

Birth and Death

Amit Patel

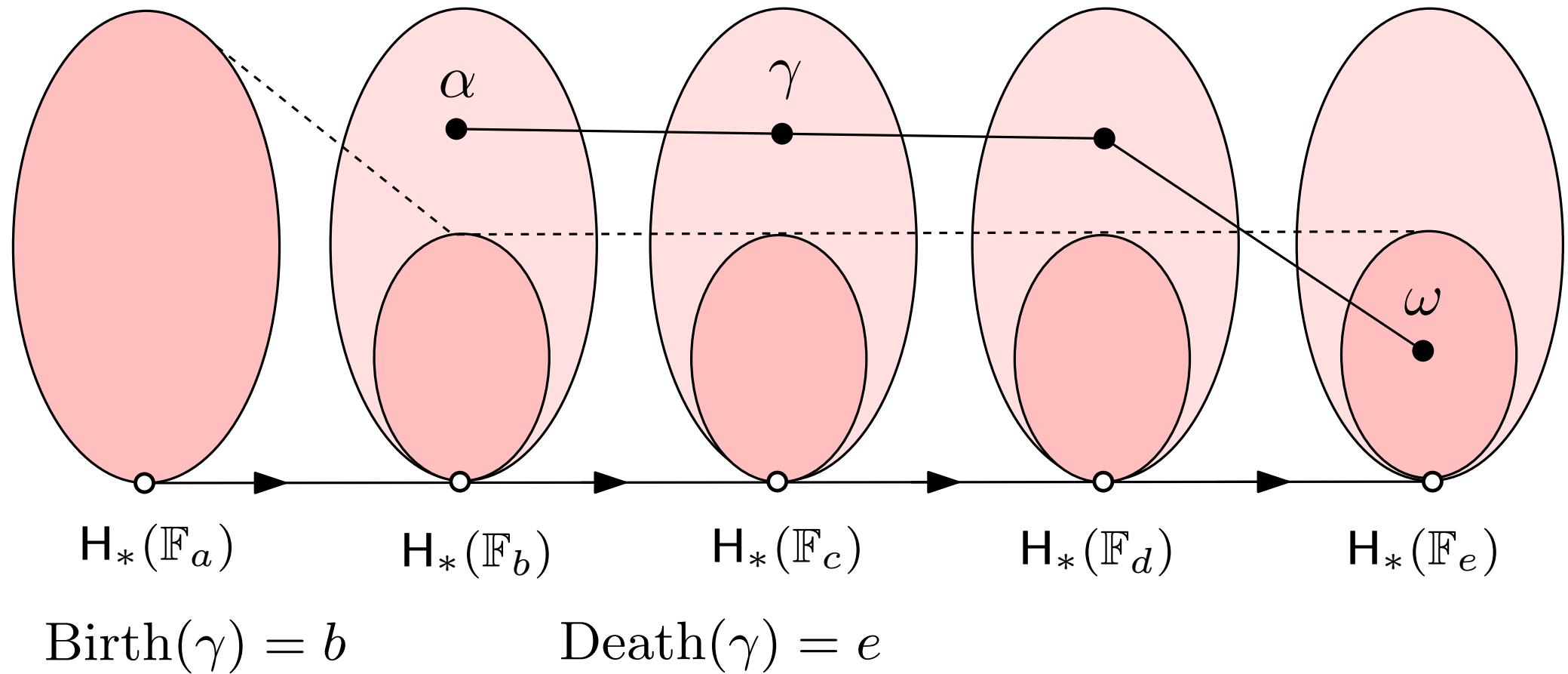
Rutgers University - Math

Persistence

Given $f : \mathbb{X} \rightarrow R$ and $a \in R$, let $\mathbb{F}_a = f^{-1}(\infty, a]$

When is $\gamma \in H_*(\mathbb{F}_a)$ born and when does it die?

$$\cdots < a < b < c < d < e < \cdots$$

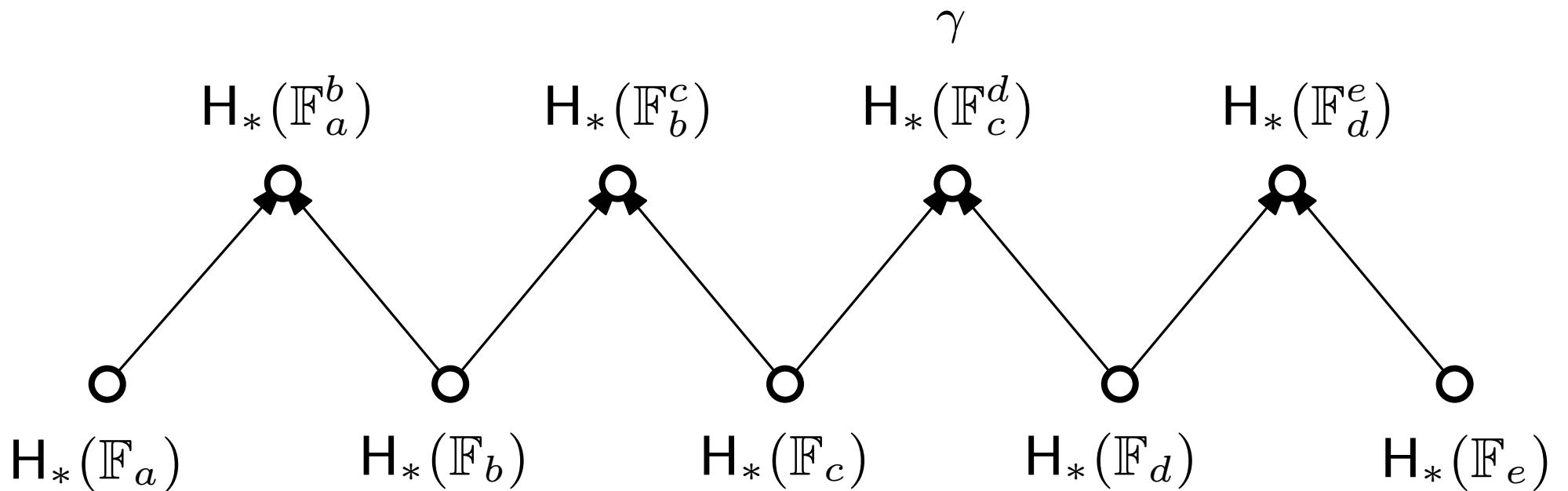


Level Set Zigzag Persistence [CdSM'09]

Given $f : \mathbb{X} \rightarrow R$ and $[c, d] \subset R$, when is $\gamma \in H_*(f^{-1}[c, d])$ born and when does it die?

For $[a, b] \subset R$, let $\mathbb{F}_a^b = f^{-1}[a, b]$ For $a \in R$, let $\mathbb{F}_a = f^{-1}(a)$

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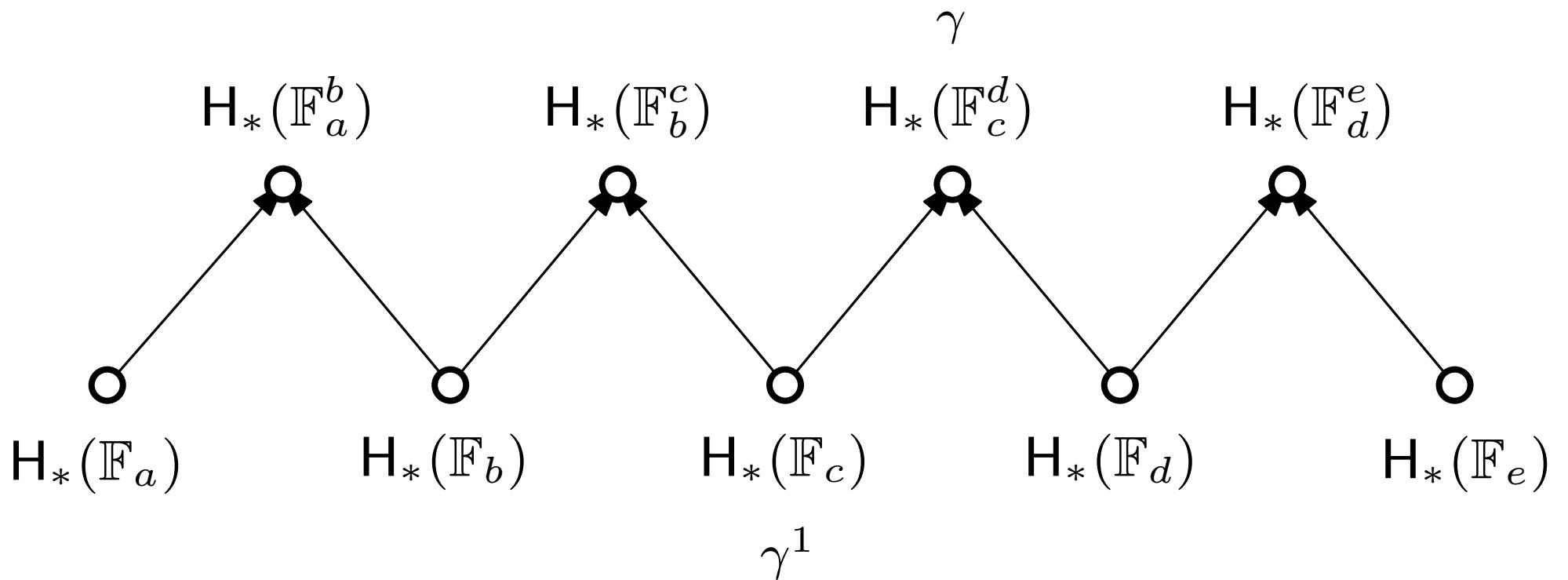


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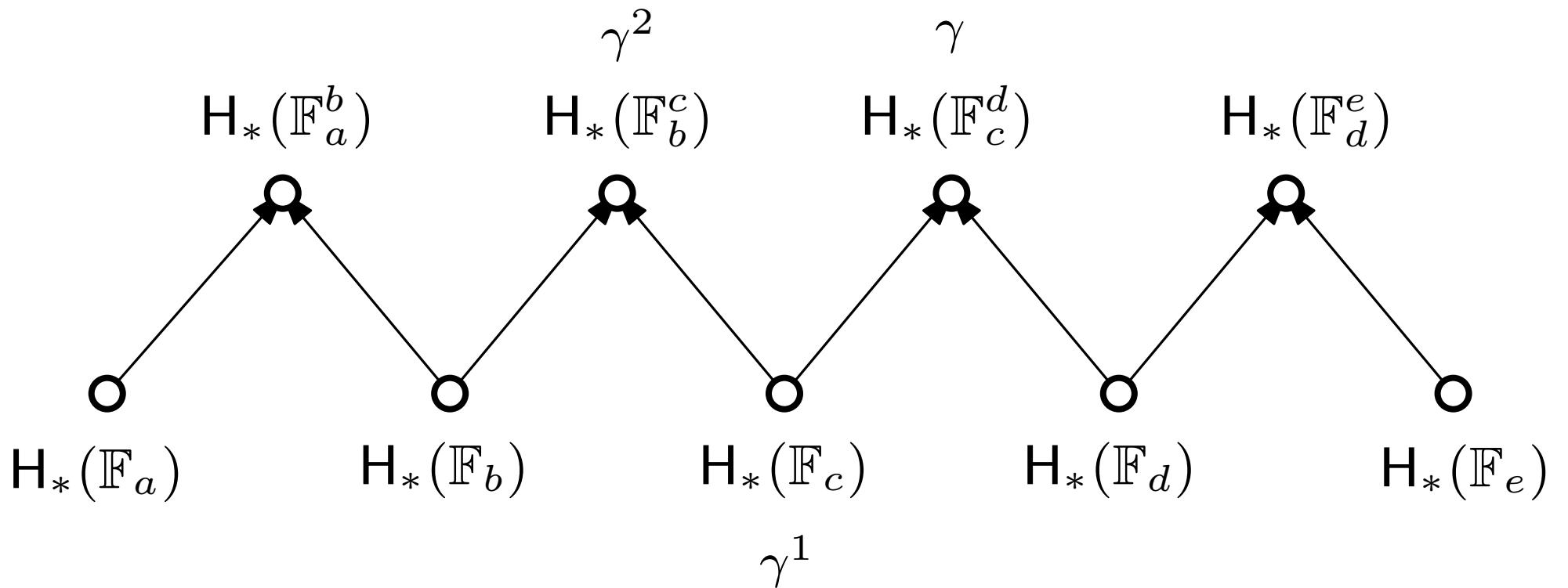


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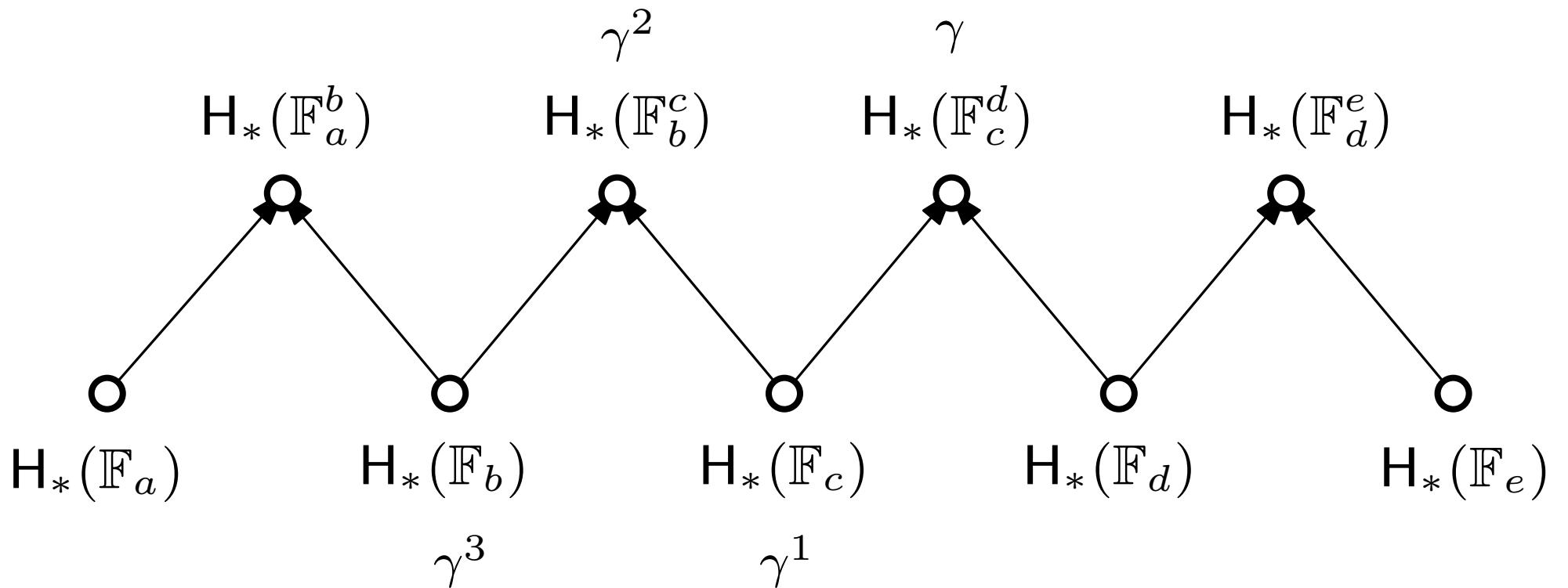


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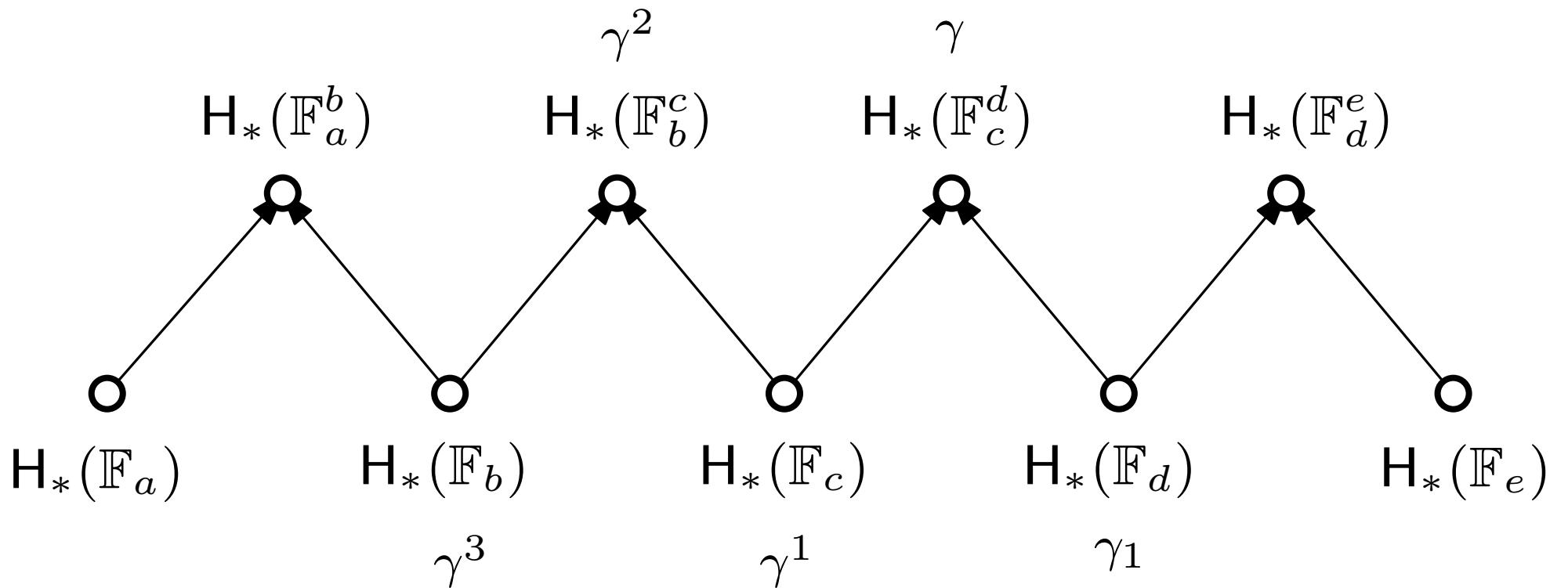


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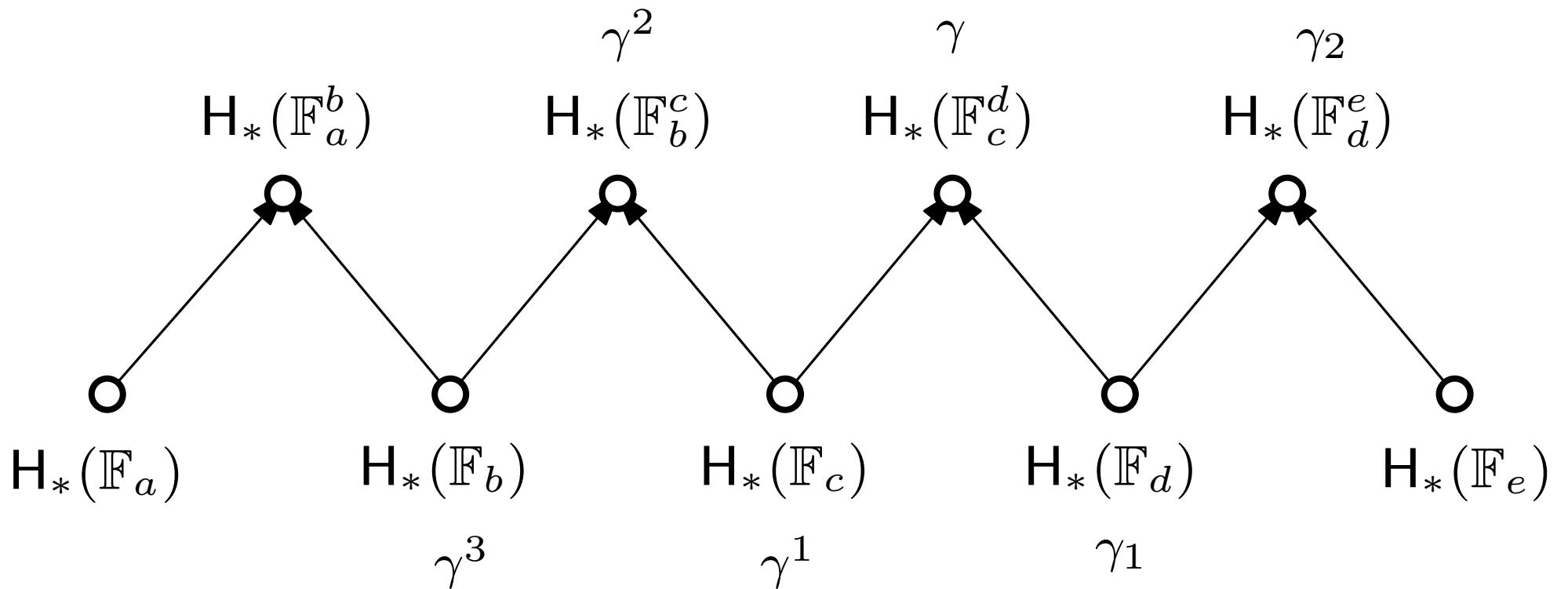


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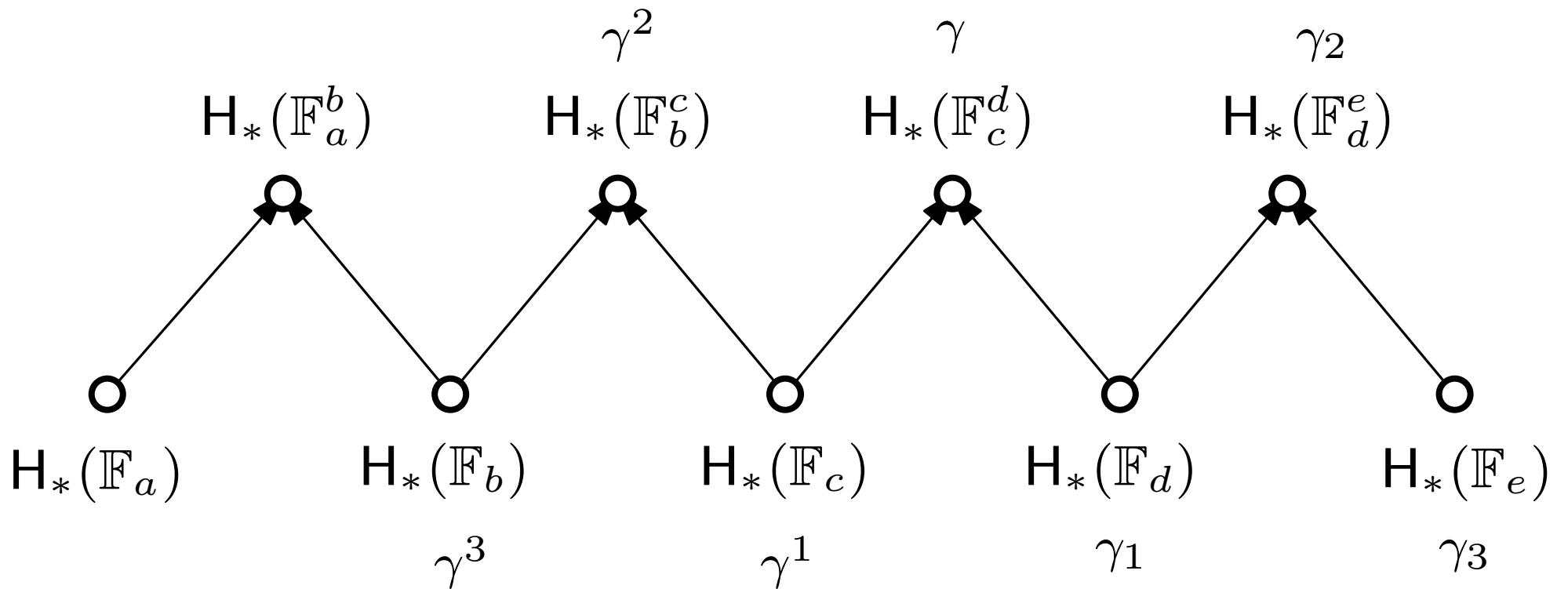


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Higher Dimensions

Given $f : \mathbb{X} \rightarrow M$, where M is an orientable Riemannian m -manifold and $U \subseteq M$ an open set, when is $\gamma \in \mathsf{H}_{m+p}^c(\mathbb{F}_U)$ born and when does it die? Assume f is proper.

For any $r \geq 0$, let $U^r = \{x \in M \mid \text{Dist}(x, U) \leq \varepsilon\}$

For $0 < r_1 < r_2 < r_3 < \dots$, we have

$$\dots \supseteq U^3 \supseteq U^2 \supseteq U^1 \supseteq U = U \subseteq U^1 \subseteq U^2 \subseteq U^3$$

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$$\downarrow \Phi$$

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Closed Homology

What is $\mathsf{H}_*^c(\mathbb{F}_U)$, for an open set $U \subseteq M$?

Let $\{K_i\}_{i \in I}$ be the collection of compact sets $K_i \subseteq f^{-1}(U)$

For $K_i \subseteq K_j$, we have $f_{ij} : \mathsf{H}_*(\mathbb{X}, \mathbb{X} - K_j) \rightarrow \mathsf{H}_*(\mathbb{X}, \mathbb{X} - K_i)$

$$\mathsf{H}_*^c(\mathbb{F}_U) := \varprojlim_{i \in I} \mathsf{H}_*(\mathbb{X}, \mathbb{X} - K_i)$$

$$= \{\alpha \in \prod_{i \in I} \mathsf{H}_*(\mathbb{X}, \mathbb{X} - K_i) \mid \alpha_i = f_{ij}(\alpha_j) \text{ for } K_i \subseteq K_j\}$$

For each compact K_i , there is $\pi_i : \mathsf{H}_*^c(U) \rightarrow \mathsf{H}_*(\mathbb{X}, \mathbb{X} - K_i)$

For an open set $V \subseteq U$, there is a unique morphism

$$u : \mathsf{H}_*^c(U) \rightarrow \mathsf{H}_*^c(V)$$

The Φ Map

$$\begin{array}{ccccccc}
 \rightarrow & \mathsf{H}_{m+p}^c(\mathbb{F}_{U^3}) & \rightarrow & \mathsf{H}_{m+p}^c(\mathbb{F}_{U^2}) & \rightarrow & \mathsf{H}_{m+p}^c(\mathbb{F}_{U^1}) & \rightarrow \mathsf{H}_{m+p}^c(\mathbb{F}_U) \\
 & & & & & & \downarrow \Phi \\
 \leftarrow & \mathsf{H}_p(\mathbb{F}_{U^3}) & \leftarrow & \mathsf{H}_p(\mathbb{F}_{U^2}) & \leftarrow & \mathsf{H}_p(\mathbb{F}_{U^1}) & \leftarrow \mathsf{H}_p(\mathbb{F}_U)
 \end{array}$$

For now, assume $U \subseteq M$ is path connected

There is a consistent choice of a generator

$$\mu_x \in \mathsf{H}^n(M, M - \{x\}), \text{ for each } x \in M$$

$$\begin{array}{ccc}
 \mathsf{H}_{m+p}^c(\mathbb{F}_U) & \xrightarrow{\pi_x} & \mathsf{H}_{m+p}(\mathbb{X}, \mathbb{X} - f^{-1}(x)) \\
 & & \swarrow \cong \\
 \mathsf{H}_p(\mathbb{F}_U) & \xleftarrow{\frown f^*(\mu_x)} & \mathsf{H}_{m+p}(\mathbb{F}_U, \mathbb{F}_U - f^{-1}(x))
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Birth and Death Restricts Nicely

For $V \subseteq U \subseteq M$ open and path connected

If $b = \text{Birth}(\gamma)$ and $d = \text{Death}(\gamma)$, then $b \leq \text{Birth}(\gamma')$ and $d \leq \text{Death}(\gamma')$.

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Stability

Let $h : \mathbb{X} \times [0, 1] \rightarrow M$ be a homotopy connecting $f = h_0$ to $g = h_1$

The *homotopy speed* of h is

$$\delta = \inf_{\varepsilon \geq 0} \text{Dist}(f(x), h_t(x)) \leq \varepsilon t \quad \forall x \in X, \forall t \in [0, 1]$$

Let $V \subset U \subseteq M$ be open sets such that $V^\delta \subseteq U$. Then

$$\begin{array}{ccccccc}
 & & & & & \gamma & \\
 \rightarrow H^c_{m+p}(\mathbb{F}_{U^3}) & \rightarrow H^c_{m+p}(\mathbb{F}_{U^2}) & \rightarrow H^c_{m+p}(\mathbb{F}_{U^1}) & \rightarrow H^c_{m+p}(\mathbb{F}_U) & \longrightarrow & & \\
 \downarrow & \downarrow & \downarrow & & & \downarrow \gamma' & \\
 \rightarrow H^c_{m+p}(\mathbb{G}_{V^3}) & \rightarrow H^c_{m+p}(\mathbb{G}_{V^2}) & \rightarrow H^c_{m+p}(\mathbb{G}_{V^1}) & \rightarrow H^c_{m+p}(\mathbb{G}_V) & & & \\
 & & & & \downarrow \Phi & & \Phi \\
 \leftarrow H_p(\mathbb{G}_{V^3}) & \leftarrow H_p(\mathbb{G}_{V^2}) & \leftarrow H_p(\mathbb{G}_{V^1}) & \leftarrow H_p(\mathbb{G}_V) & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & & & \\
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 \end{array}$$

$$\text{Birth}(\gamma) \leq \text{Birth}(\gamma') \leq \text{Birth}(\gamma) + \delta + \varepsilon$$

$$\text{Death}(\gamma) \leq \text{Death}(\gamma') \leq \text{Death}(\gamma) + \delta + \varepsilon$$

Last Slide

An extension of the classic notion of birth and death to mappings $f : \mathbb{X} \rightarrow M$, where M is an orientable Riemannian manifold.

- For each open set $U \subseteq M$ and $\gamma \in \mathsf{H}_*^c(\mathbb{F}_U)$, we define $\text{Birth}(\gamma)$ and $\text{Death}(\gamma)$.
- Birth and death is consistent with the restriction map $\mathsf{H}_*^c(\mathbb{F}_U) \rightarrow \mathsf{H}_*^c(\mathbb{F}_V)$, for $V \subseteq U$.
- Birth and death is stable to homotopic perturbations to f .