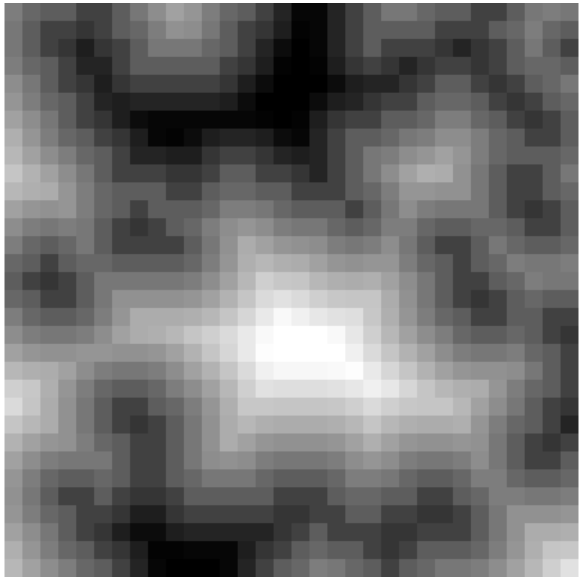
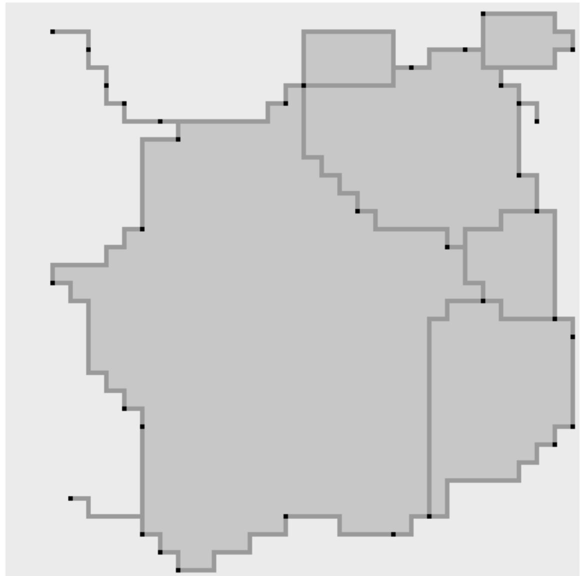
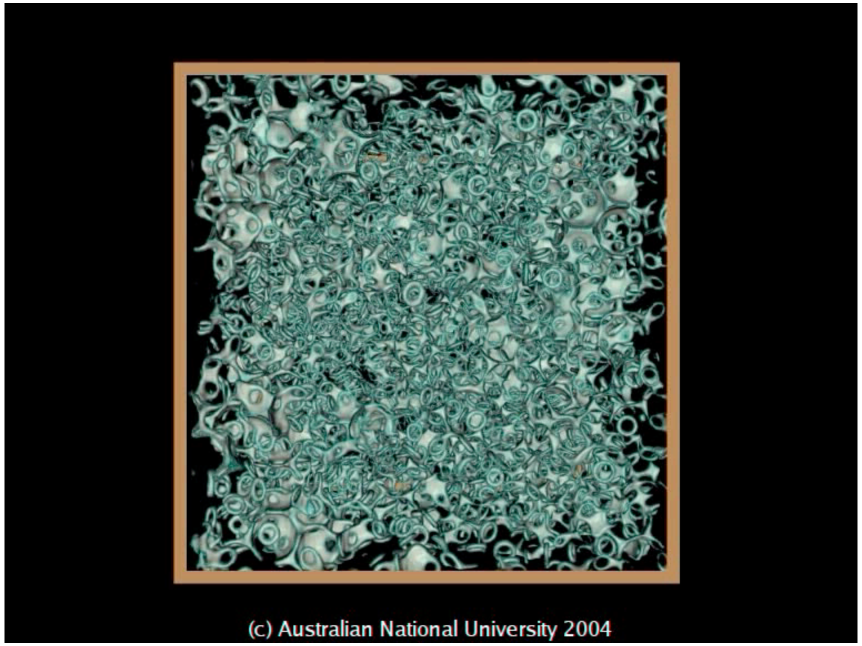
Constructing Discrete Morse Complexes from digital images

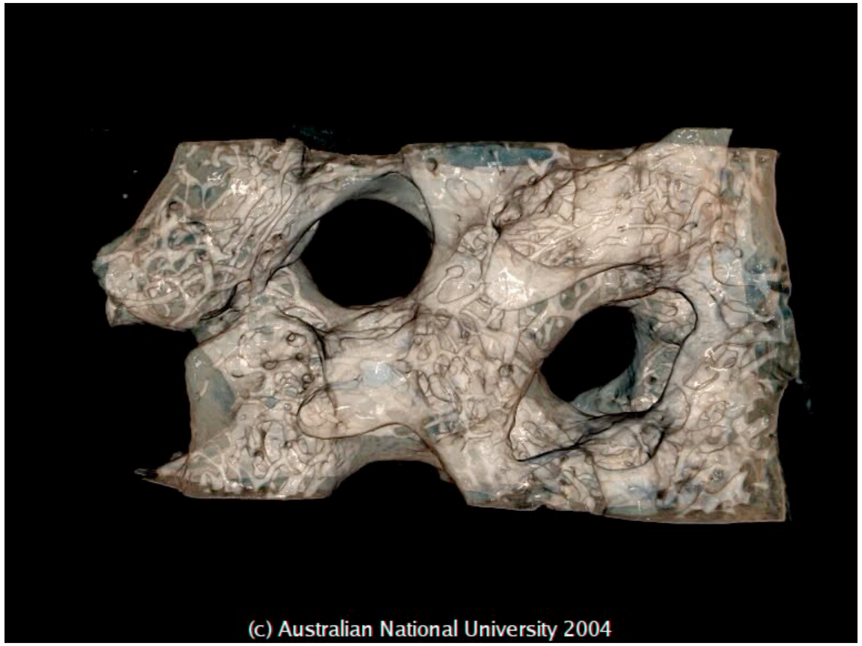


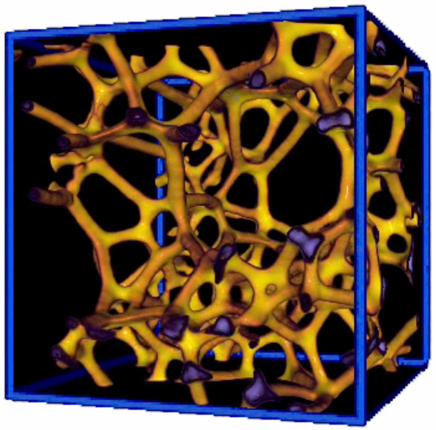


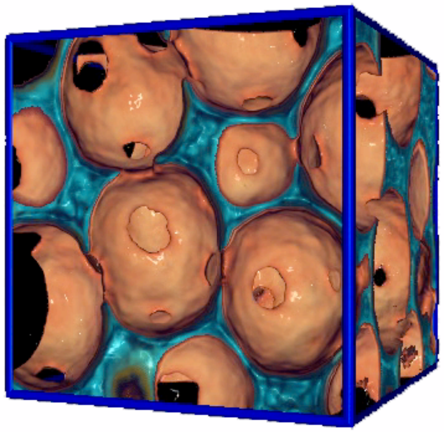
Vanessa Robins   Adrian Sheppard   Peter Wood
Applied Maths, RSPE
The Australian National University

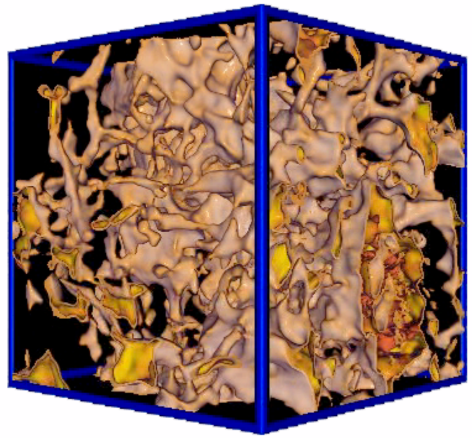
X-ray CT images



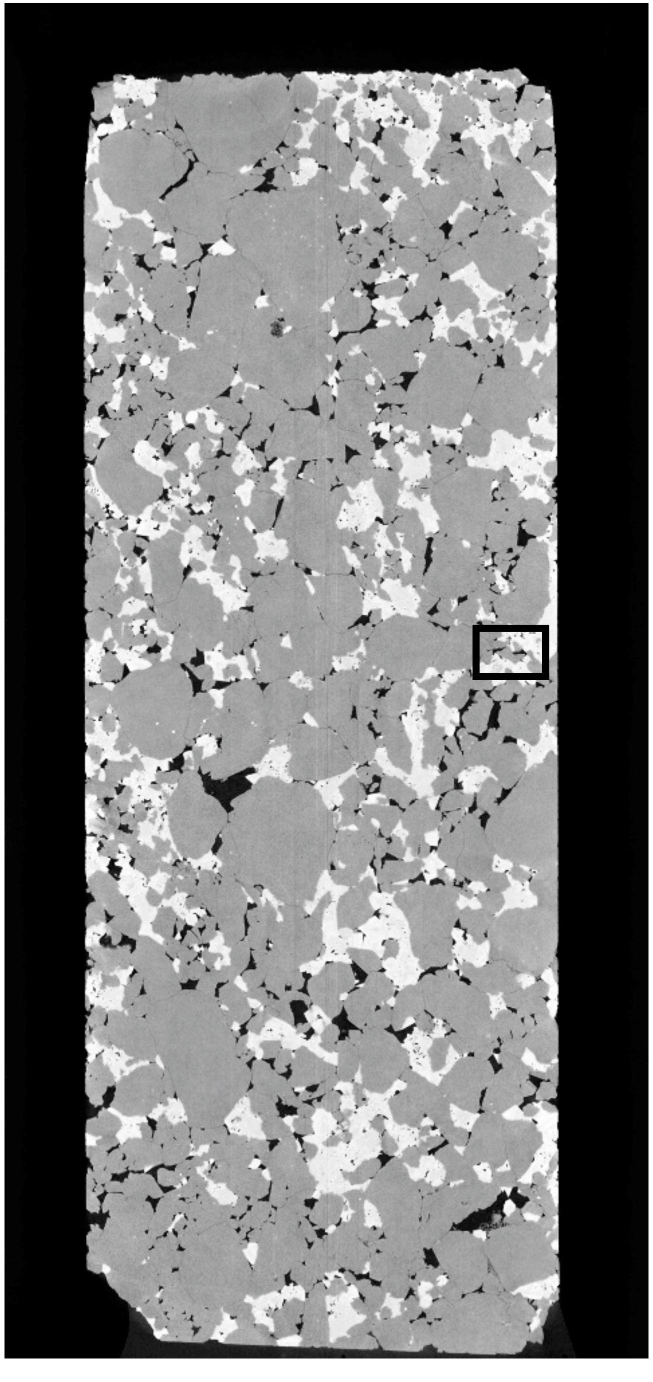


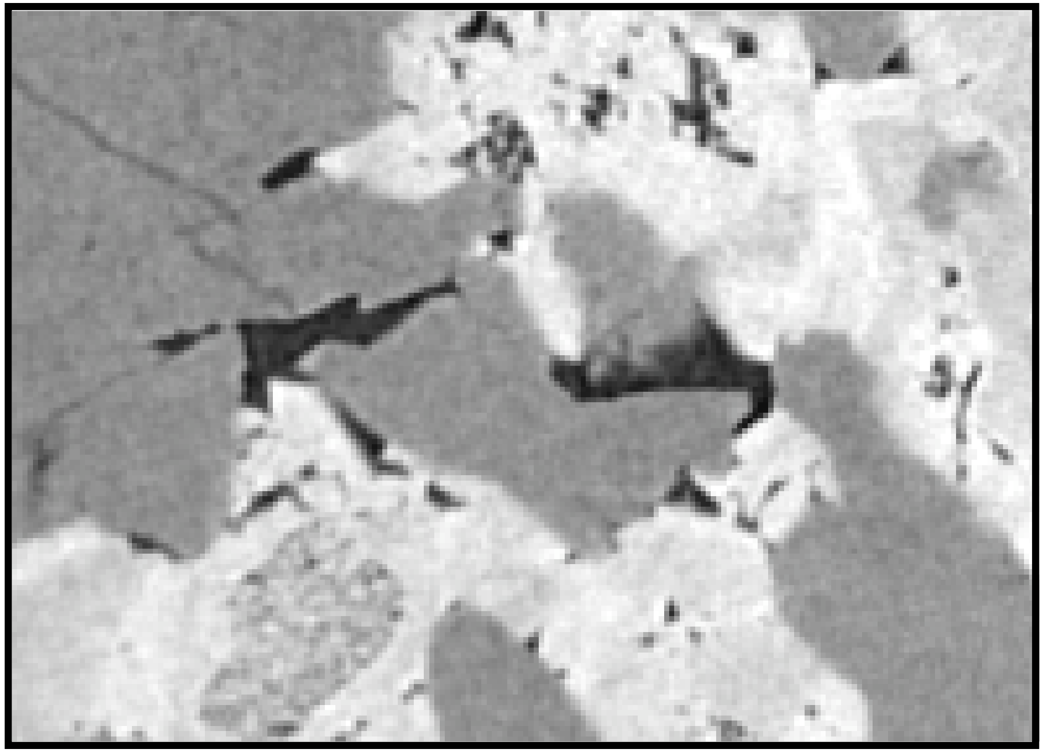






Rock core sample



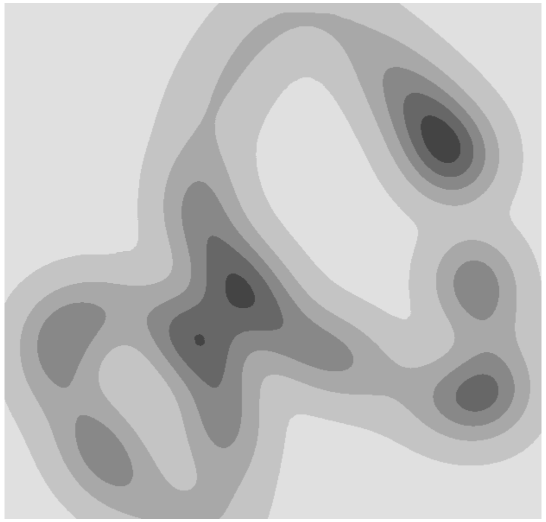


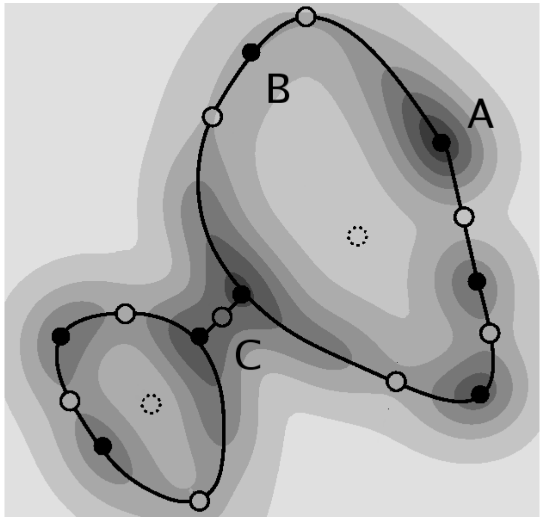
Challenges

* Large 3D images, complex structure, noisy.
* Image segmentation - decide which voxels represent different materials.
* Object characterisation via Euclidean distance transforms and medial surfaces.
* Our goal is to apply Morse theory and persistent homology to these problems.

Morse Theory

* Examine the level sets of a 'nice' function.
* Changes in the topology occur at the critical points of the function.
* The type of critical point determines the type of topological change.
* unstable manifolds form a chain complex.

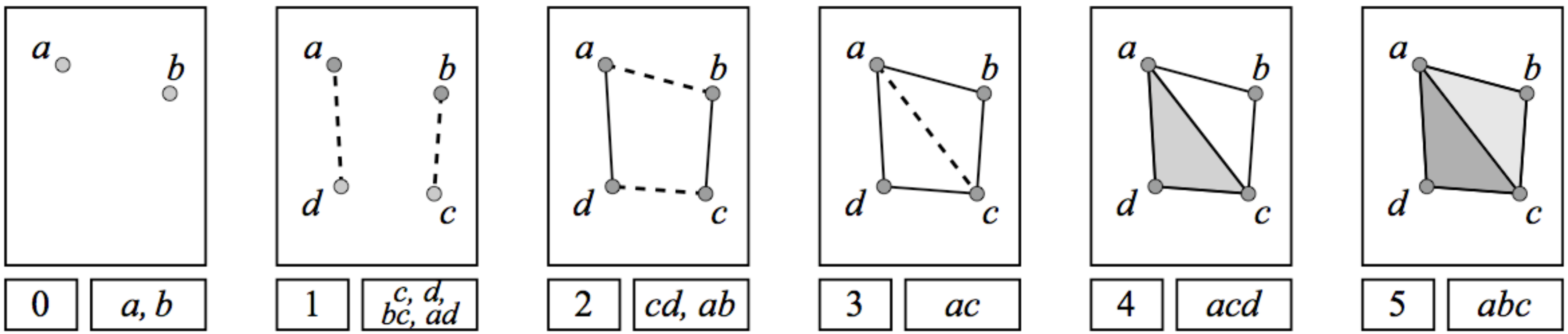




○ min: 0-cell
○ saddle: 1-cell
○ max: 2-cell

Persistent homology

- tracks changes in the topology of a filtration.
- pairs creation and destruction events.



b ac ab

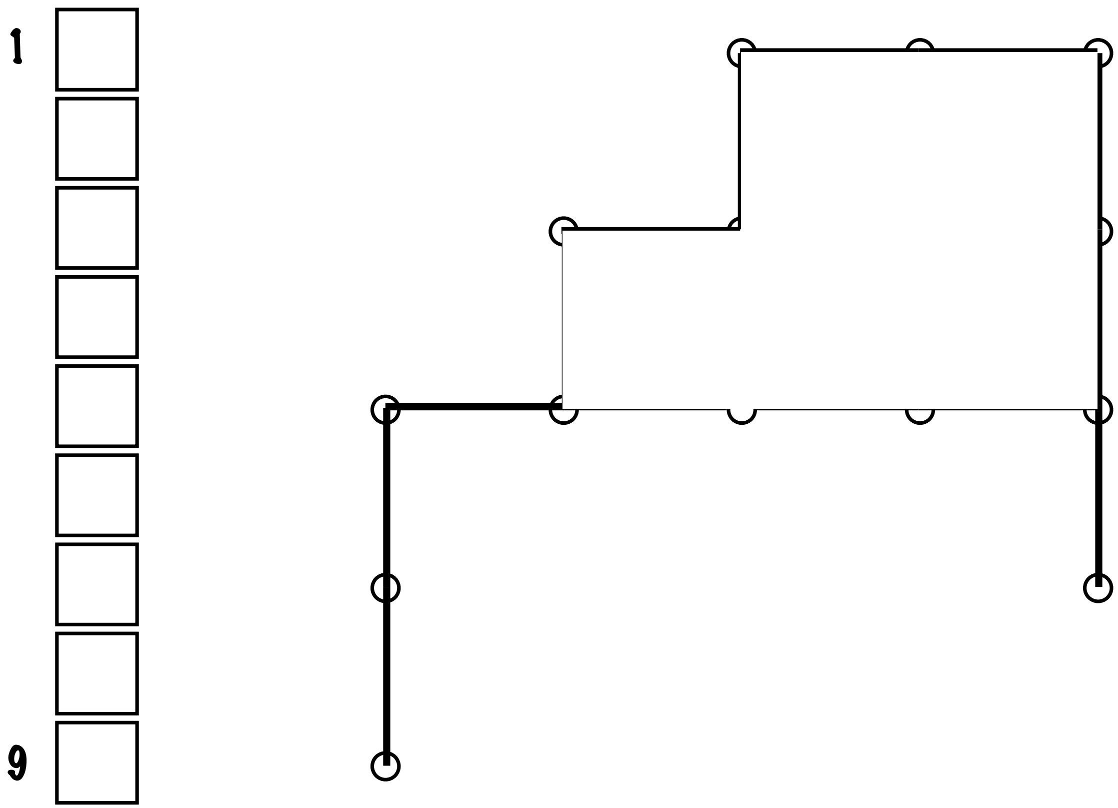
image from Zomorodian "Computational Topology" (2009)

Greyscale images as cubical complexes

* A grayscale image is a function g: D → R where D ⊂ Z^3 (usually D = {1,...,i}^{3}).
* We treat voxels as the vertices of a cubical complex, K(D).
* The lower level cut on the cubical complex is

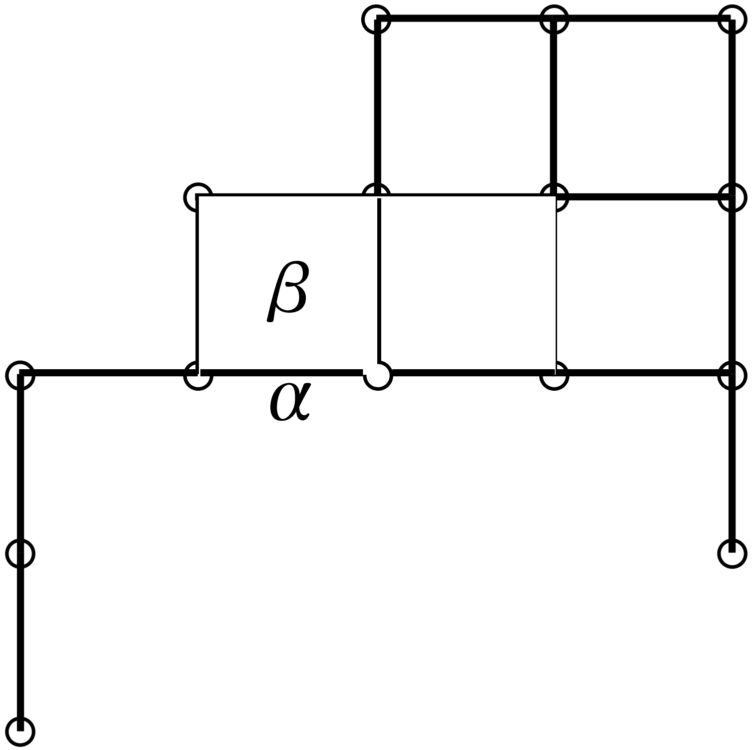
$$\mathcal{K}_t = \{\alpha \in \mathcal{K} \mid g(x) \le t,\ \forall x \in \alpha\}.$$

Greyscale images as cubical complexes



Simple homotopy

Whitehead 1939



* a free face is one with no other cofaces
* a free pair eg. (α , β)
* an elementary collapse is the removal of a free pair
* an expansion is the inverse of a collapse

Discrete Morse Theory

* A function on a cell complex, f: K → R is a discrete Morse function if, for each cell, exactly one of the following holds:

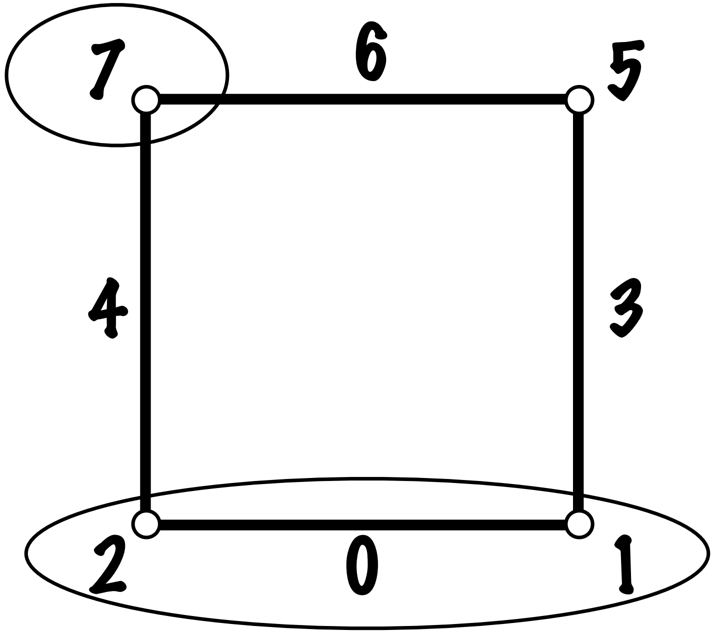
  1. α has a single coface (dim = $p+1$) with lesser or equal value;
  2. α has a single face (dim = $p-1$) with greater or equal value;
  3. all faces of α take strictly lower values and all cofaces of α take strictly higher values.

* Cases 1 and 2 define pairs in a discrete gradient vector field, V.

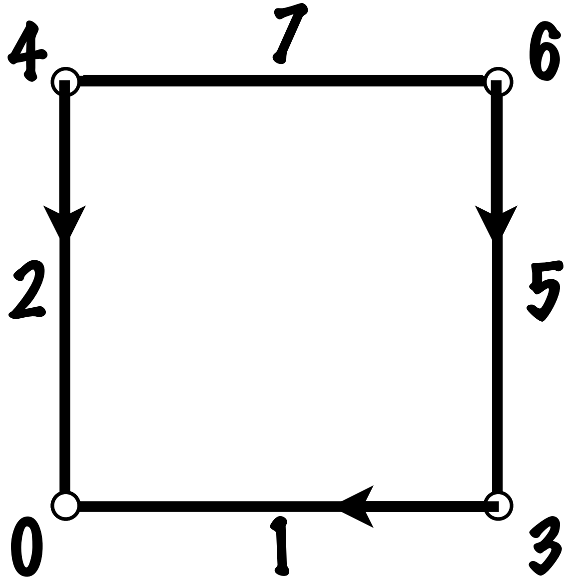
* Case 3 defines a critical cell.

Forman 1998

Discrete Morse Theory



✘ not a Morse function

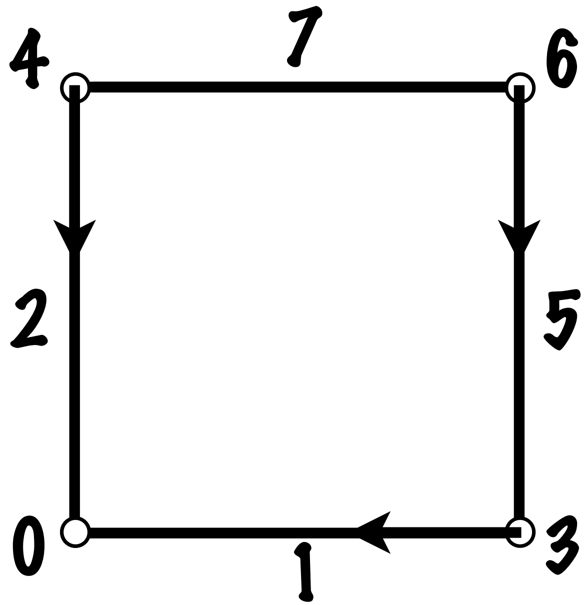


✔ a Morse function

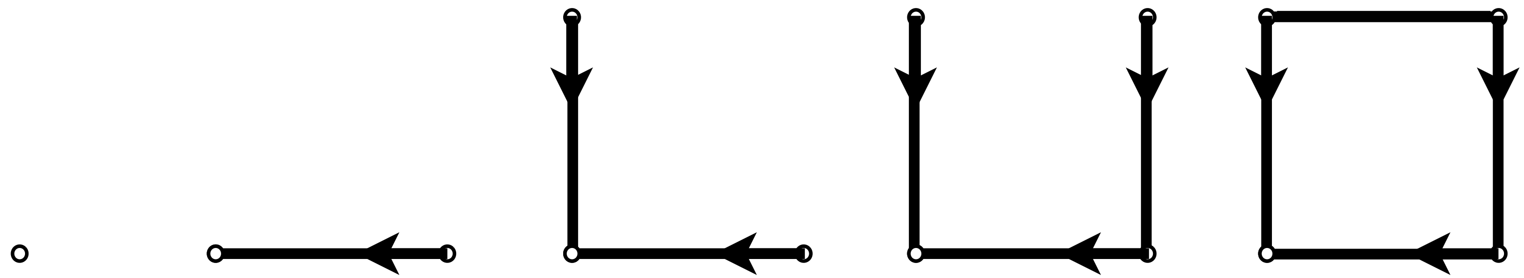
Discrete Morse Theory

* Given a complex K and dmf f, a level cut at value c is

$$\mathcal{K}(c) = \bigcup_{f(\alpha) \leq c} \bigcup_{\gamma \leq \alpha} \gamma.$$



K(0) K(1) K(2) K(5) K(7)

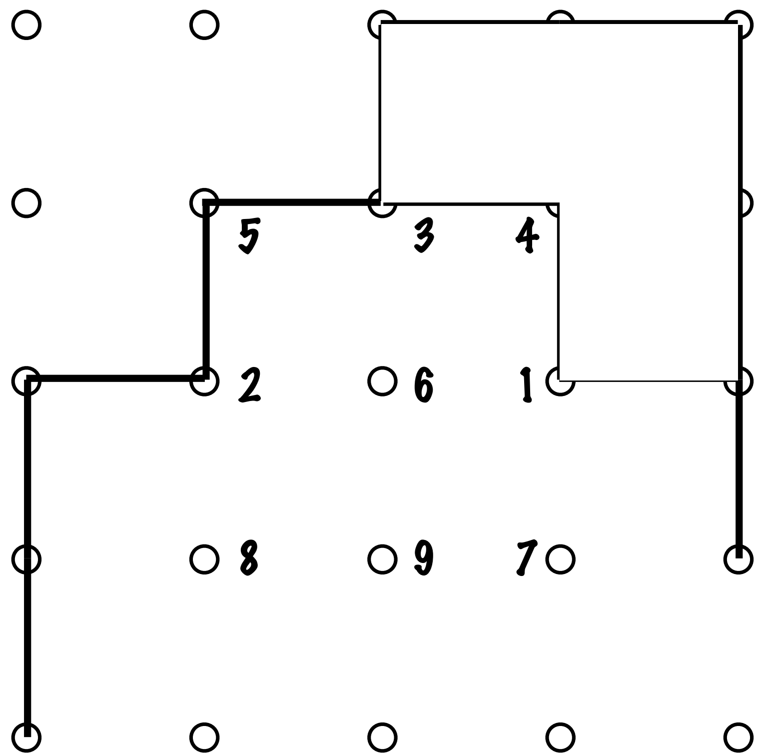


Discrete vector fields

* A discrete vector field is a list of pairs of cells (a,b) such that a is a facet of b and each cell appears at most once in this list.
* A discrete gradient vector field has no closed paths.
* A discrete morse function defines a gradient vector field.
* Gradient paths between critical cells define a chain complex that computes homology.

Greyscale images again

* The image is a function defined only on the vertices.
* Assume each voxel takes a different value.
* Grow cell complex one lower star at a time.
* Grow each lower star with simple homotopy expansions where possible.

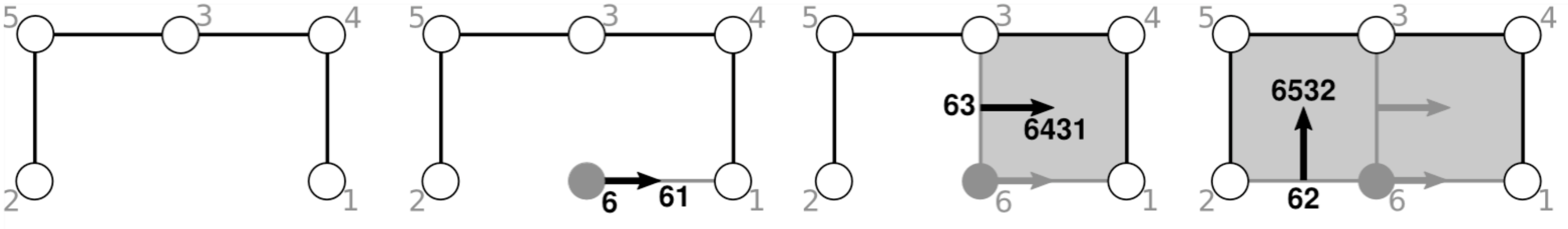


Algorithm 1. ProcessLowerStars(D, g)
Input: D digital image voxels.
Input: g grayscale values on voxels.
Output: C critical cells.
Output: V discrete vector field $V[\alpha^{(p)}] = \beta^{(p+1)}$.

for $x \in D$ **do**
 if $L(x) = \{x\}$ **then** {x is a local minimum}
  add x to C
 else
  $\delta :=$ the 1-cell in $L(x)$ such that $G(\delta)$ is minimal
  $V[x] := \delta$
  add all other 1-cells from $L(x)$ to PQzero.
  add all cells $\alpha \in L(x)$ to PQone such that $\alpha > \delta$
  and num_unpaired_faces(α) $= 1$
  while PQone $\neq \emptyset$ or PQzero $\neq \emptyset$ **do**
   while PQone $\neq \emptyset$ **do**
    $\alpha :=$ PQone.pop_front
    if num_unpaired_faces(α) $= 0$ **then**
     add α to PQzero
    else
     V[pair(α)] $:= \alpha$
     remove pair(α) from PQzero
     add all cells $\beta \in L(x)$ to PQone such that
     ($\beta > \alpha$ or $\beta >$ pair(α)) and
     num_unpaired_faces(β) $= 1$
    end if
   end while
   if PQzero $\neq \emptyset$ **then**
    $\gamma :=$ PQzero.pop_front
    add γ to C
    add all cells $\alpha \in L(x)$ to PQone such that
    $\alpha > \gamma$ and num_unpaired_faces(α) $= 1$
   end if
  end while
 end if
end for

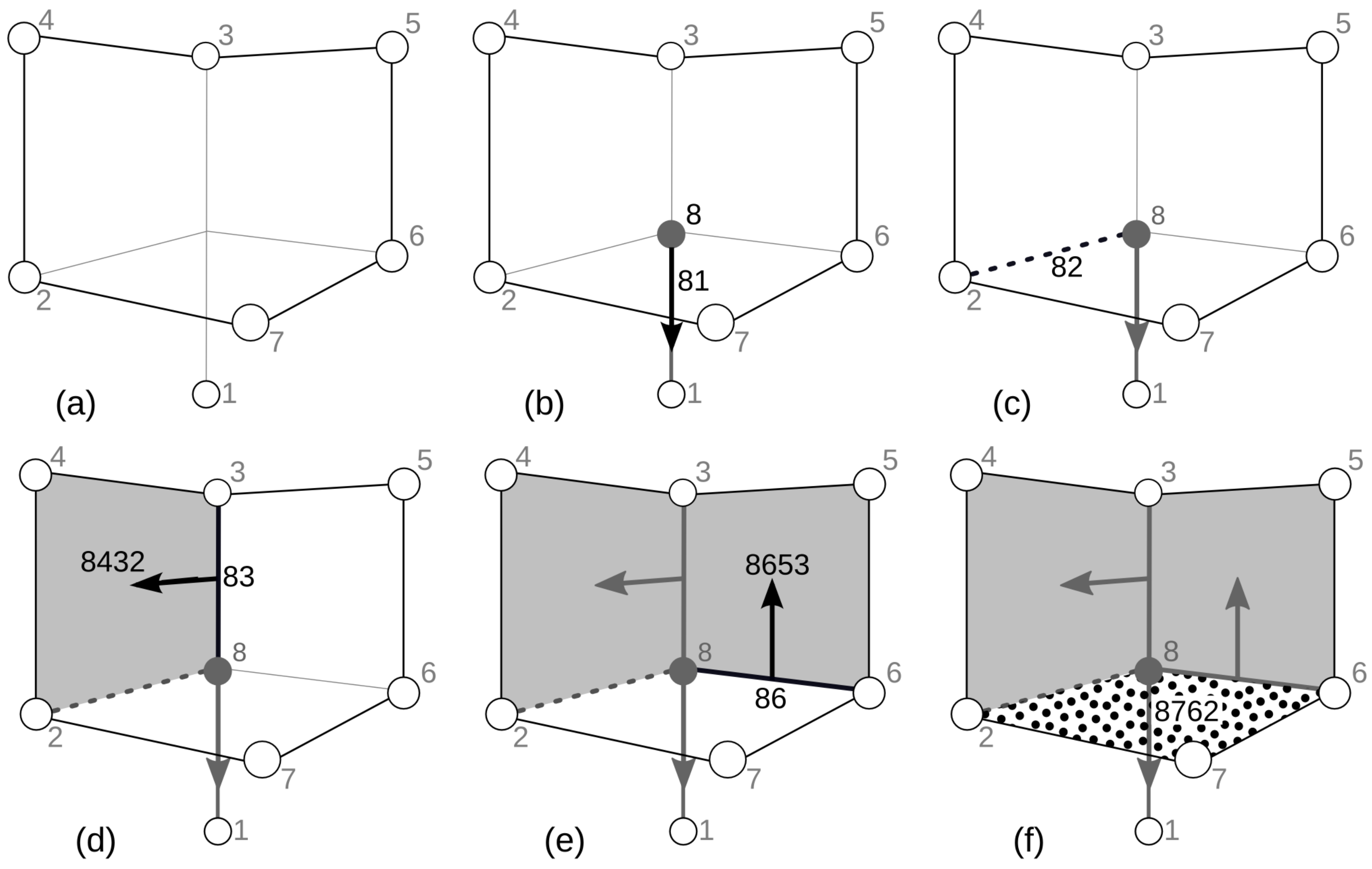
IEEE TPAMI vol.33 p.1646 (2011)

ProcessLowerStar

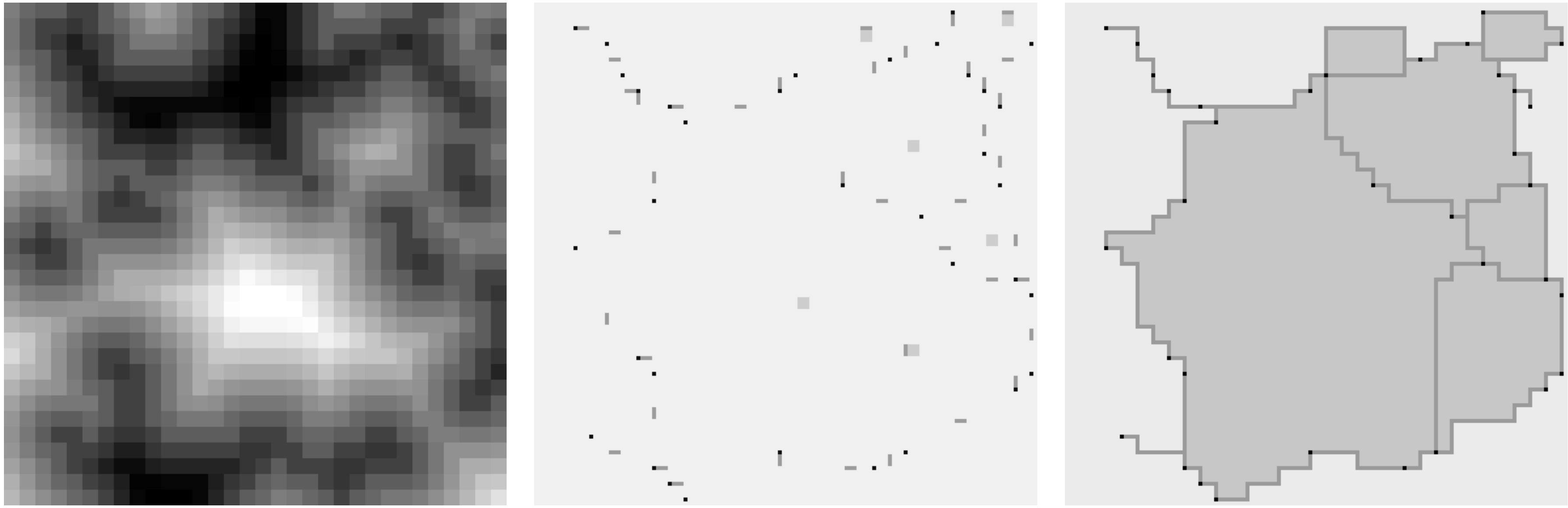


- The algorithm uses two priority queues of cells: PQone and PQzero with a ranking function, G.
- G(cell) = decreasing list of the cell's vertex values
- PQzero: contains cells that have no unpaired faces
- PQone: has cells with exactly one unpaired face
- Outputs: cell pairs in V and critical cells in C

ProcessLowerStar



The Morse complex

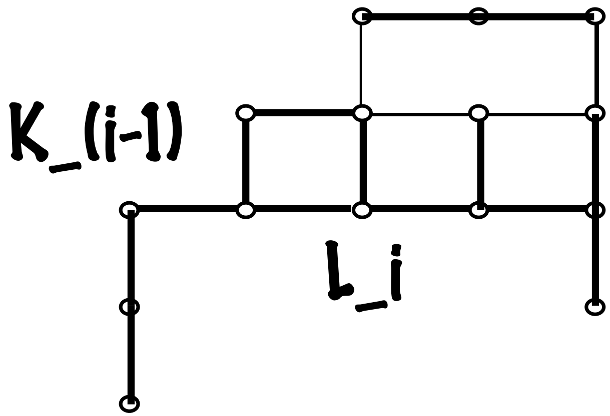


- critical cells are `adjacent' if there is a V-path connecting them.
- use the value of the critical cell to define a filtration on the morse cell complex and compute persistent homology.

Proof of correctness

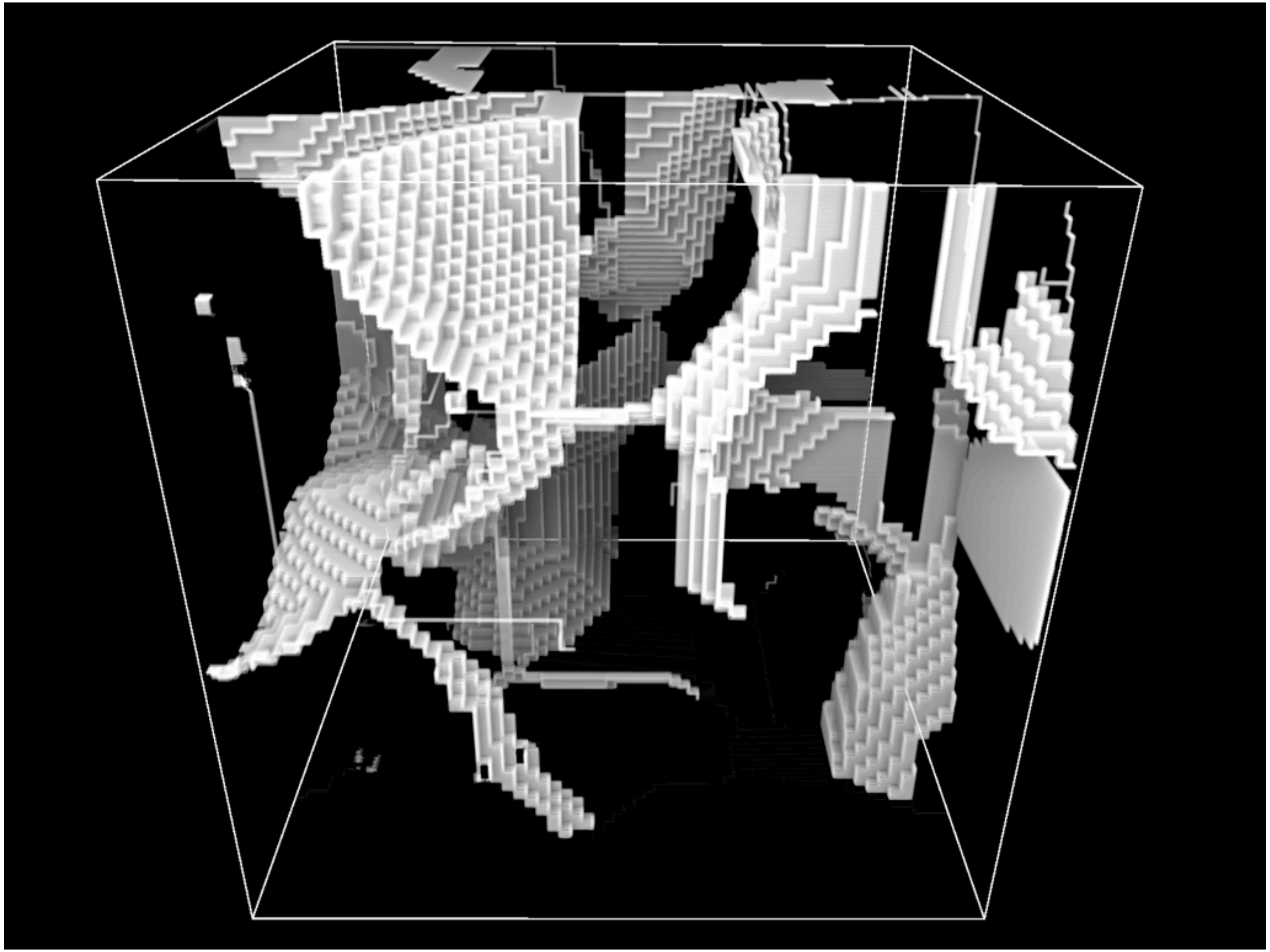
Theorem 11. *Given a grayscale digital image, $g : D \to \mathbb{R}$, and its associated cubical complex, $\mathcal{K}$, let x_i be the ith voxel in the ordering by grayscale values, $\mathcal{K}_i$ be the lower level cut at the value $g(x_i)$, and L_i be the lower star of x_i. Then, for each critical p-cell $\gamma^{(p)} \in L_i$ identified by* ProcessLowerStars *either*

1. *γ creates a new p-cycle in $H_p(\mathcal{K}_i)$ or*
2. *γ fills in a $(p-1)$-cycle from $H_{p-1}(\mathcal{K}_{i-1})$.*



* Use the Meyer-Vietoris sequence to show that any change in homology between K_(i-1) and K_i is signalled by the presence of a non-bounding cycle in the lower link.
* Show ProcessLowerStar is equivalent to an algorithm acting on the lower link.
* Prove that the algorithm cannot create `negative' critical cells within the lower link.
* see IEEE TPAMI vol.33 p.1646 (2011) August feature article

Applications



Future directions

* What does ProcessLowerStar do in higher dimensions? (c.f. Couprie and Bertrand TPAMI 2009).
* Morse cancellation and simplification.
* Using topological information in image segmentation.
* Dynamic CT - fluid flow as a Morse filtration.