A$ is its cellular decomposition
  - It is a homeomorphism invariant
- The Euler characteristic generalizes counting...

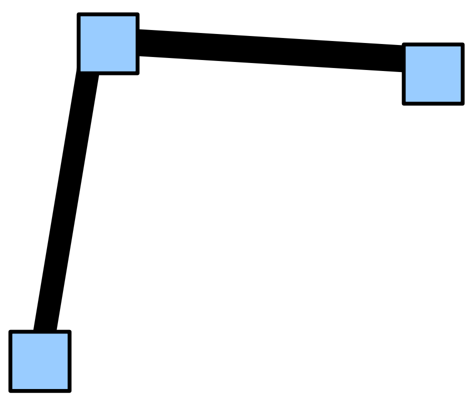
Euler characteristic

$\chi($  $) = 3$

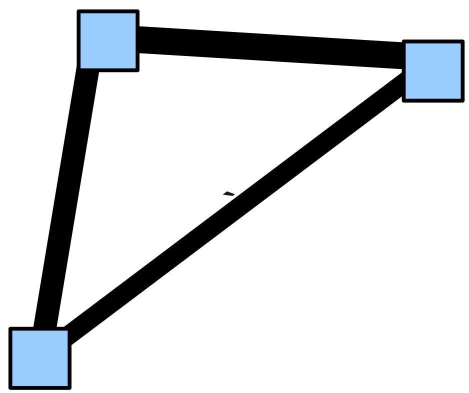
Euler characteristic

$\chi($  $) = 2$

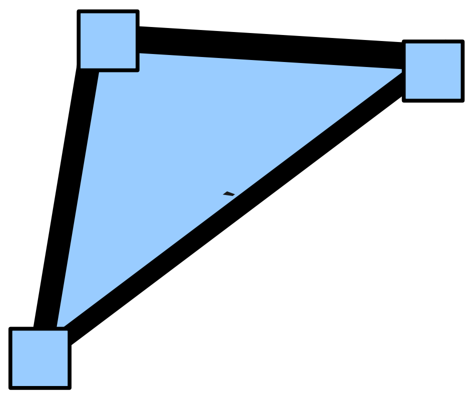
Euler characteristic

$\chi($  $) = 1$

Euler characteristic

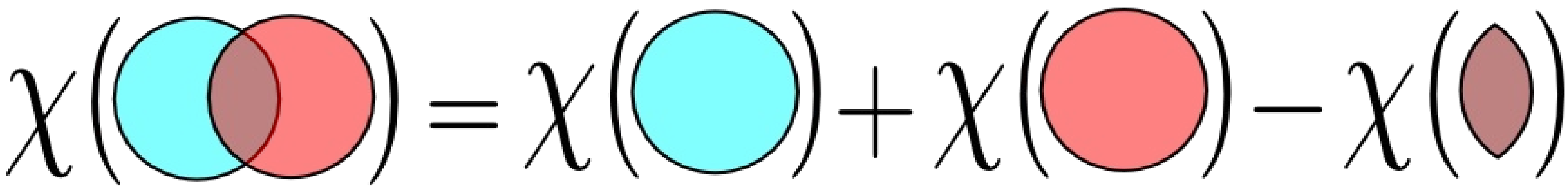
$\chi($  $) = 0$

Euler characteristic

$\chi($  $) = 1$

The Euler measure

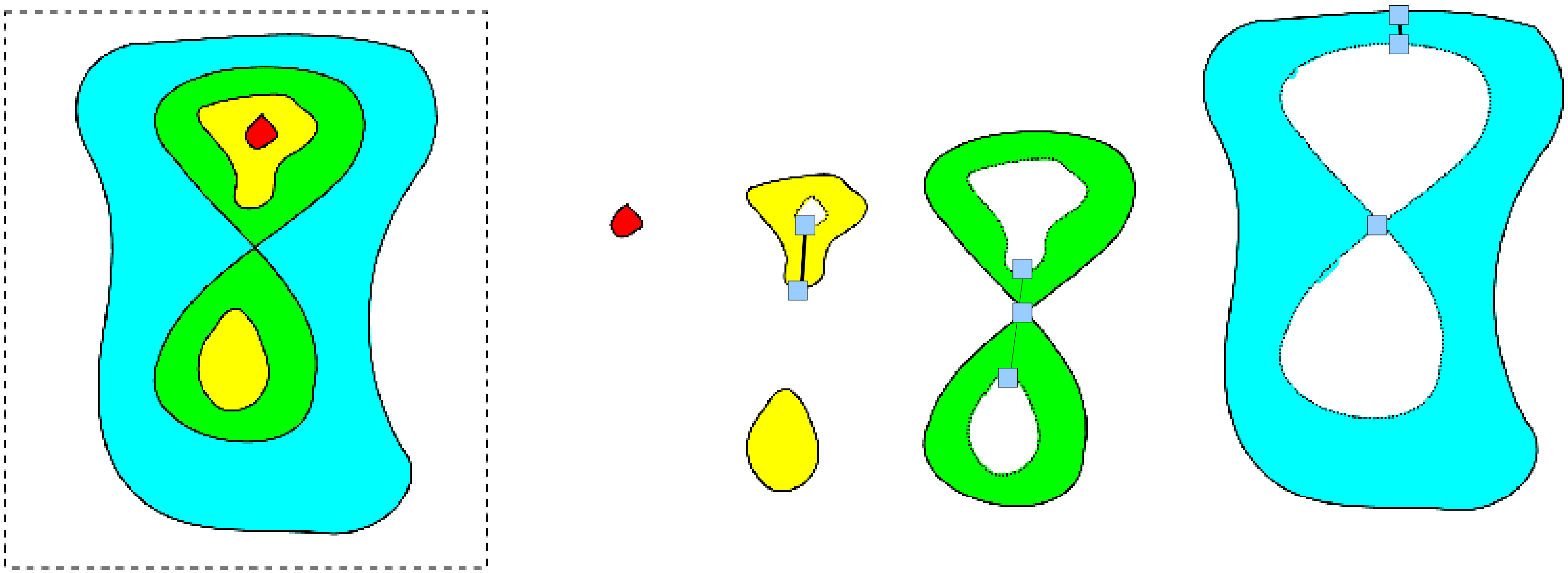
- An important property of the Euler characteristic is that it is a valuation
- In particular, it satisfies an inclusion-exclusion principle



- This lets us define an integration theory based around the Euler characteristic as a measure

$$\int h \, d\chi = \sum_{s=-\infty}^{\infty} s\chi(h^{-1}(s))$$

The Euler integral

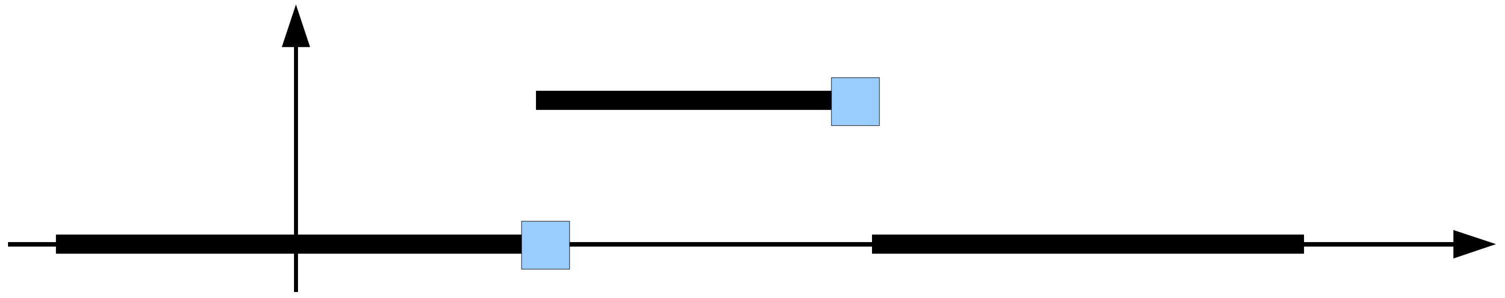


Euler characteristic	χ	1	1	-1	0
Function value	h	4	3	2	1

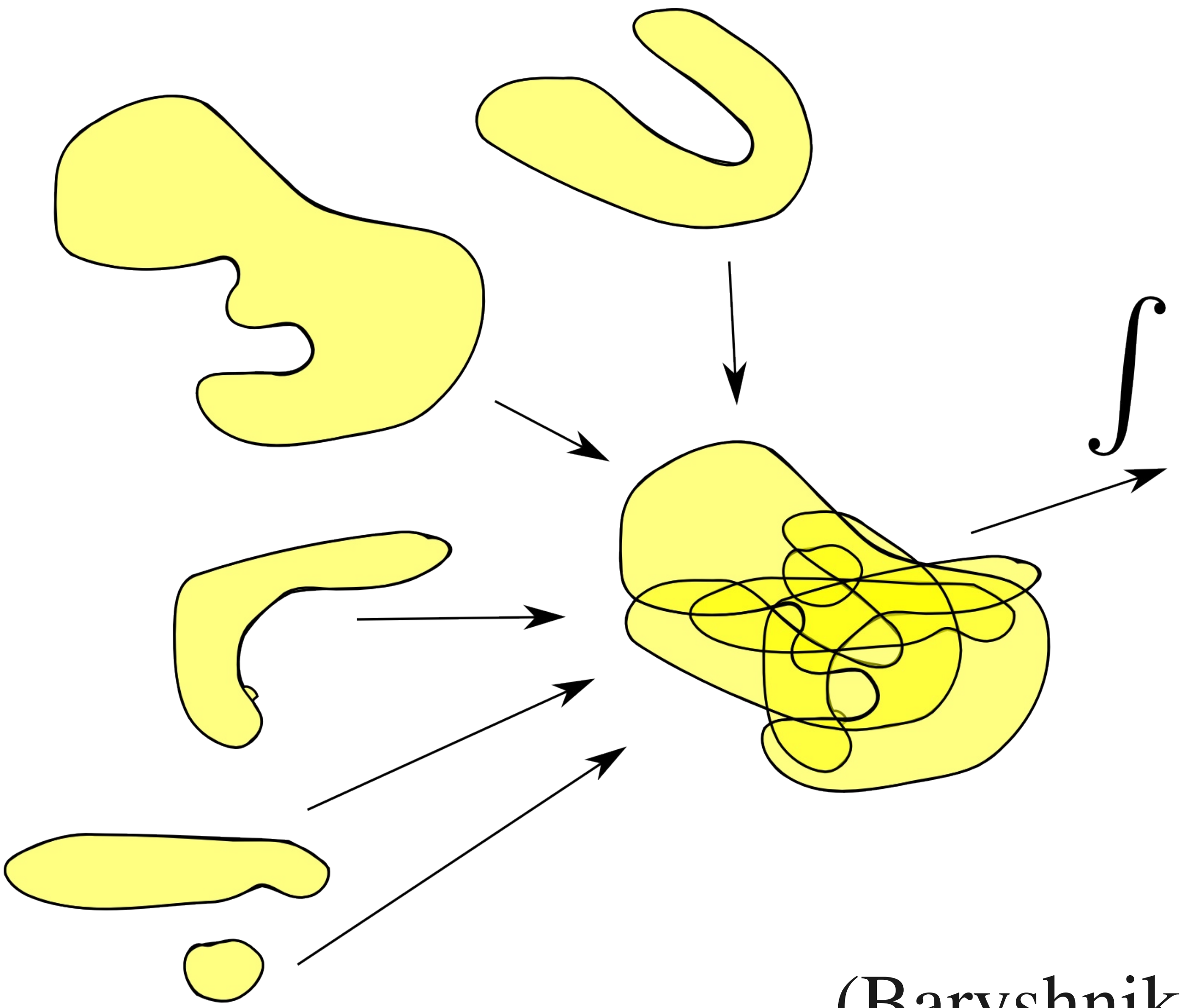
$$\int h \, d\chi = 4 + 3 - 2 + 0 = 5$$

Properties of the Euler integral

- It is linear over $\mathrm{CF}(\mathbb{R}^n;\mathbb{Z})$
  - This is false for a very important reason if one extends to definable functions (c.f. Baryshnikov & Ghrist 2010)
- It is finitely summable
- It is not monotonic
- It has nontrivial measure-zero sets



Target enumeration



$$\int h \, d\chi = 5$$

(Baryshnikov & Ghrist 2007)

Discretization analysis

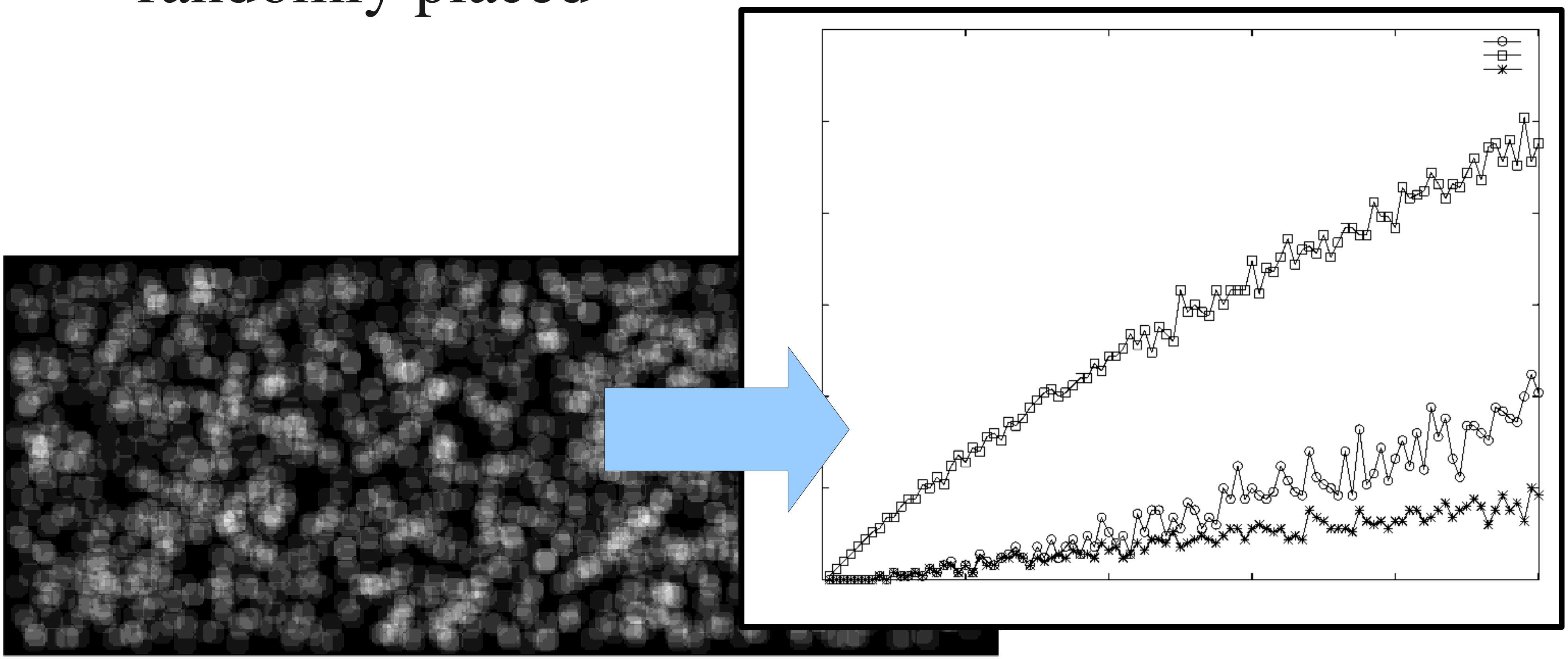
- What if there isn't a sensor at every point?
- Discretization errors are not straightforward
  - This is a *completely open* field!

radius	Error Type 0	Error Type 1	Error Type 3	radius	Error Type 0	Error Type 1	Error Type 3
1	0	8	0	51	168	584	80
2	0	24	0	52	152	600	88
3	0	40	0	53	112	584	96
4	0	56	0	54	136	576	88
[illegible]	0	72	0	55	120	632	80
6	0	88	0	56	200	600	96
7	0	104	0	57	176	632	112
8	8	112	8	58	248	632	112

(joint with Sam Krupa)

Discretization analysis

- Statisical analysis of asymptotics of the Euler characteristic integral for many identical targets, randomly placed

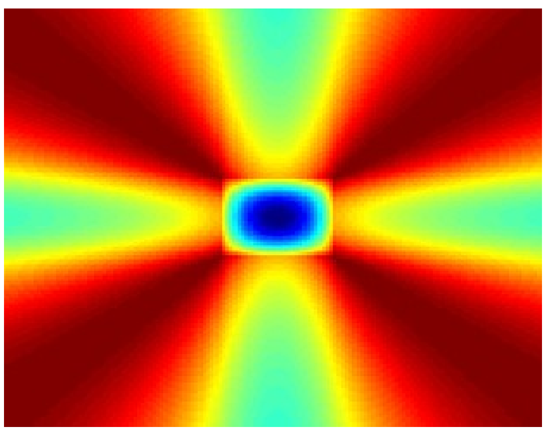


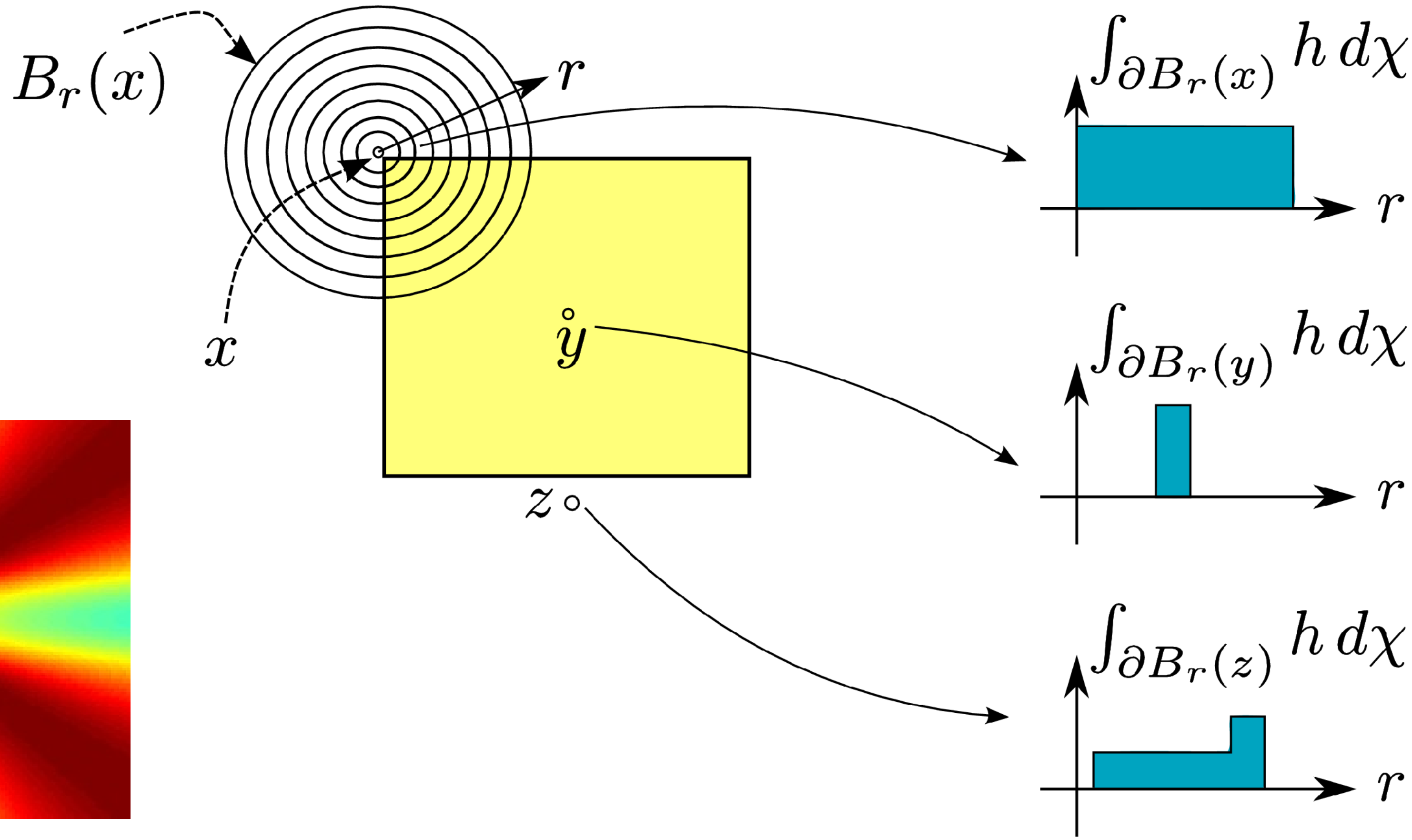
Euler integral transforms

Penn
UNIVERSITY of PENNSYLVANIA

Michael Robinson

The Euler-Bessel transform

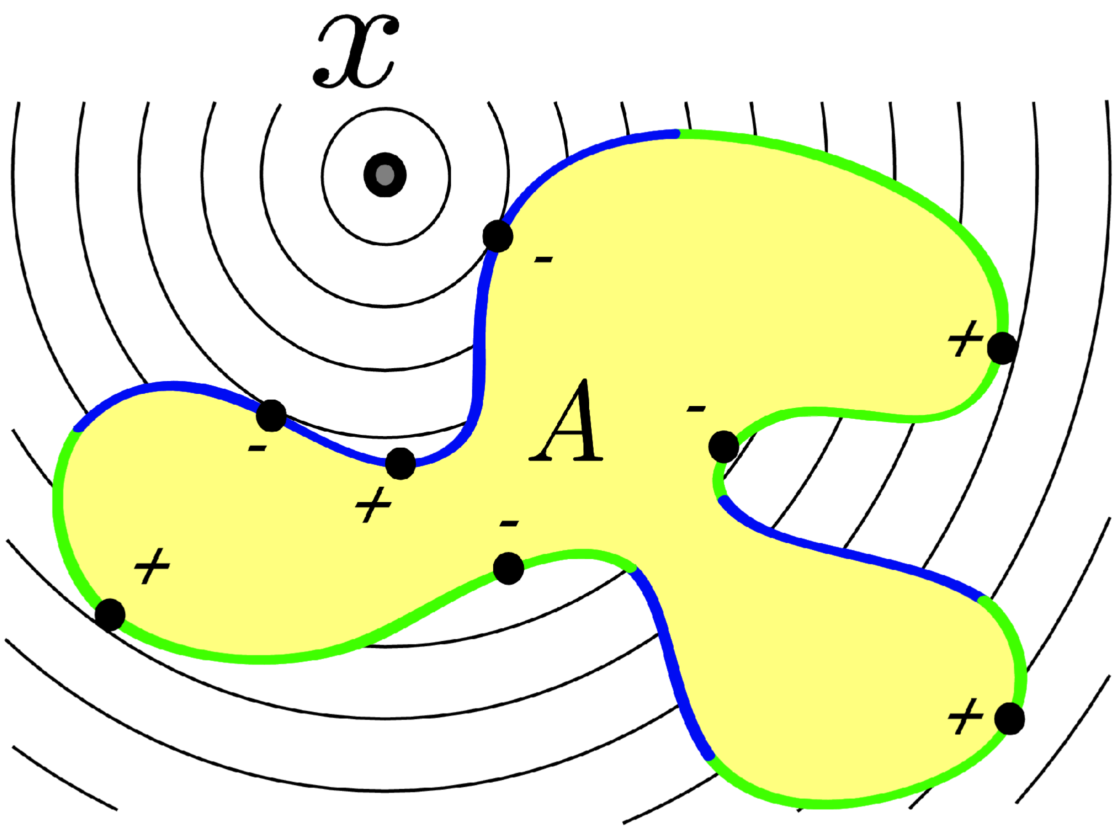




$$(Bh)(x) = \int_0^\infty \int_{\partial B_r(x)} h\, d\chi\, dr$$

The Euler-Bessel transform

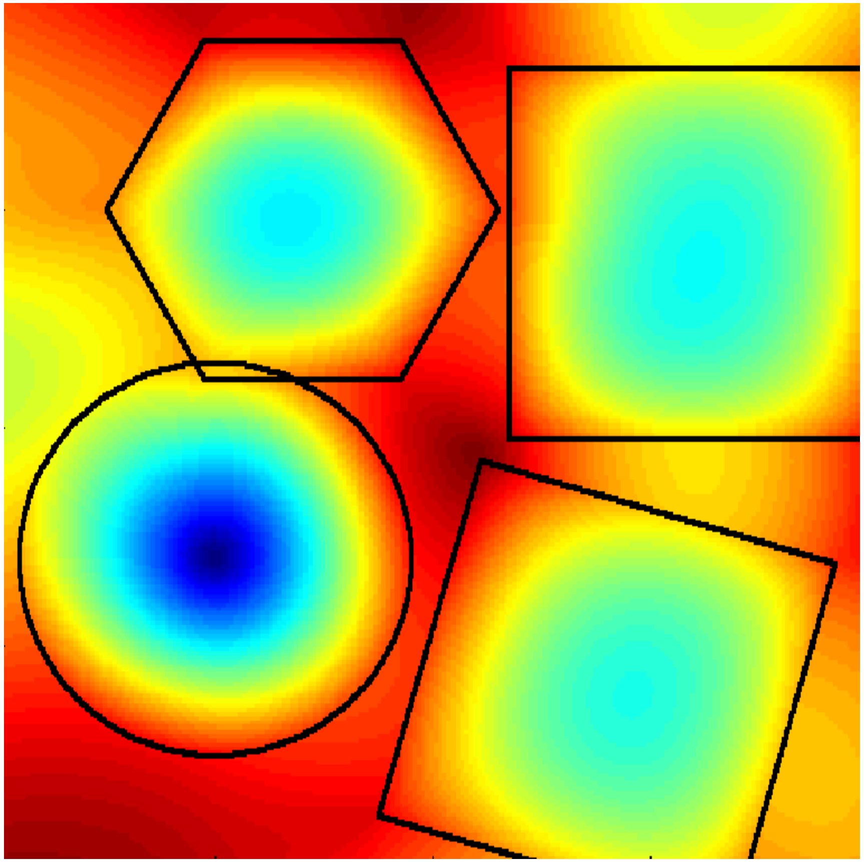
- <u>Theorem</u>: (Ghrist & R.) The Euler-Bessel transform also concentrates the measure on a set of critical points, counted with sign



<u>Benefit</u>: Computationally efficient simulations

Why might we care?

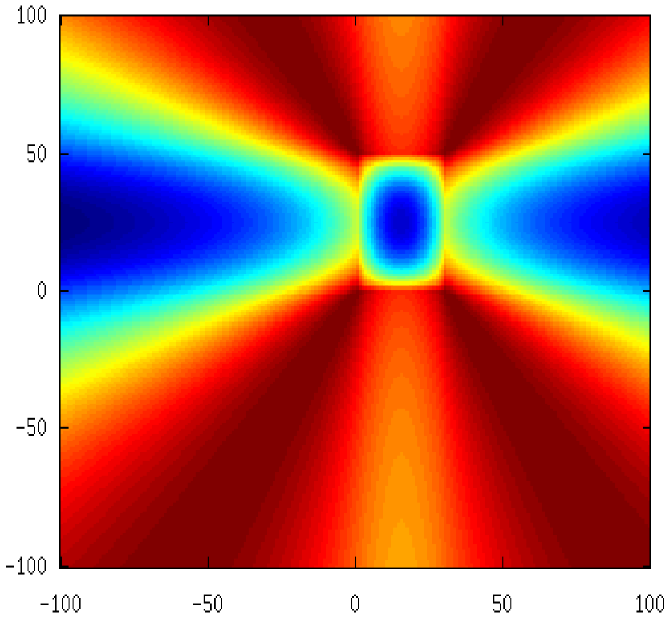
- The Euler-Bessel transform localizes targets
  - It usually has a minimum at each target center
- It detects target shape
  - The minimum isn't as pronounced if the target isn't circular



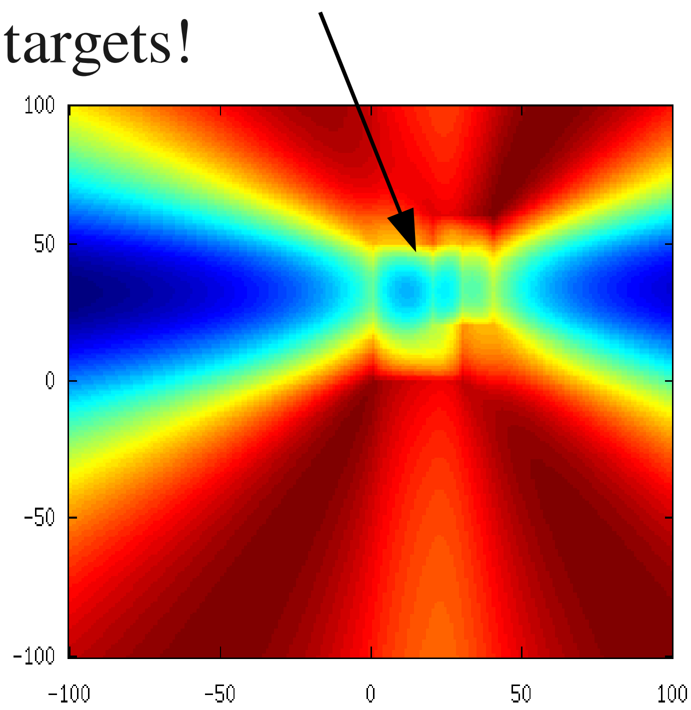
Target sidelobes

- When contours and targets are not matched, one gets sidelobes

More local minima than targets!

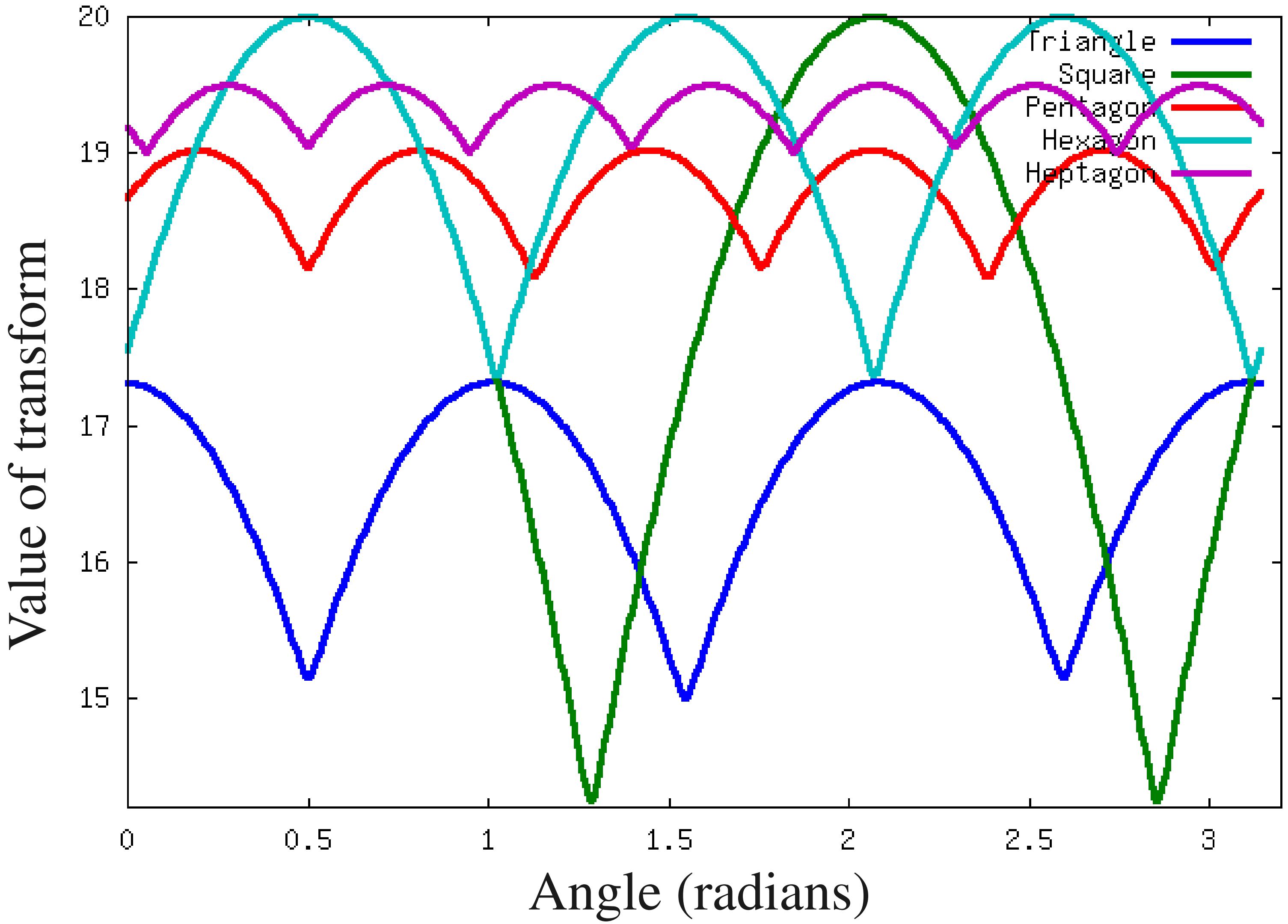


Circular contours, rectangular target



Circular contours, two rectangular targets, partially overlapping

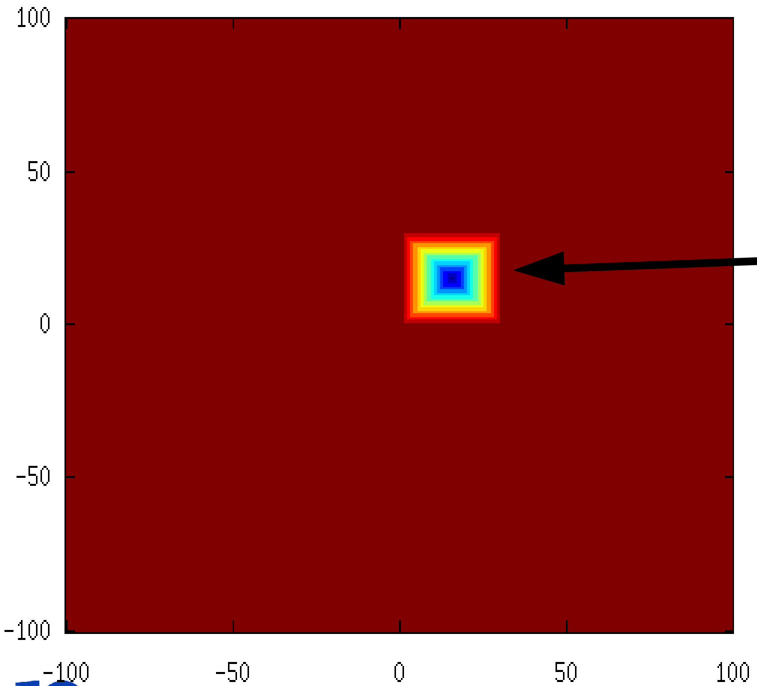
Sidelobes in the far-field: *n*-gons



Other contours

- Sidelobes are mitigated by matched filtering
- Swap out for other contours of integration

$$(Bh)(x) = \int_0^\infty \int_{\partial B_r(x)} h \, d\chi \, dr$$



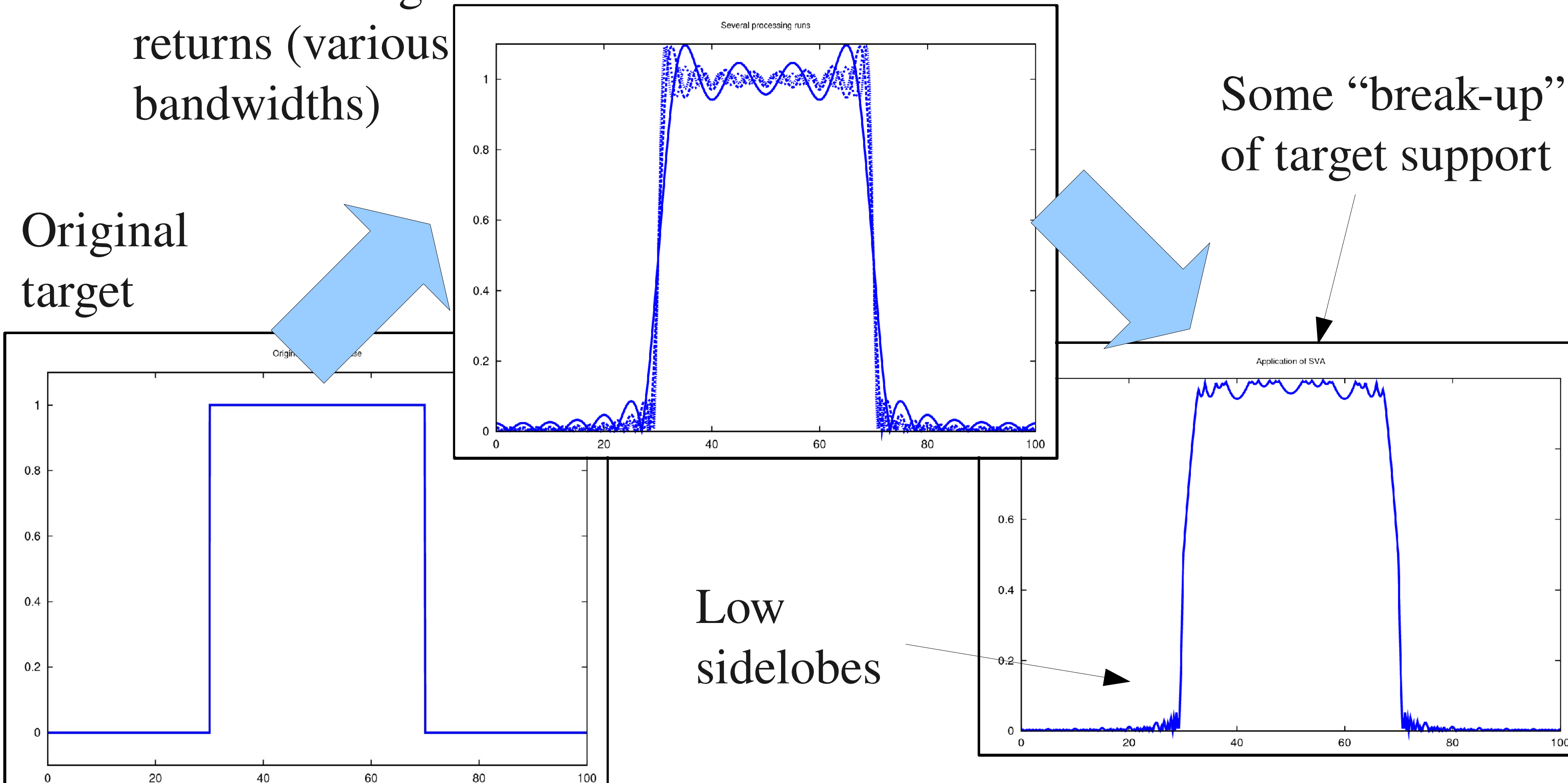
Square contours
Square target

SVA* (Fourier case)

Processed target returns (various bandwidths)

Original target

Some “break-up” of target support



Low sidelobes

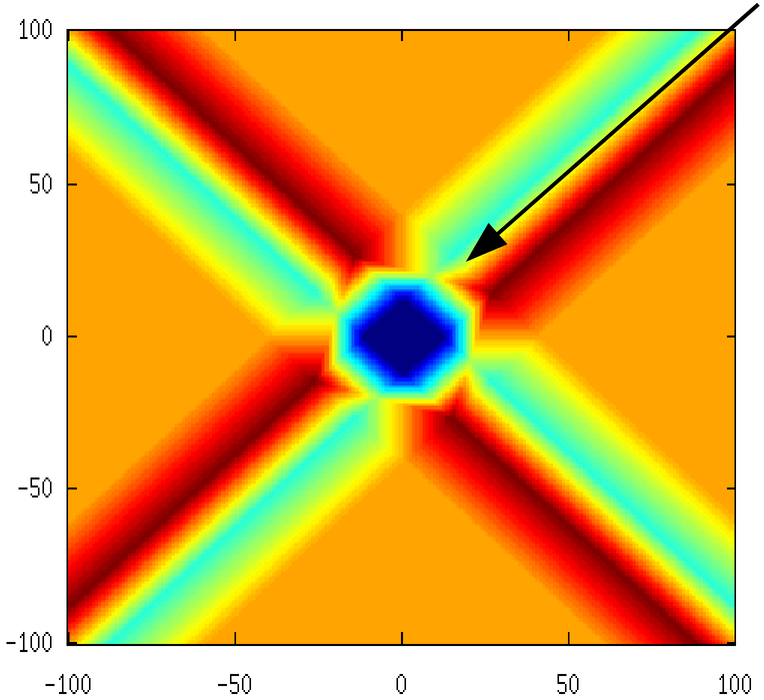
Final result

* **S**patially **V**ariant **A**podization

Euler-Bessel SVA results

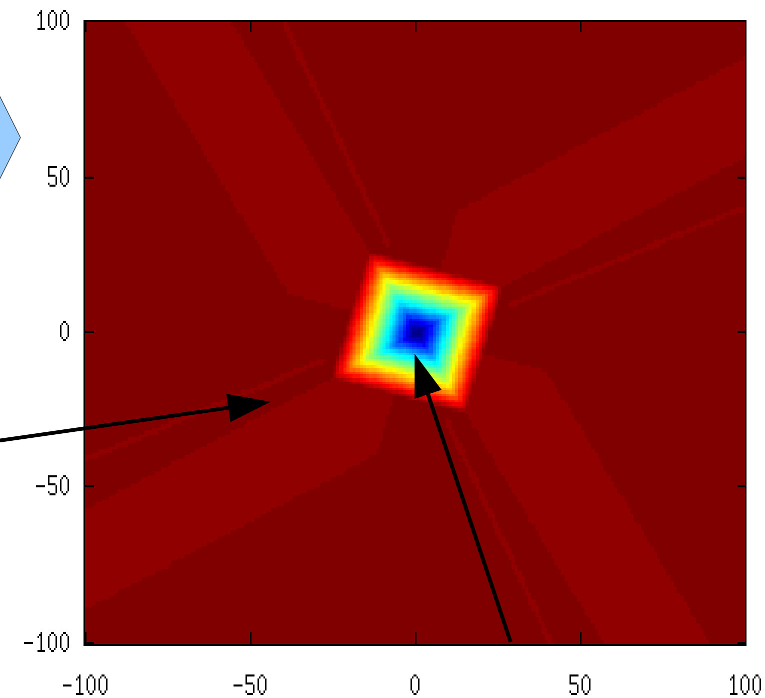
Rotated square target

Square integration contours rotated to various positions, SVA applied to results



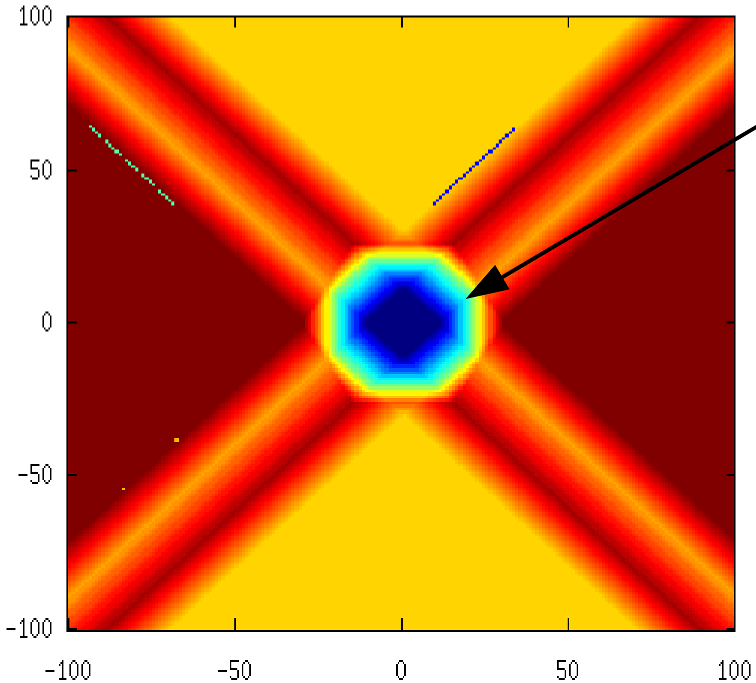
Square integration contours used, aligned with coordinate frame

Sidelobes almost completely eliminated

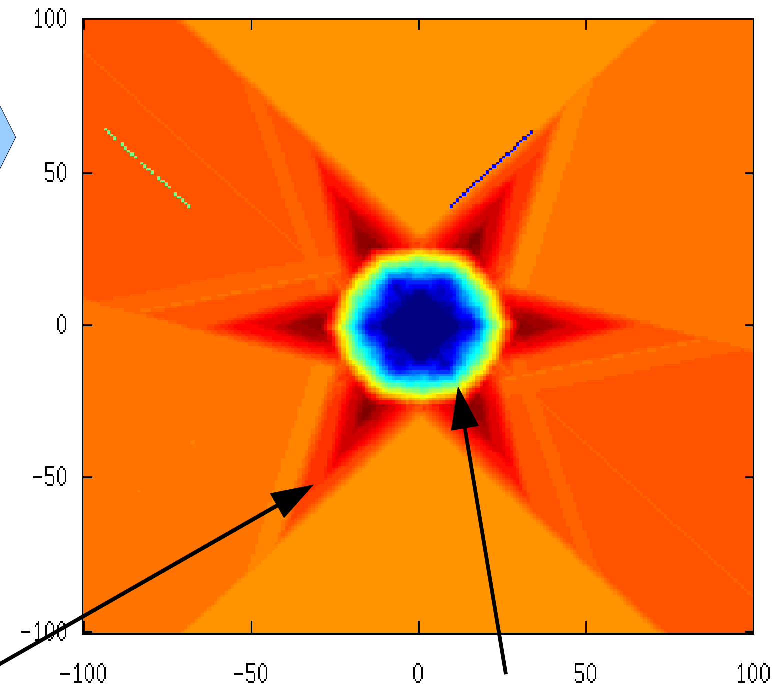


No target break-up

More EBSVA results



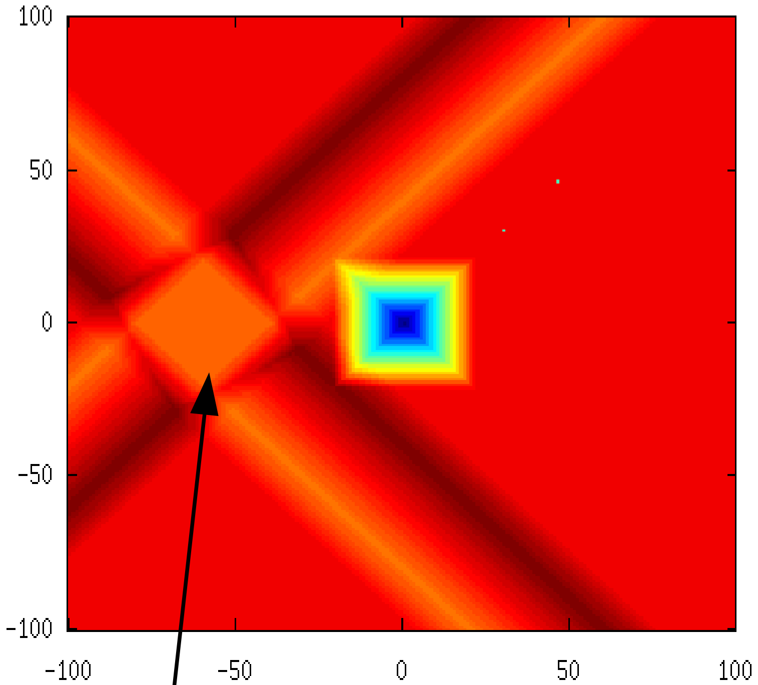
Regular
hexagonal
target



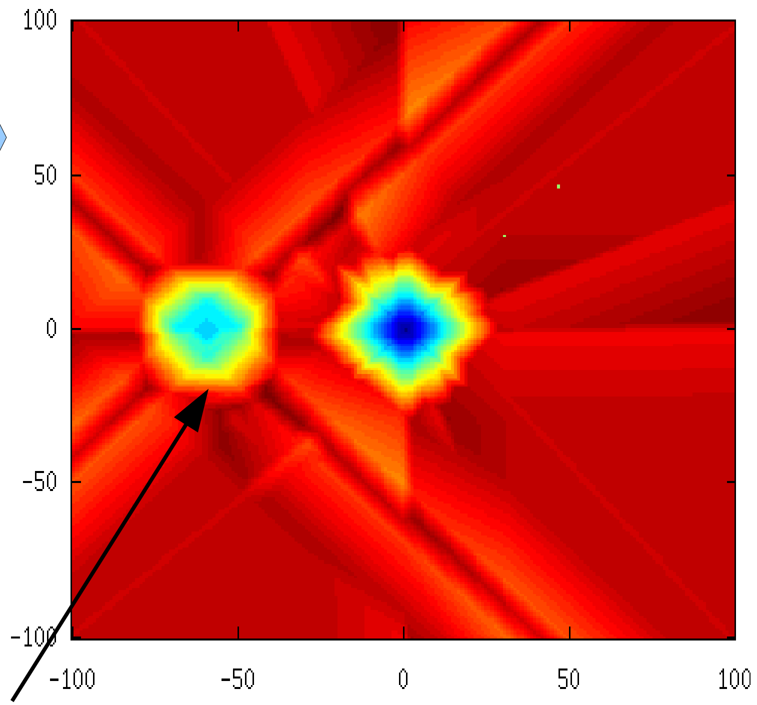
Response outside of
some remaining
sidelobes is stabilized

Minimal target break-up

EBSVA with two targets



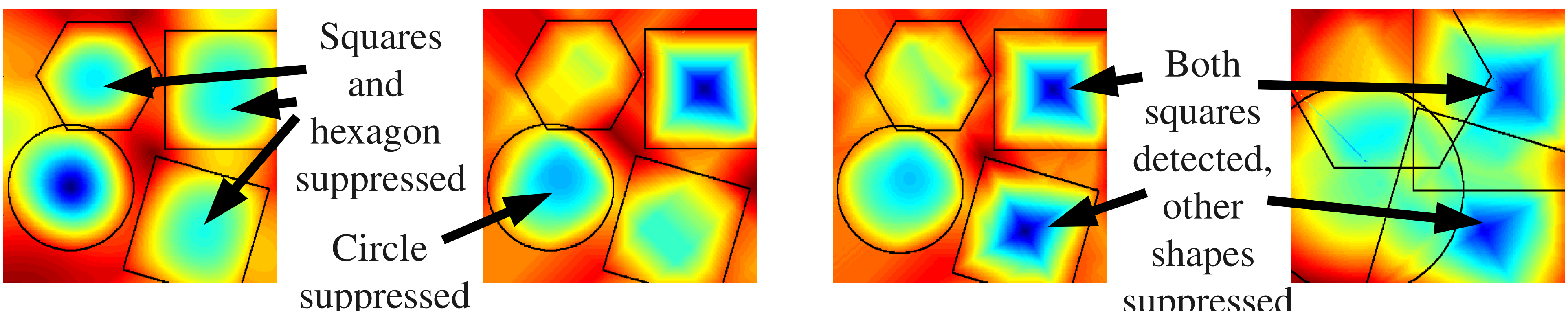
Second target, rotated



SVA recovers target, with some SNR loss

Shape filtering

- By selecting the norm and tailoring the use of SVA, it is possible to create *shape filters*
  - They are $\mathbb{Z}$-linear
  - They are insensitive to position and size of targets
  - They can be made insensitive to orientation if desired



Circular contours

Square contours, aligned with x,y-axes

Square contours, using SVA to remove rotational depedence

Can still detect square supports with overlap

Wavelet transforms

Euler “orthogonality”

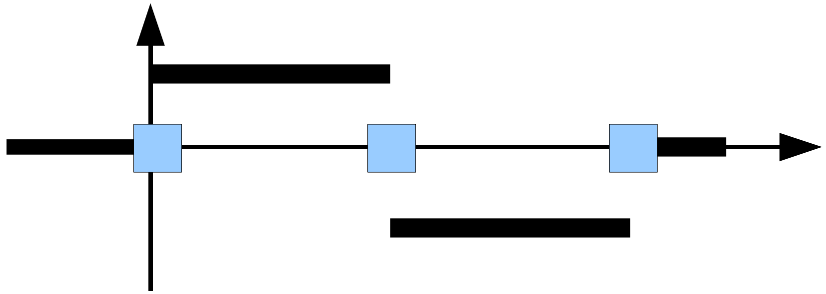
- Definition: Euler “inner product”

$$(f,g)_\chi = \int f\, g\, d\chi$$

- It's not an inner product:
  - Not positive (semi)-definite
- Additionally, no nice convergence properties due to finite summability of the Euler measure
- So don't expect to get **complete** ortho**normal** sets
  - There do exist orthogonal sets, though

Euler Haar wavelets (Type 1)

- Definition: Mother Haar wavelet (ω^1_{00})



We define the child wavelets ω^1_{st} in the usual way

Extend this definition to $\mathbb{R}^n$ via tensor products

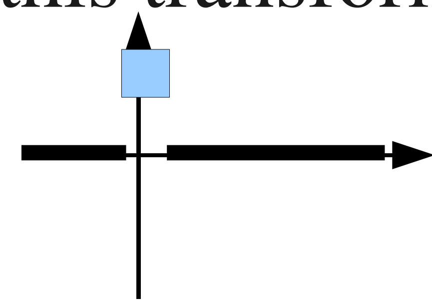
- The set of these wavelets is $(.,.)_\chi$-orthogonal, but for each wavelet ω, $(\omega,\omega)_\chi = -2$

Good news/Bad news

- This is enough to define a practical transform, working on $f \in CF(\mathbb{R}^n;\mathbb{Z})$

$$f \mapsto (f, \omega_{st})_\chi \text{ (for } s,t \in \mathbb{Z})$$

- Sadly, this transform is *not injective*!



$$\left(\quad , \omega^1_{st} \right)_\chi = 0$$

- This doesn't present difficulties if you are only interested in “images”
  - But it does indicate we need more wavelets

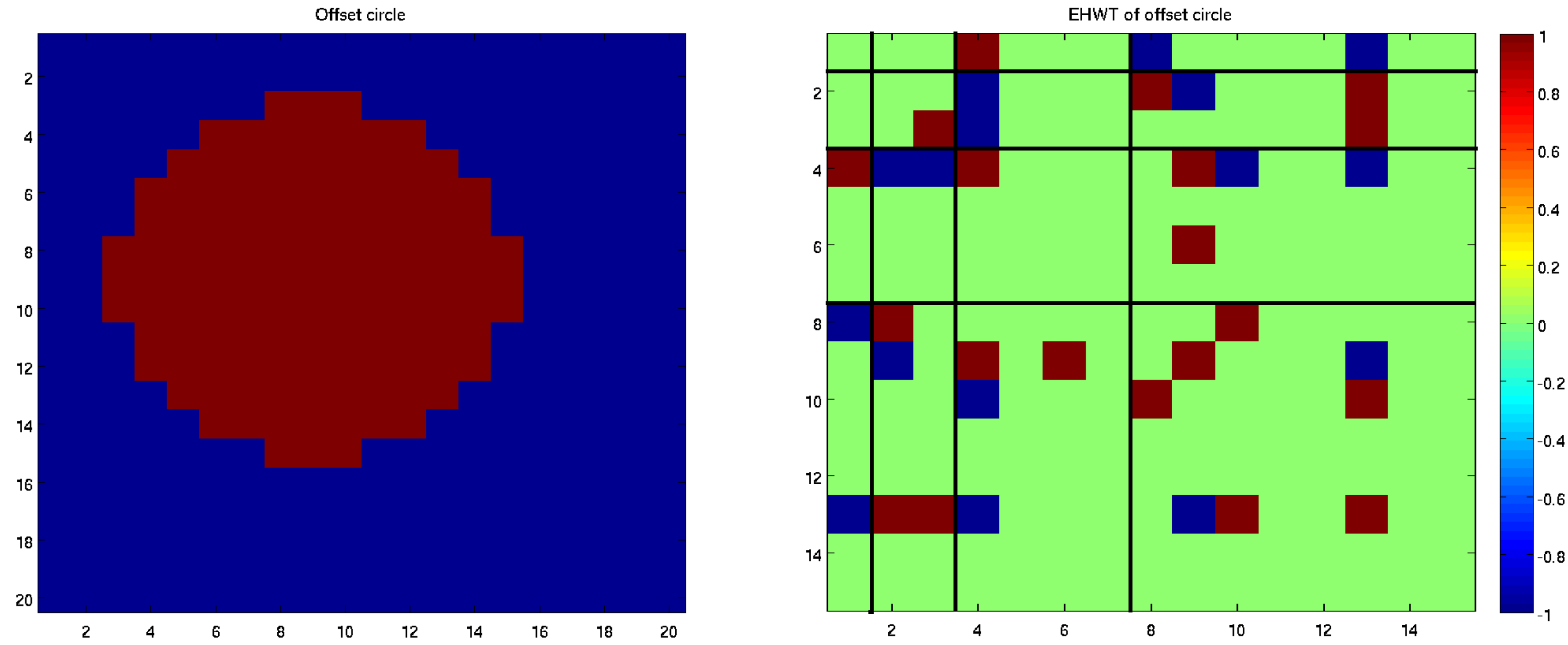
Euler Haar wavelets (Type 0)

- A second kind of wavelet: ω^0_{st}
  - Indicator functions on dyadic points in $\mathbb{R}$
  - This extends to higher dimenions by tensor products
- Definition: Euler Haar Wavelet Transform (EHWT)

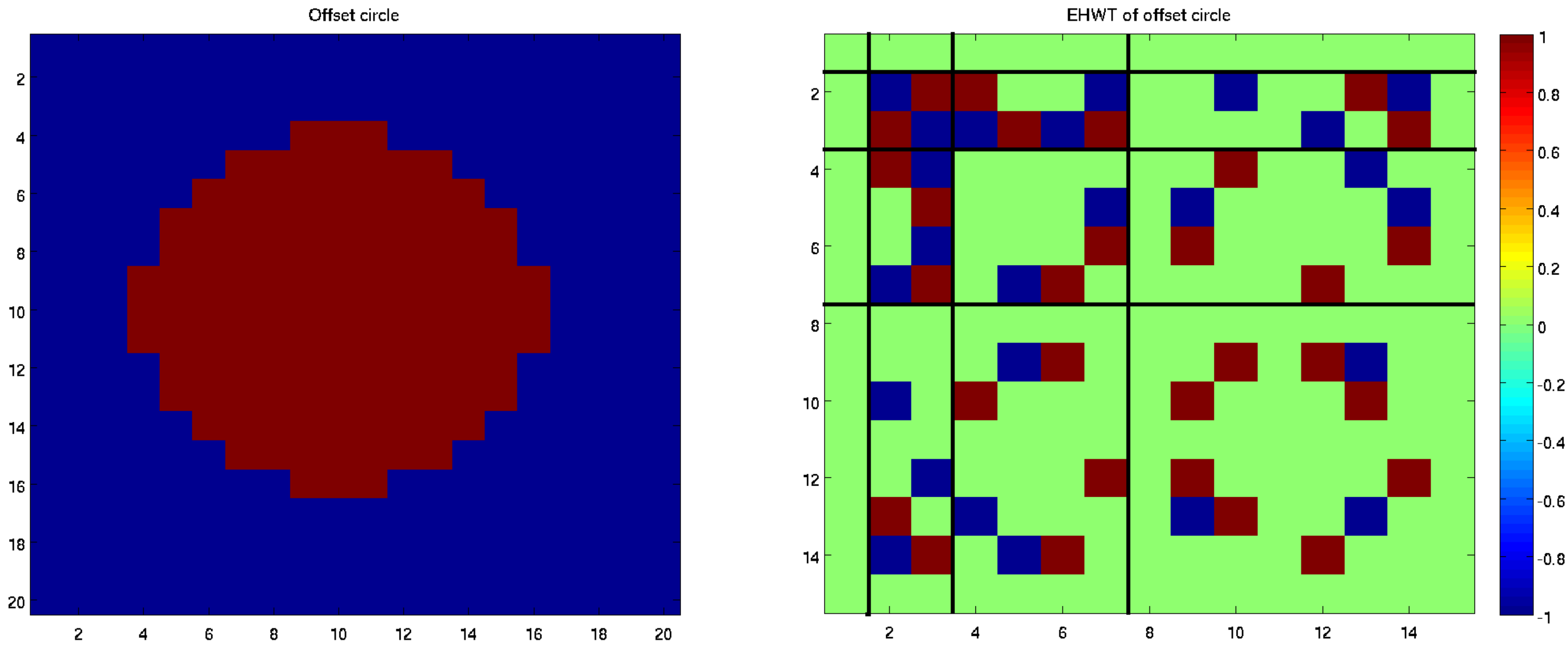
$$f \mapsto (f, \omega^p_{st})_\chi \text{ (for } s,t \in \mathbb{Z},\ p \in \{0,1\})$$

- Obviously the $\{\omega^0_{st}\}$ wavelets and the $\{\omega^1_{st}\}$ wavelets are each orthogonal sets
  - But their union is **not**
  - So there will be some redundancy in the EHWT

2-d example

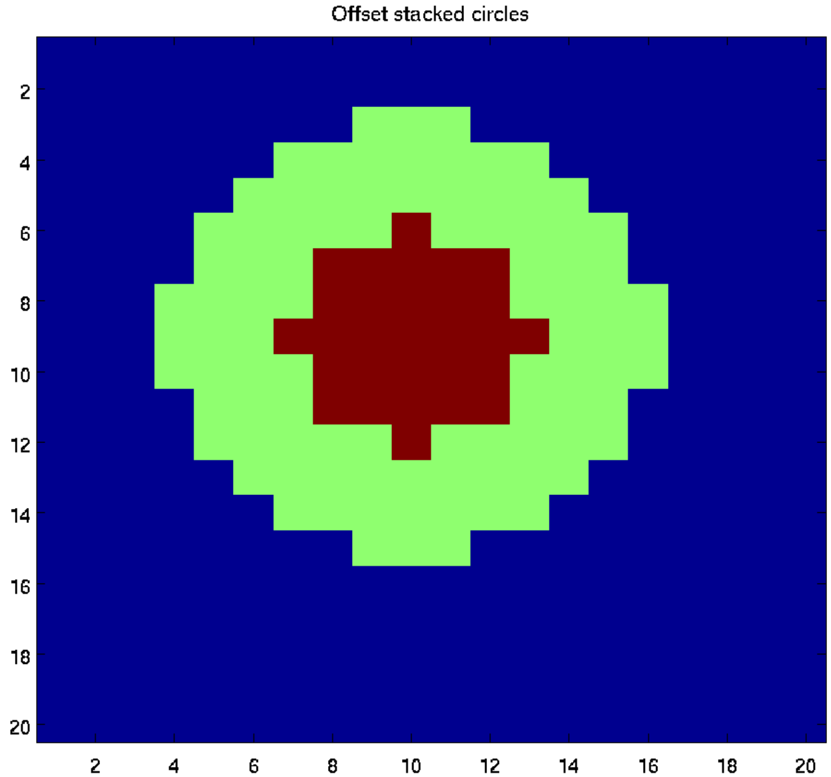

Offset circle
2
4
6
8
10
12
14
16
18
20
2
4
6
8
10
12
14
16
18
20
EHWT of offset circle
2
4
6
8
10
12
14
2
4
6
8
10
12
14
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1

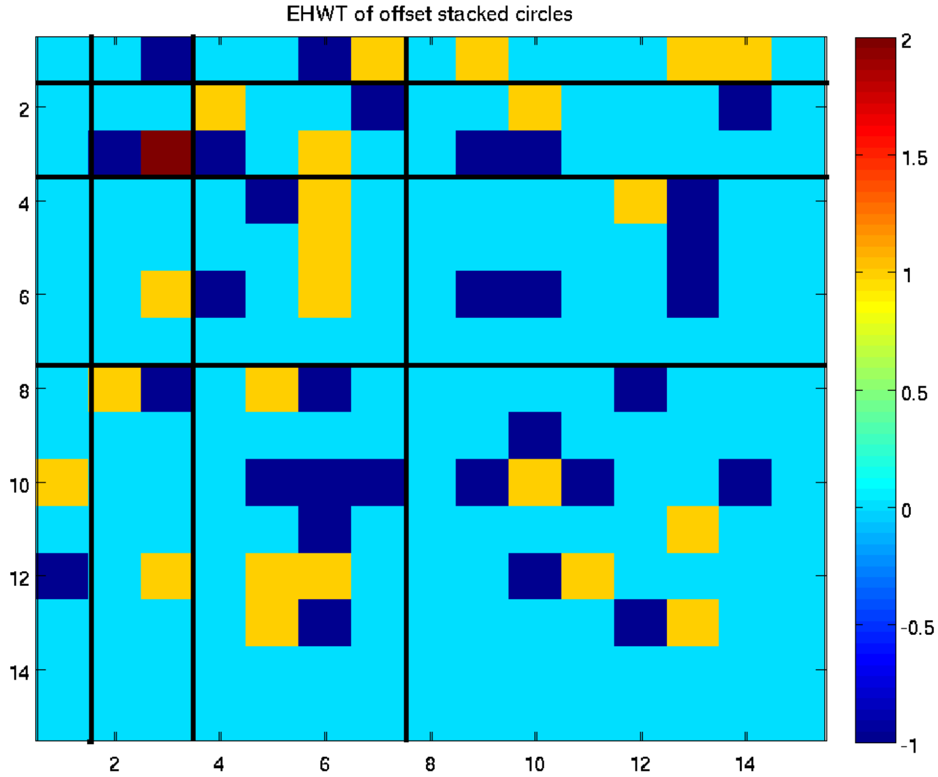
2-d example



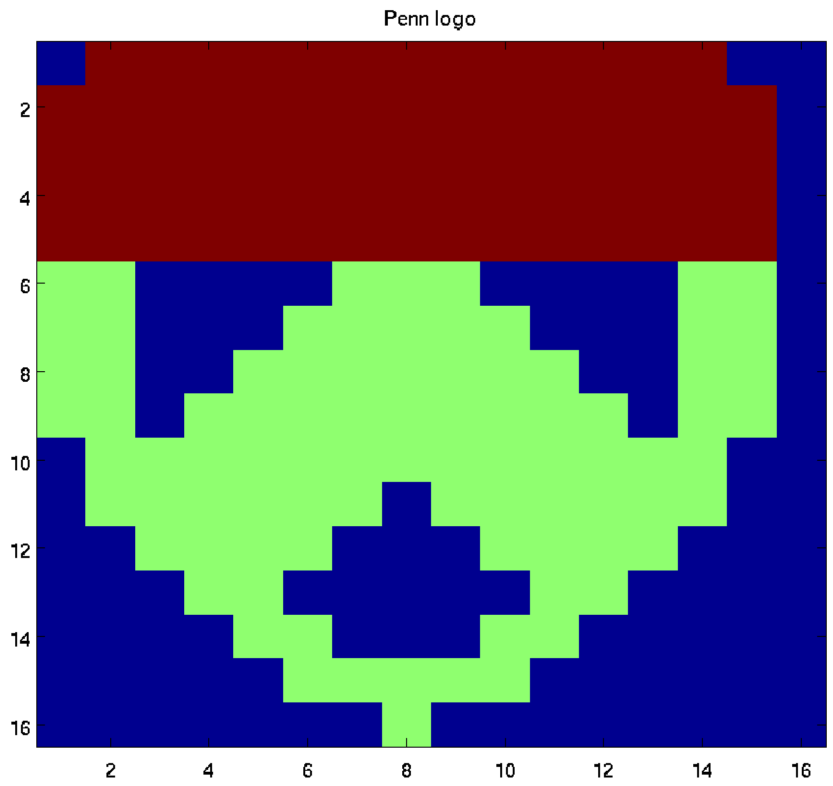
Michael Robinson

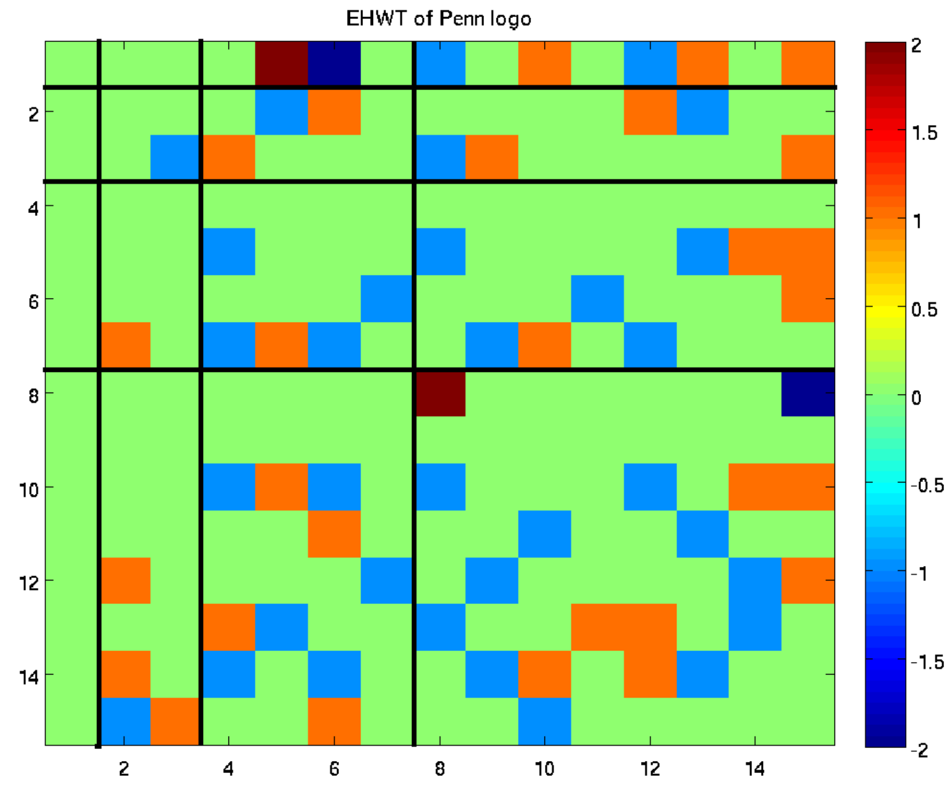
2-d example


Offset stacked circles
2
4
6
8
10
12
14
16
18
20
2
4
6
8
10
12
14
16
18
20


EHWT of offset stacked circles
2
4
6
8
10
12
14
2
4
6
8
10
12
14
2
1.5
1
0.5
0
-0.5
-1

2-d example


Penn logo
2
4
6
8
10
12
14
16
2
4
6
8
10
12
14
16


EHWT of Penn logo
2
4
6
8
10
12
14
2
4
6
8
10
12
14
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2

Sparsity and injectivity

- It appears from the examples that not all of the coefficients are necessary
  - Many of them are zero
- If the transform is injective, maybe we can use it for
  - Detection processing
  - Filtering and reconstruction
- <u>Theorem</u>: (R.) The EHWT is injective
- But the usual Fourier representation fails...

$$f \neq \sum_{st} \omega_{st} (\omega_{st}, f)_X$$

Future directions: applications

- Refine and explore shape filtering
  - Testing on examples from machine vision, radar
- Quantitative analysis of sidelobe mitigation techniques
  - In particular, how much should one anticipate that SVA will help?
  - Target break-up and occlusion effects need to be addressed
- Treat inversion for the EHWT
  - Examine its uses for filtering, compression

For more information

Michael Robinson

robim@math.upenn.edu

Preprints available from my website:

http://www.math.upenn.edu/~robim