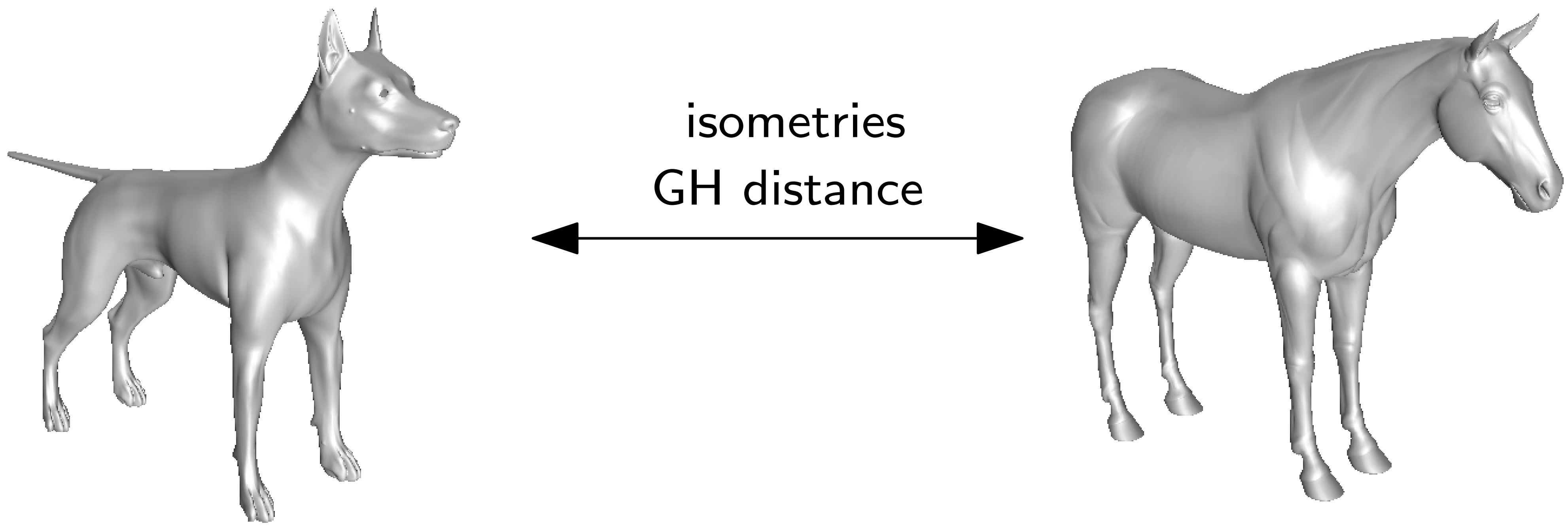
Workshop on Computational Topology – November 7-11, 2011

Stable Multi-Scale Signatures for Shapes using Topological Persistence

Steve Oudot – Geometrica group, INRIA Saclay – Île-de-France

joint work (still in progress..) with Frédéric Chazal and Alexandre Bos

Comparing Shapes via Signatures



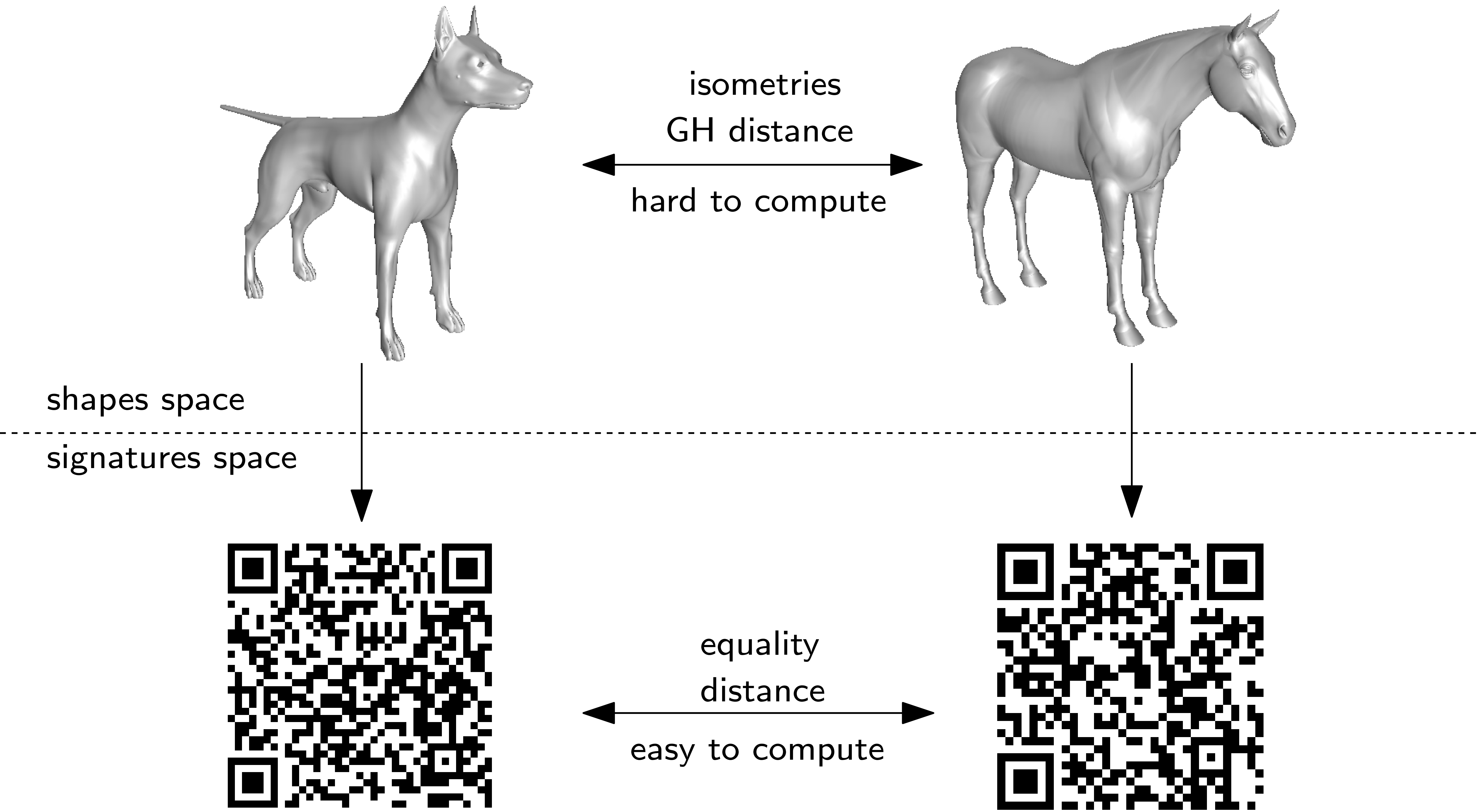
shapes space

Mathematical formulation:

shapes $\equiv$ compact metric spaces

distance between shapes $\equiv$ Gromov-Hausdorff (GH) distance

Comparing Shapes via Signatures



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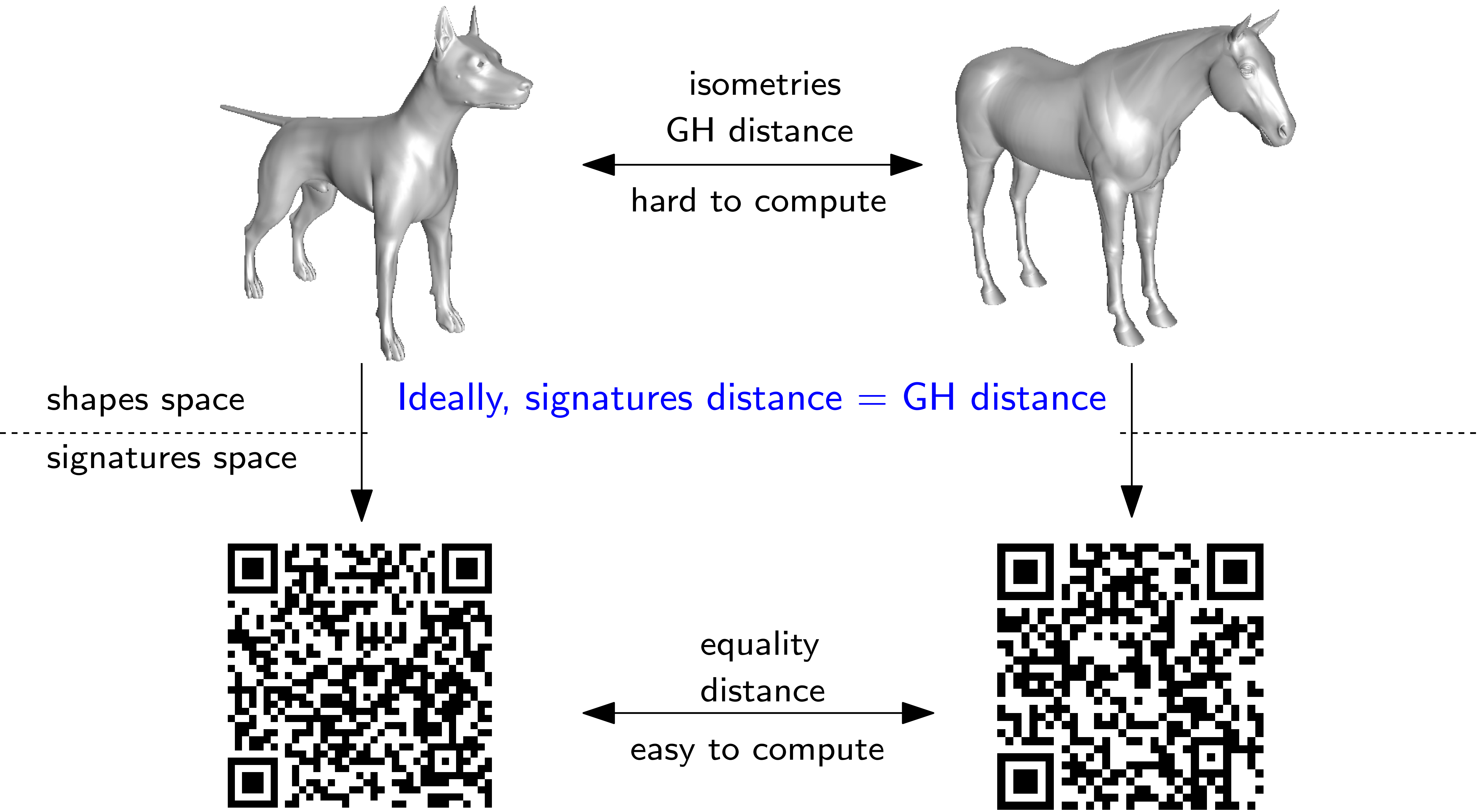

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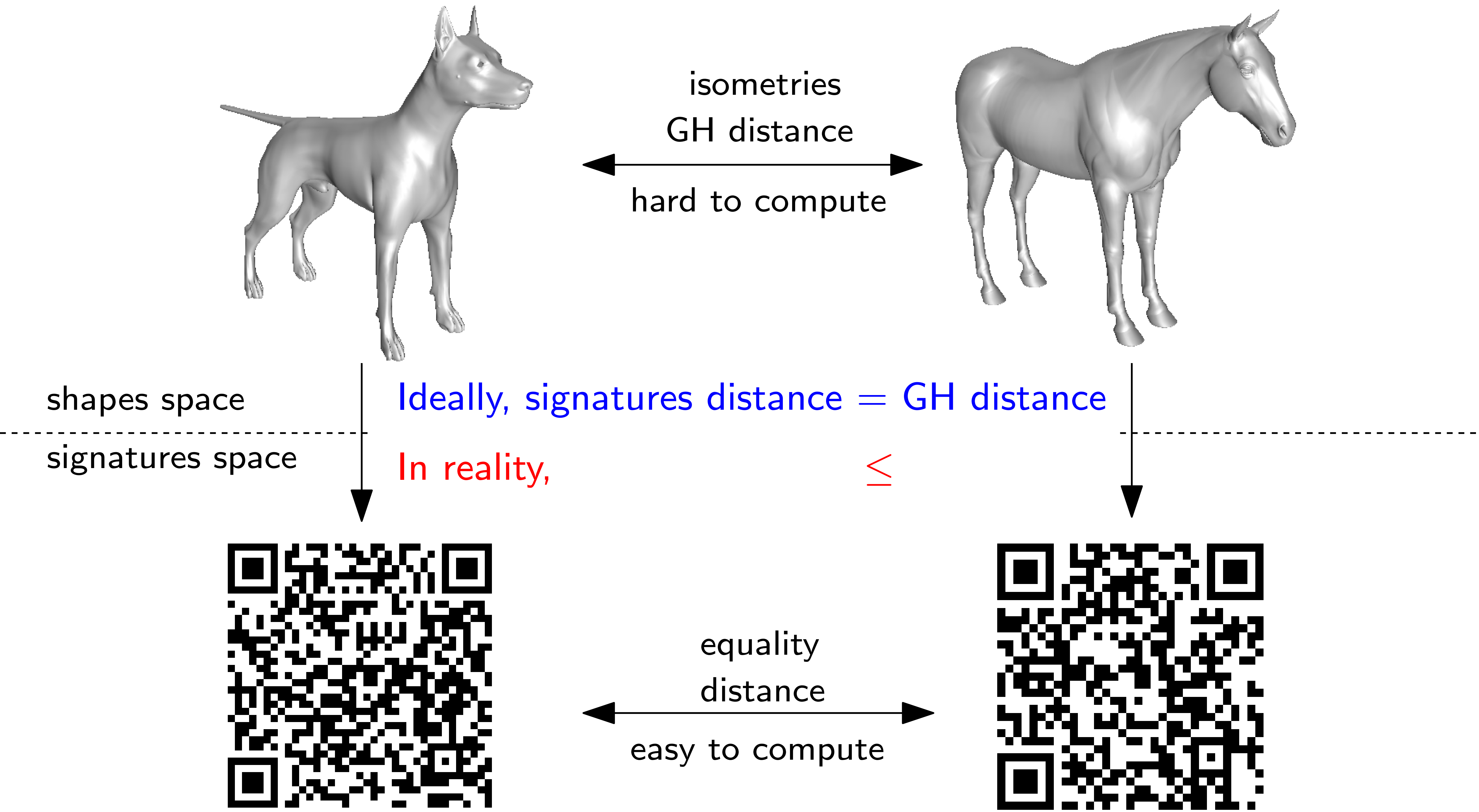
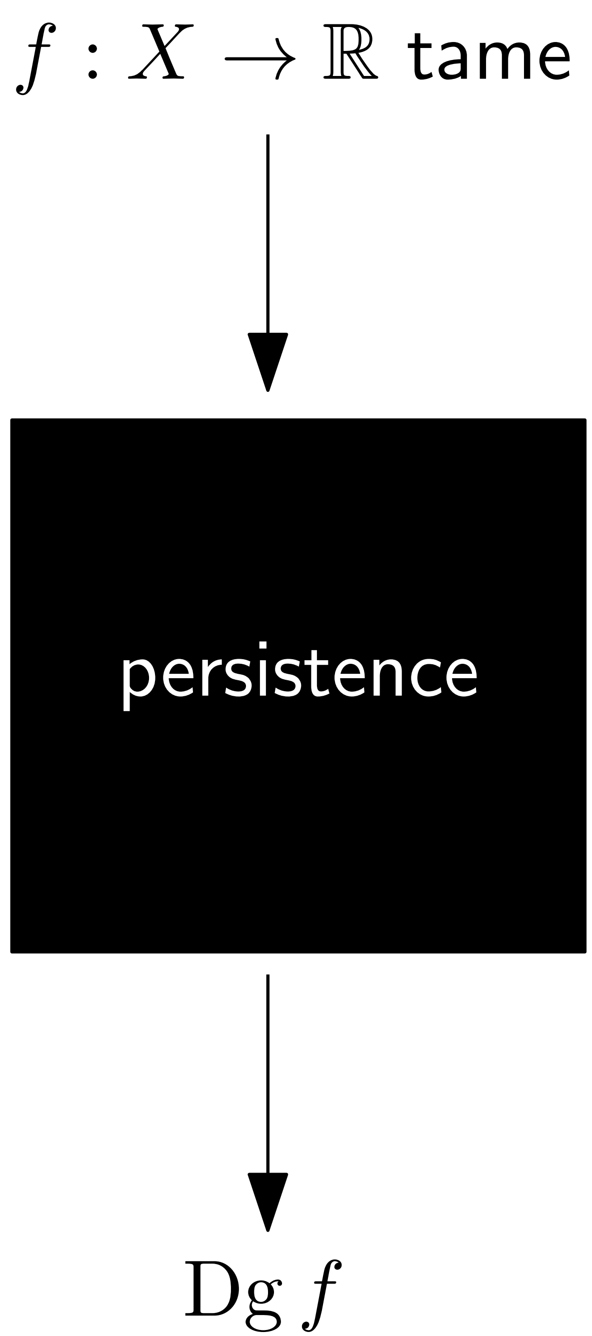


Persistence: from Signatures for Functions...



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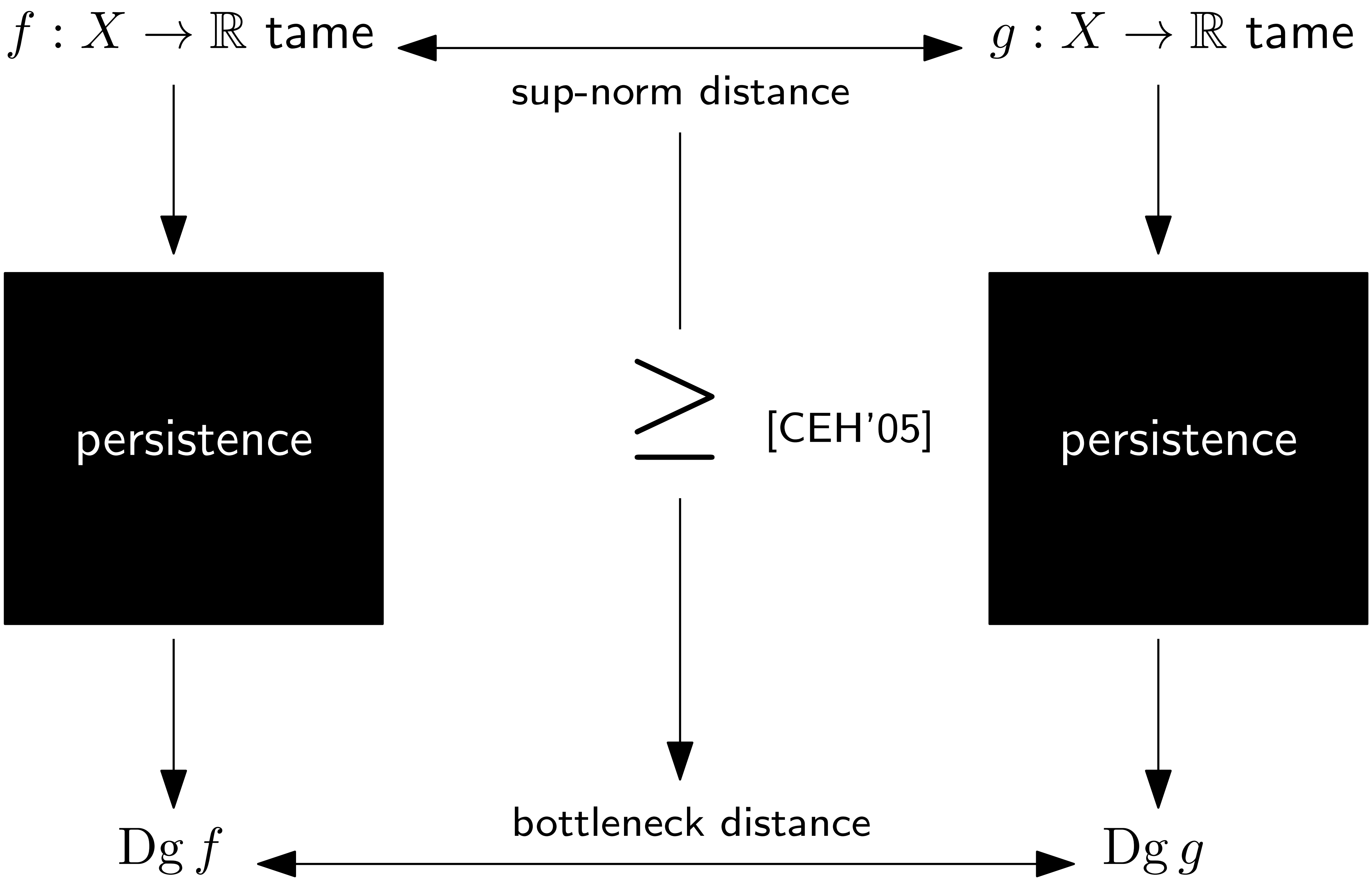

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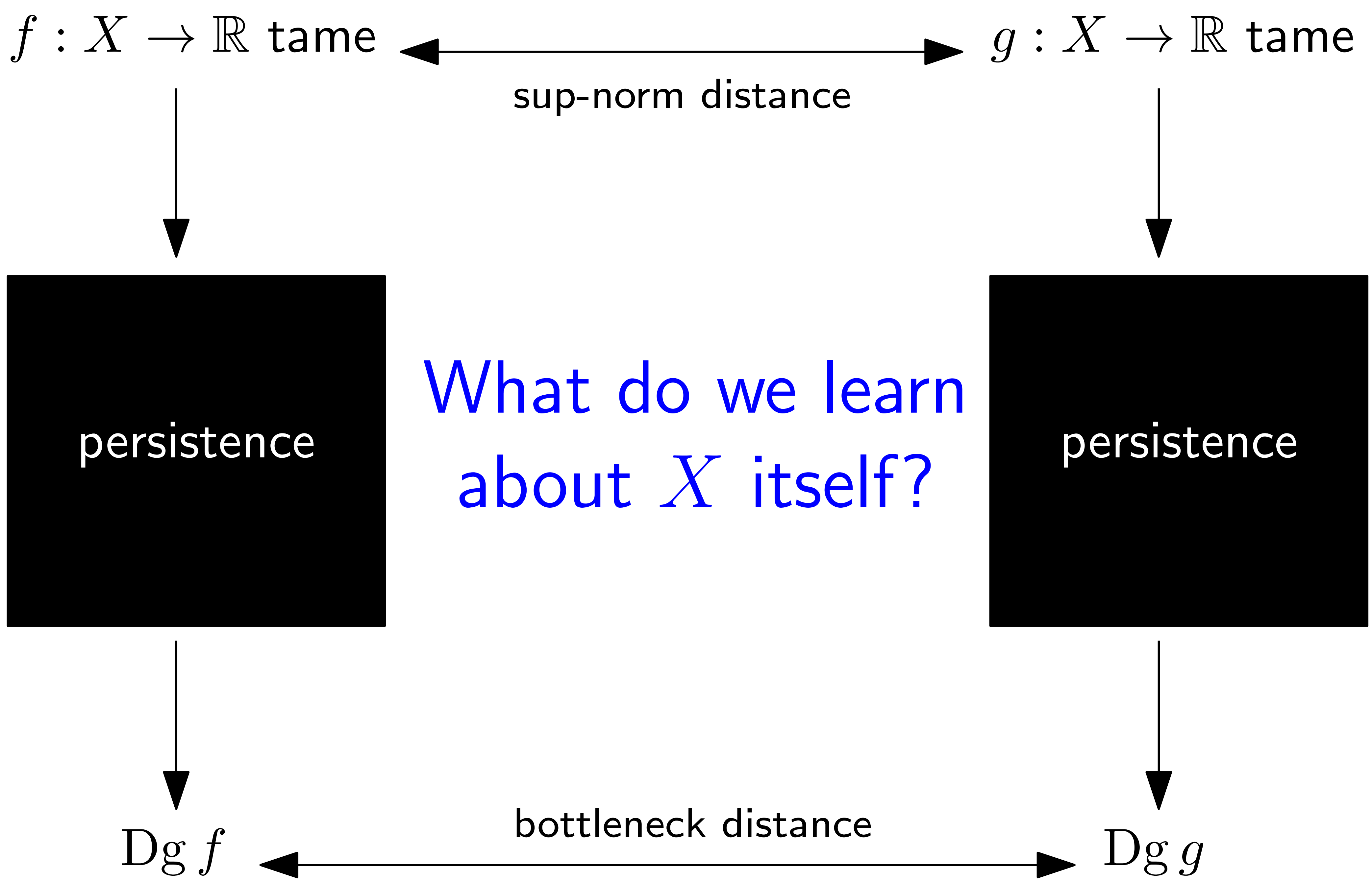

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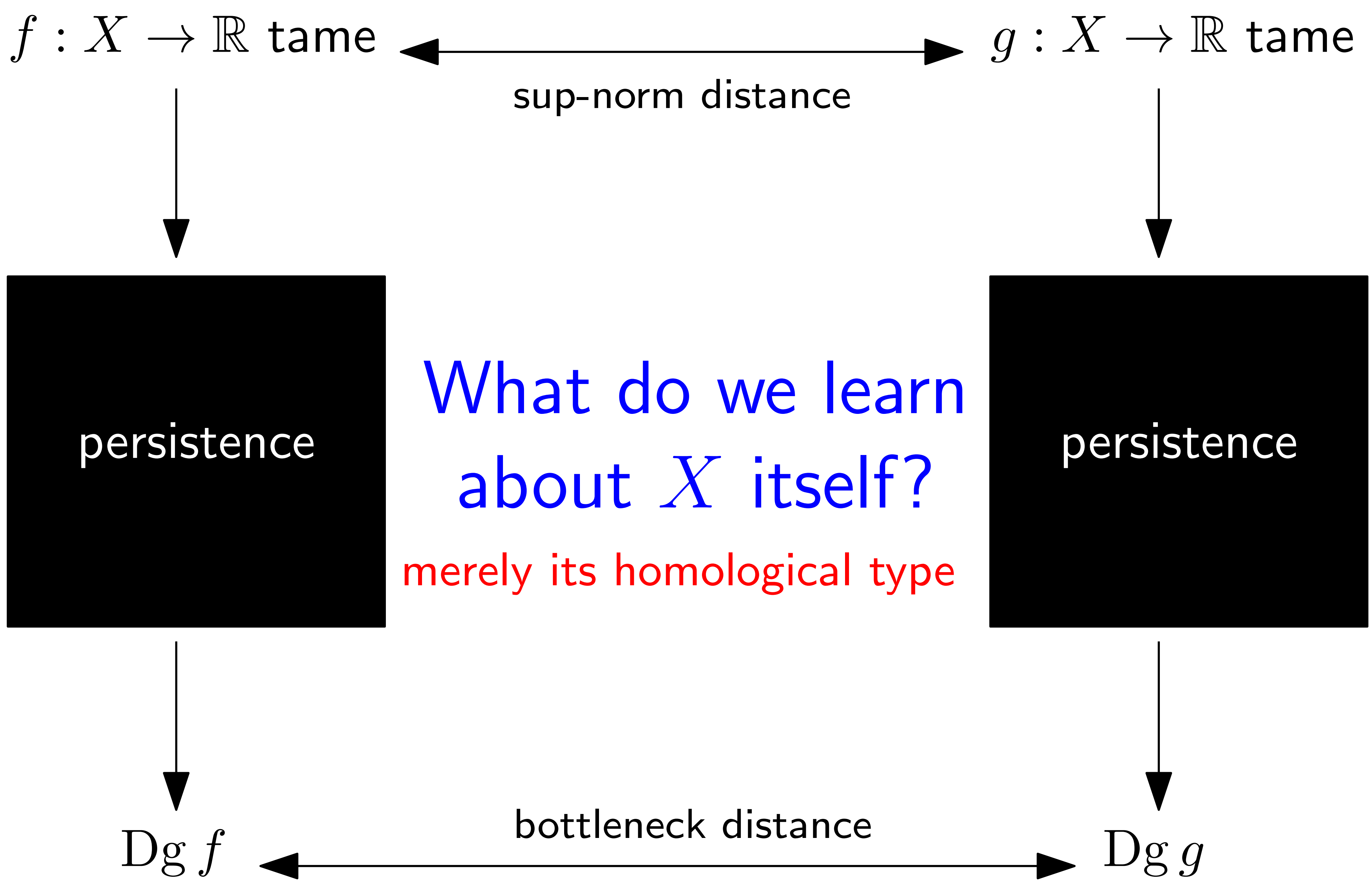

… to Signatures for Spaces

Input: a compact metric space (X, d_X)

Parameter: a Lipschitz continuous function $f : X \to \mathbb{R}$ *derived from* d_X

Signature: $\mathrm{Dg}\, f$

Our hope: that $\mathrm{Dg}\, f$ reveals part of the structure of (X, d_X)

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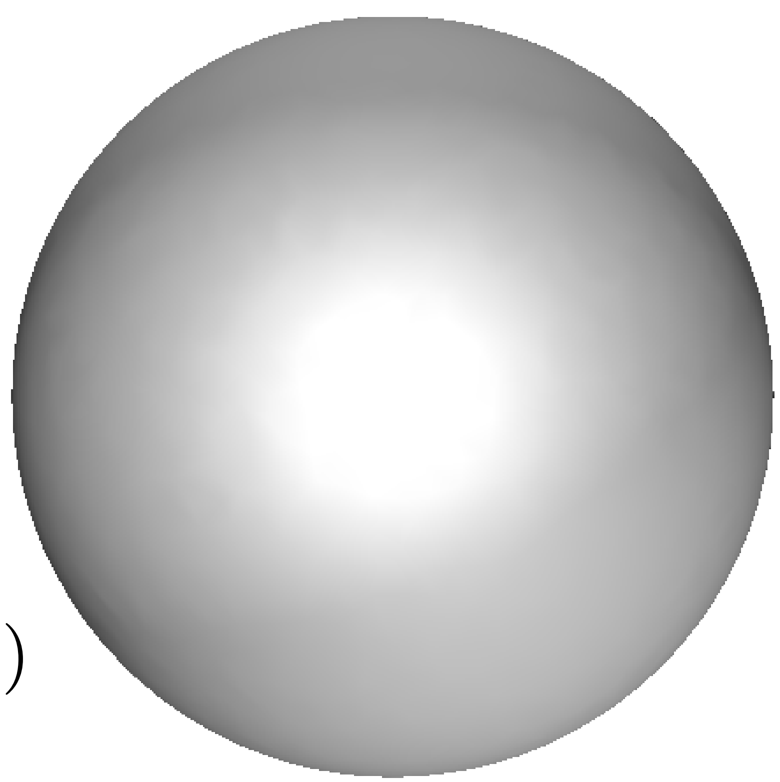
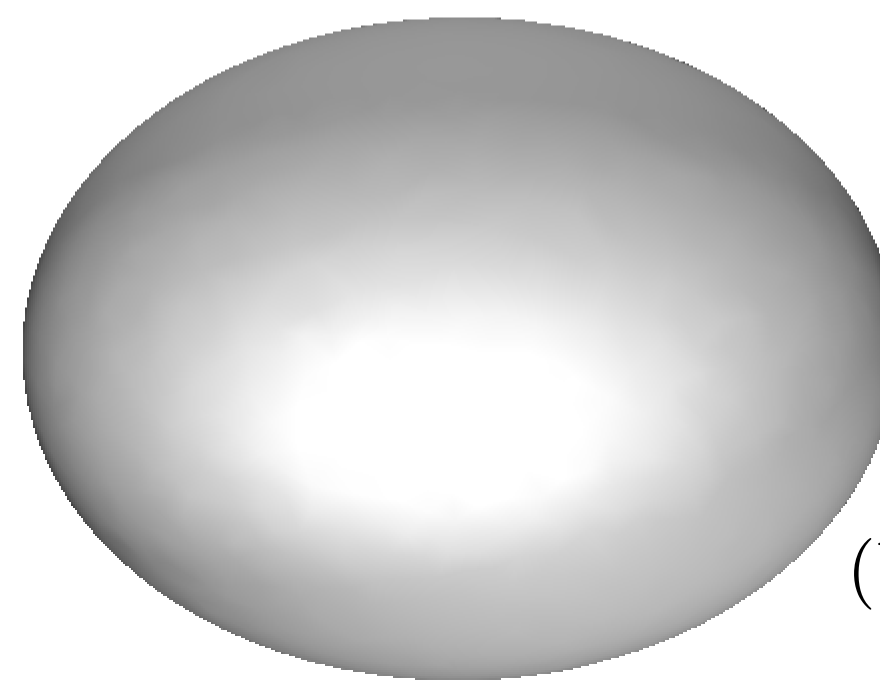
Illustration: distinguishing between a sphere and an ellipsoid:



(X, d_X)



(Y, d_Y)

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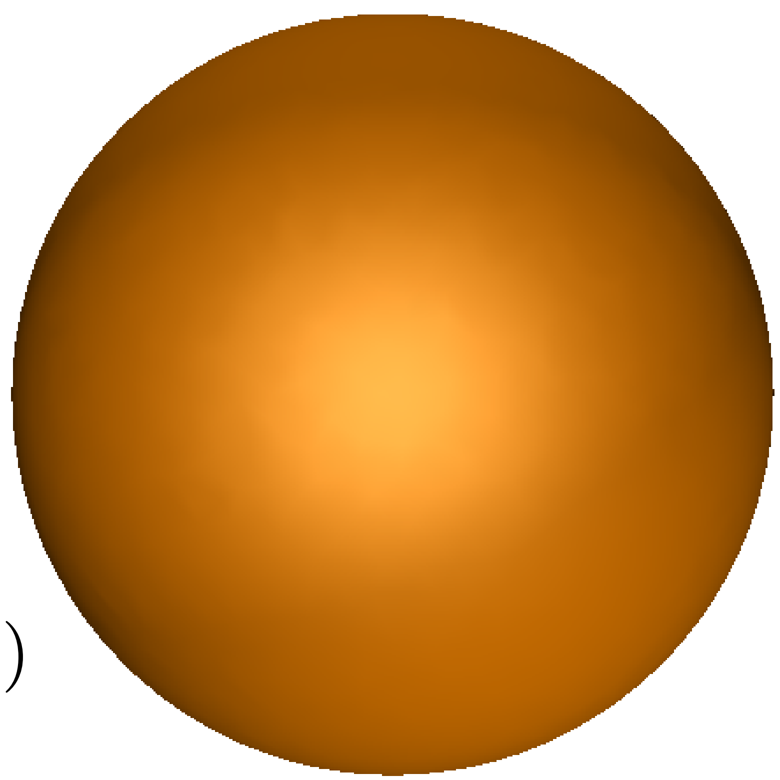
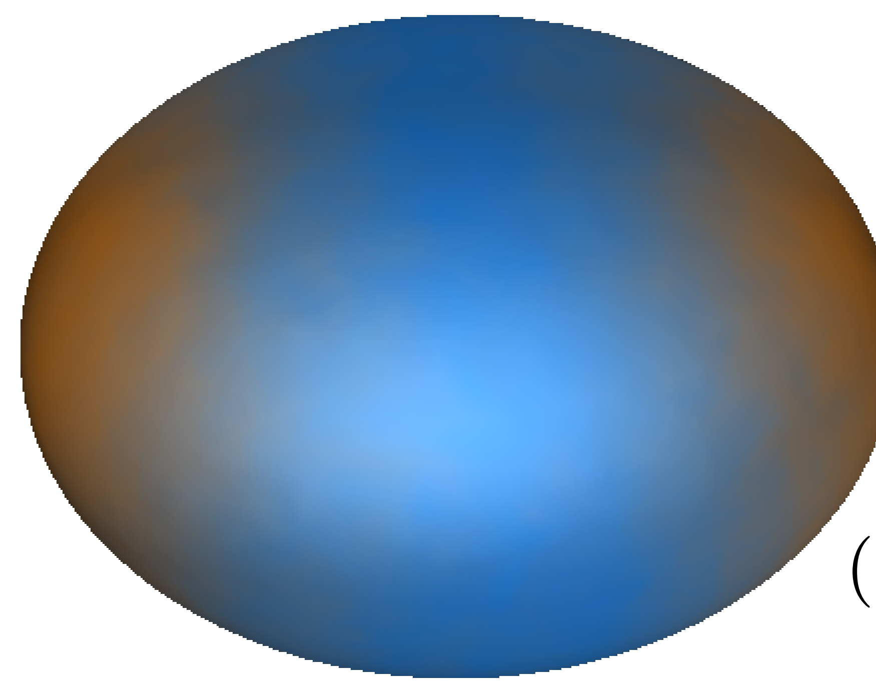
Illustration: distinguishing between a sphere and an ellipsoid:$\to$ eccentricity

(X, d_X)



$$f(x) = \max_{x' \in X} \mathrm{d}_X(x, x')$$



(Y, d_Y)

$$g(x) = \max_{y' \in Y} \mathrm{d}_X(y, y')$$

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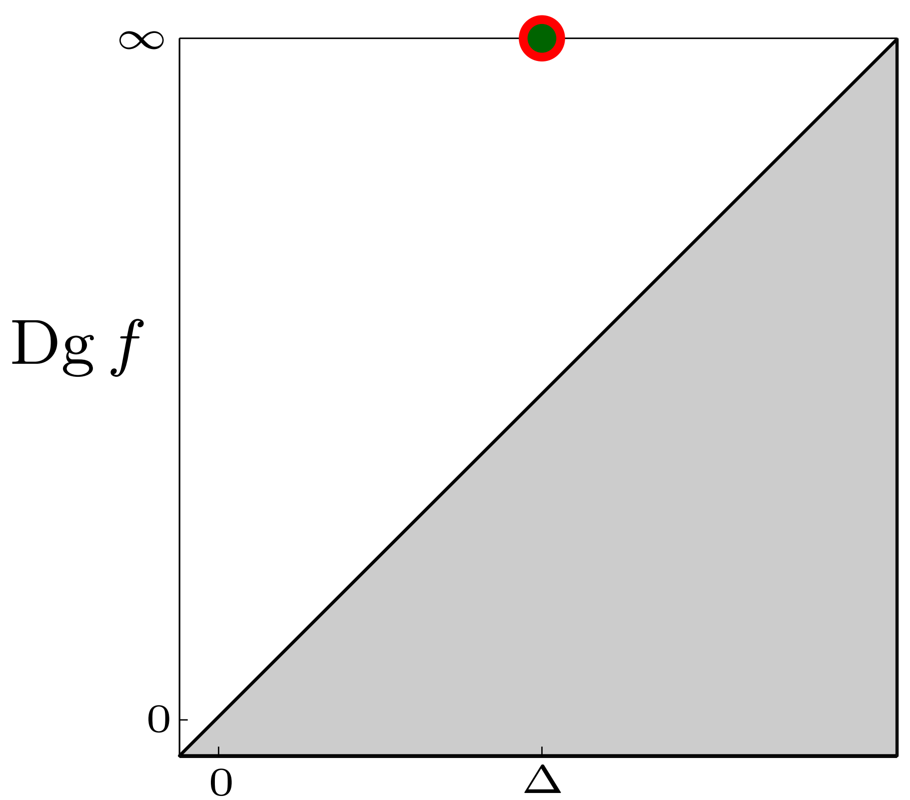
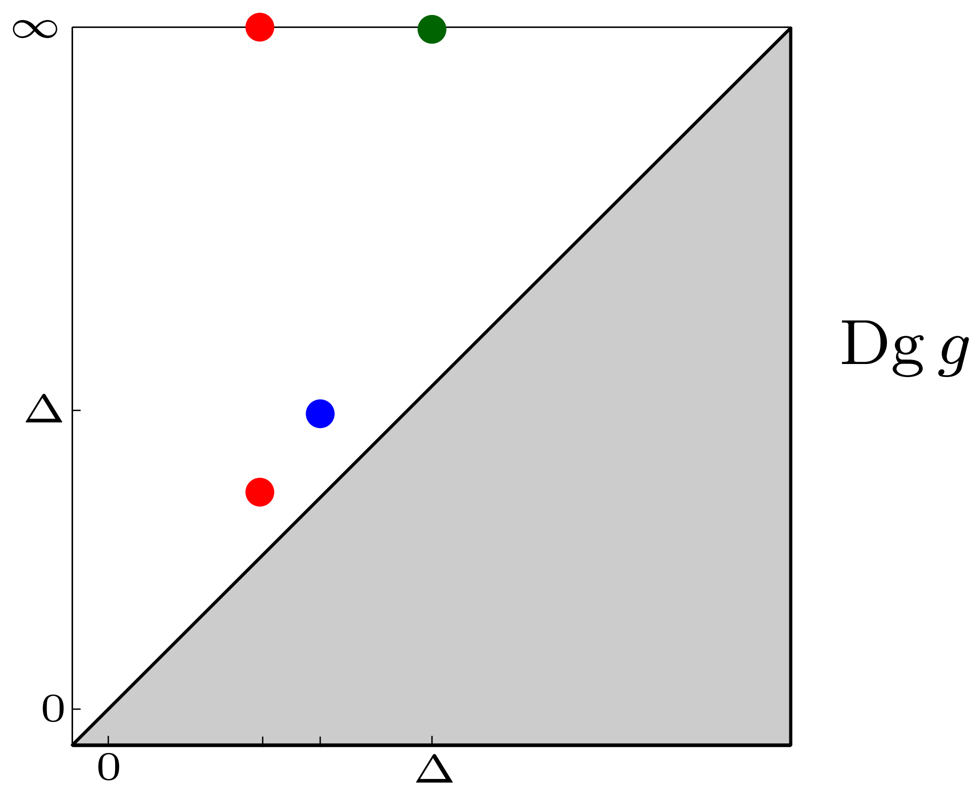


$f(x) = \max_{x' \in X} \mathrm{d}_X(x, x')$

2-dim
1-dim
0-dim



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Other examples of functions derived from d_X:

- higher-order eccentricities

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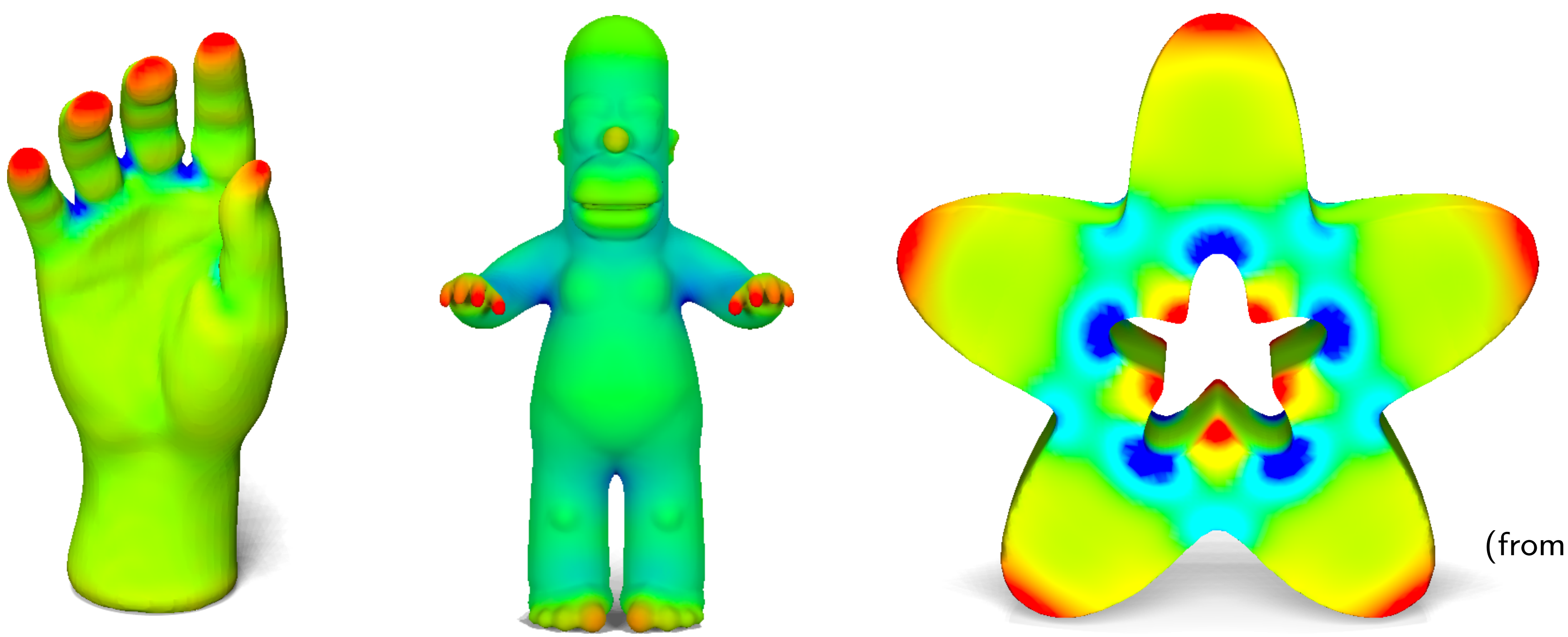
Other examples of functions derived from d_X:

- *heat-kernel signature* [Sun, Ovsjanikov, Guibas 09] (hyp: X Riemannian manifold)



(from [Sun, Ovsjanikov, Guibas 09])

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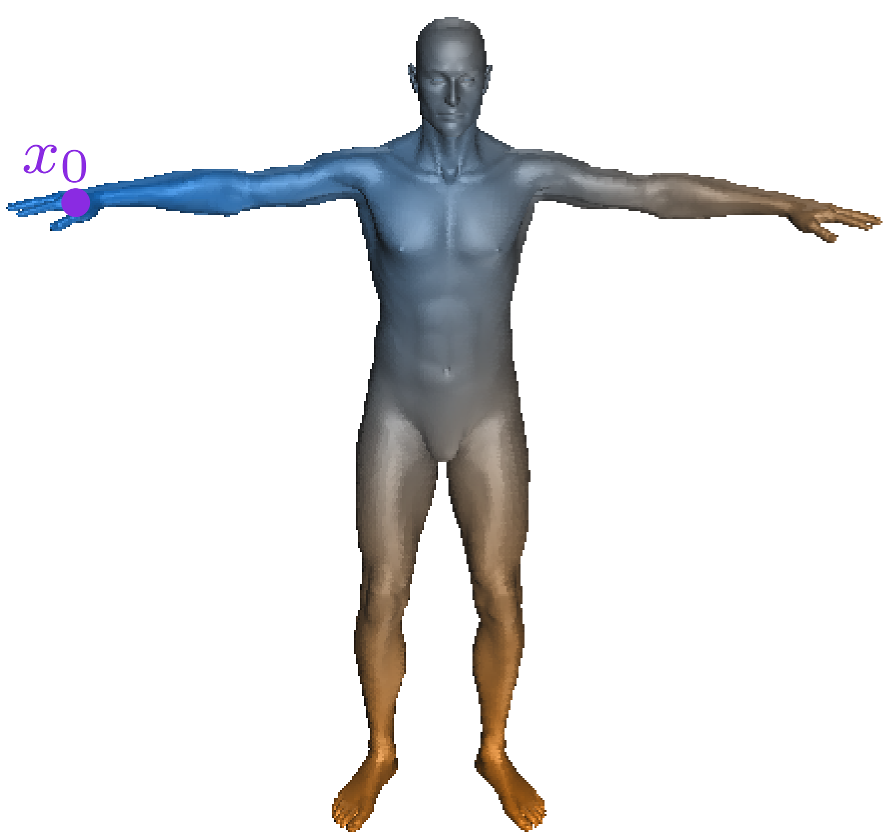
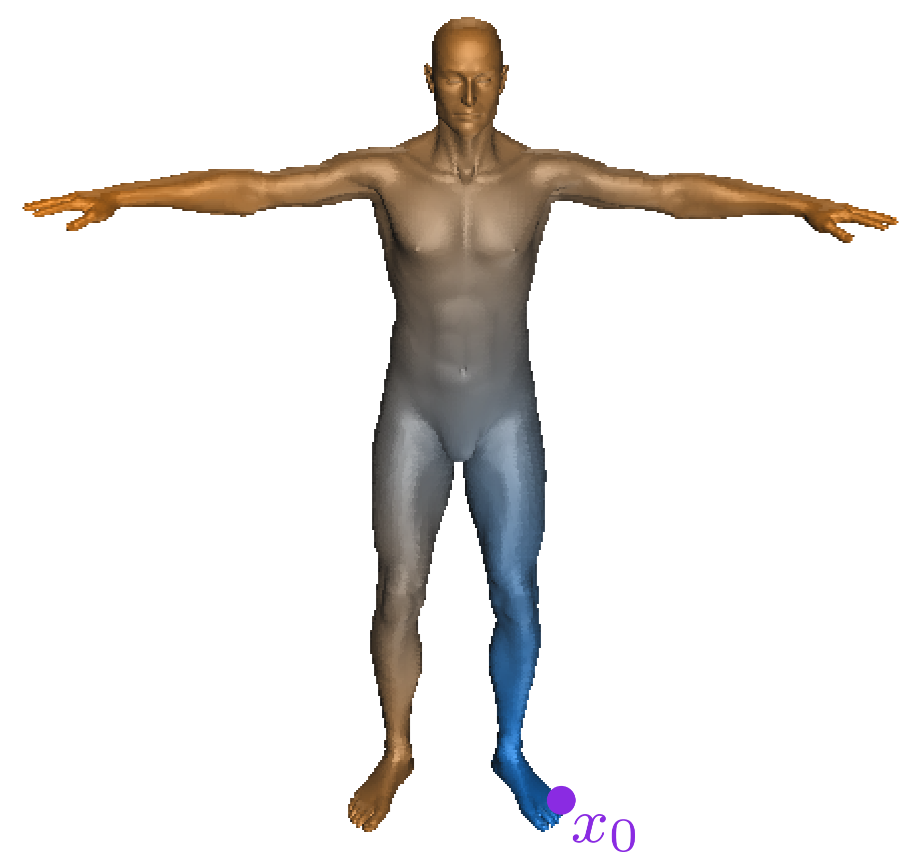
Other examples of functions derived from d_X: (parametrized by base points)

- distance to a base point $x_0 \in X$





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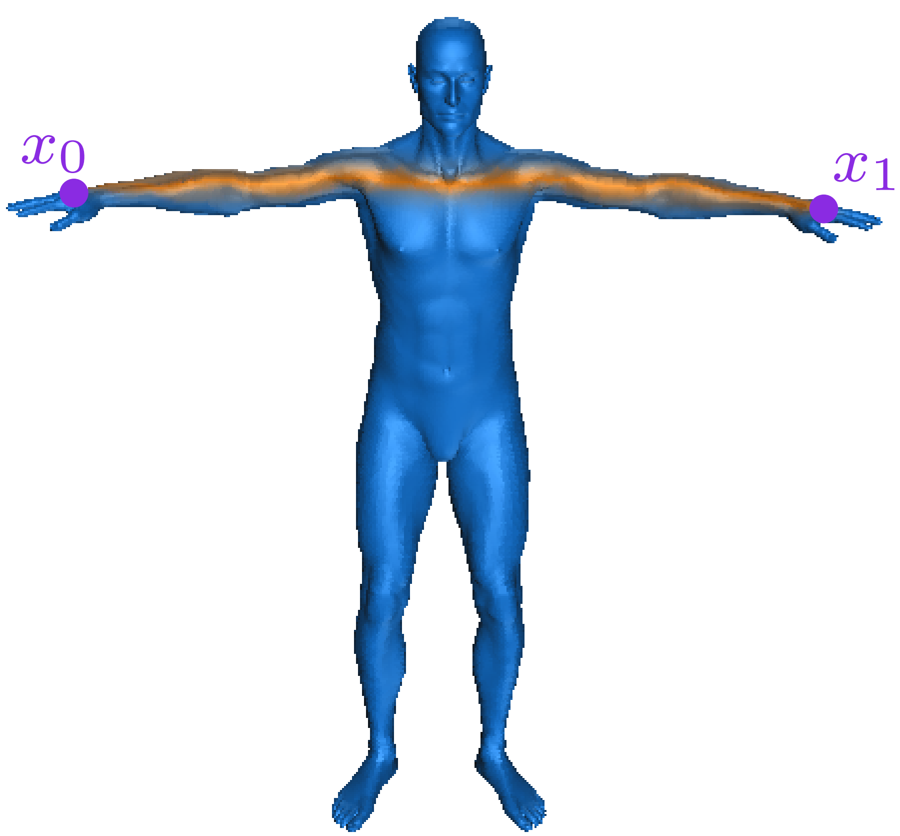
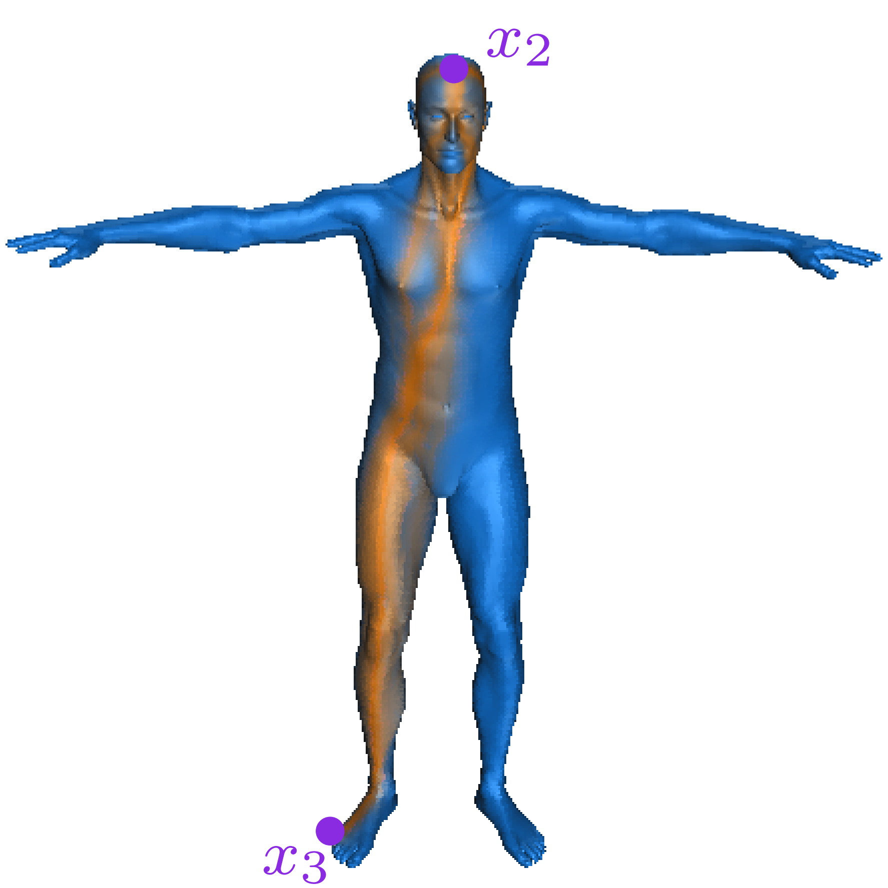
Other examples of functions derived from d_X: (parametrized by base points)

- *fuzzy geodesic* [Sun, Chen, Funkhouser 10]





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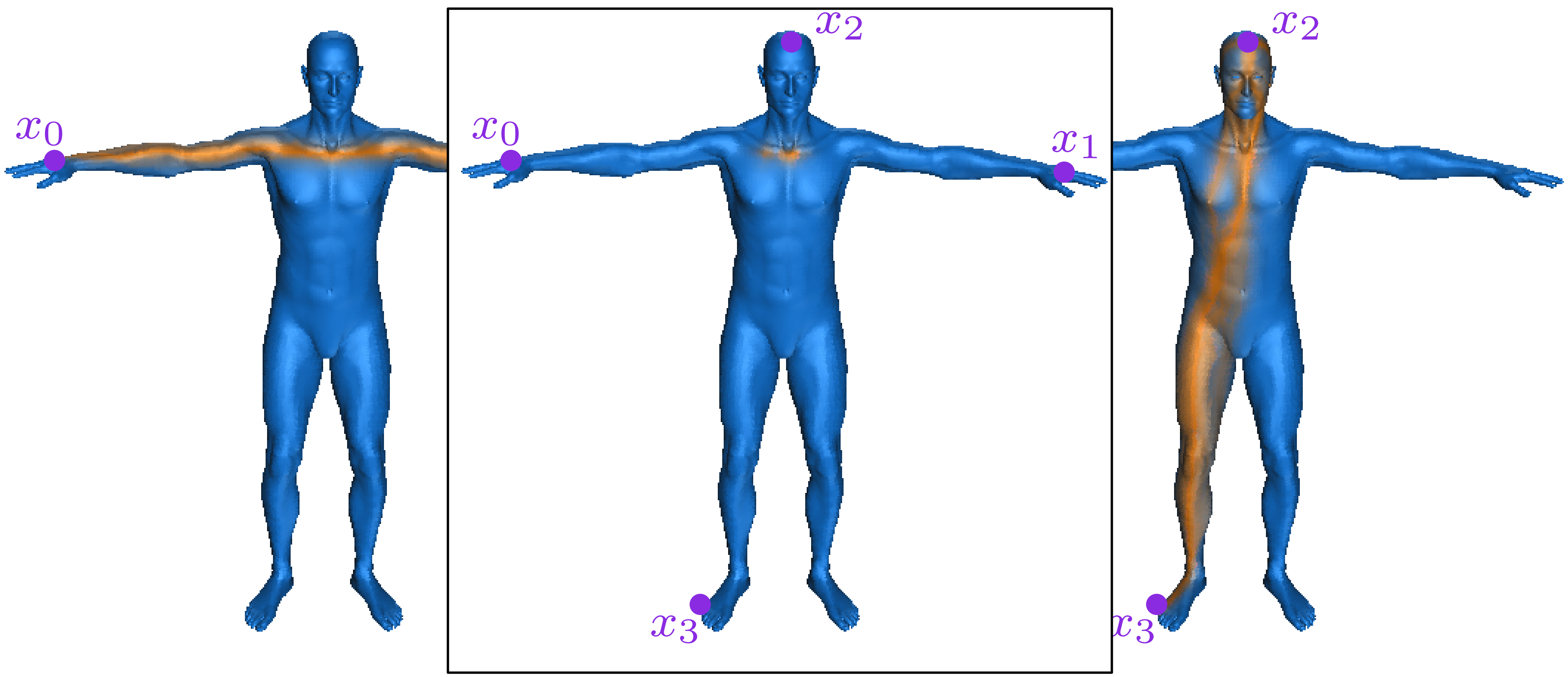
- *intersection configuration* [Sun, Chen, Funkhouser 10]



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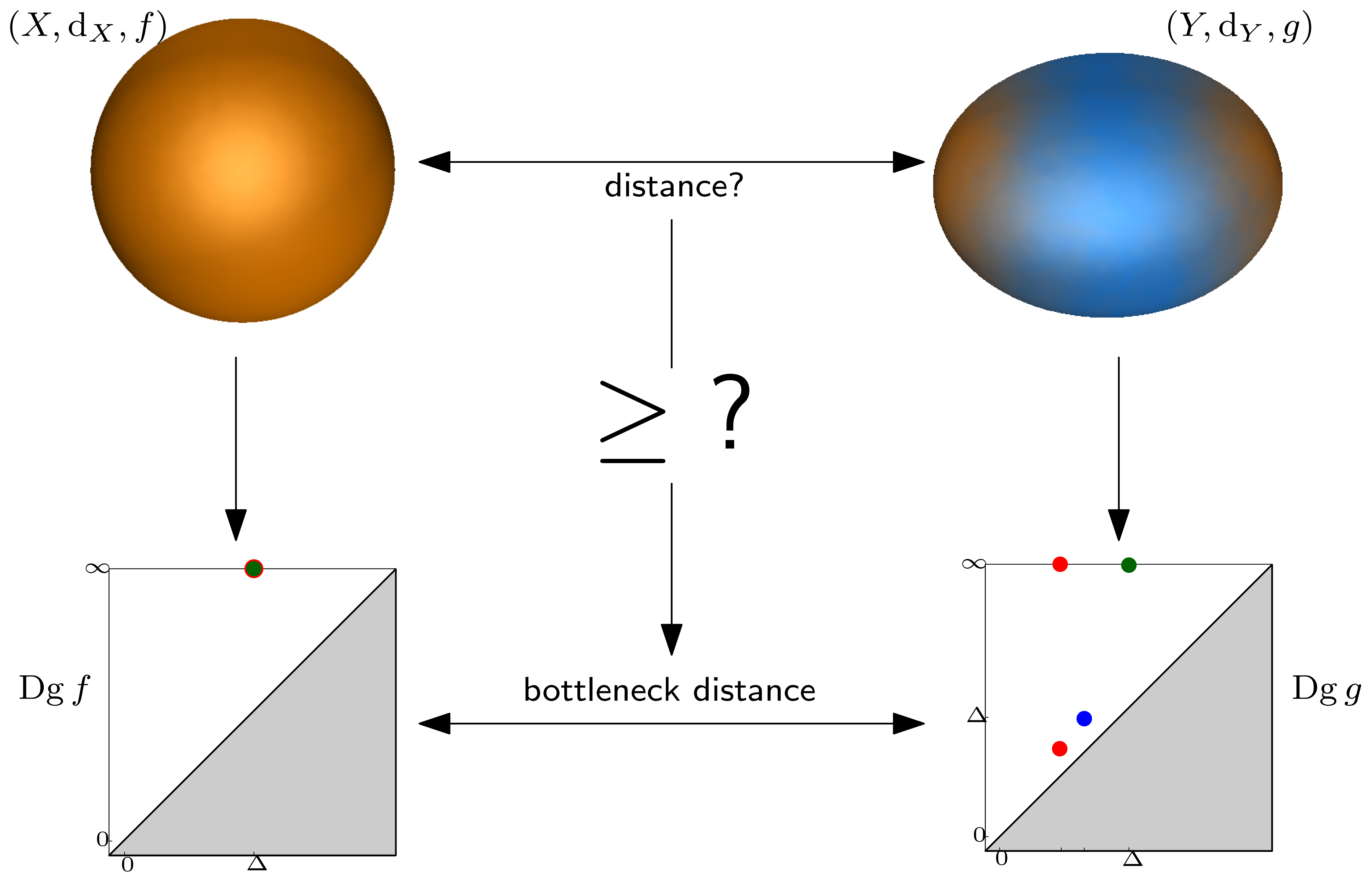
Observations:

- same spirit as size theory for shape comparison [d'Amico, Frosini, Landi 05]

- setting is more general

Stability of our Signatures



Stability of our Signatures

Definitions: Given (X, d_X, f) and (Y, d_Y, g),

- correspondence:

a set $C \subseteq X \times Y$ such that:

$\forall x \in X$, $\exists y \in Y$ s.t. $(x, y) \in C$

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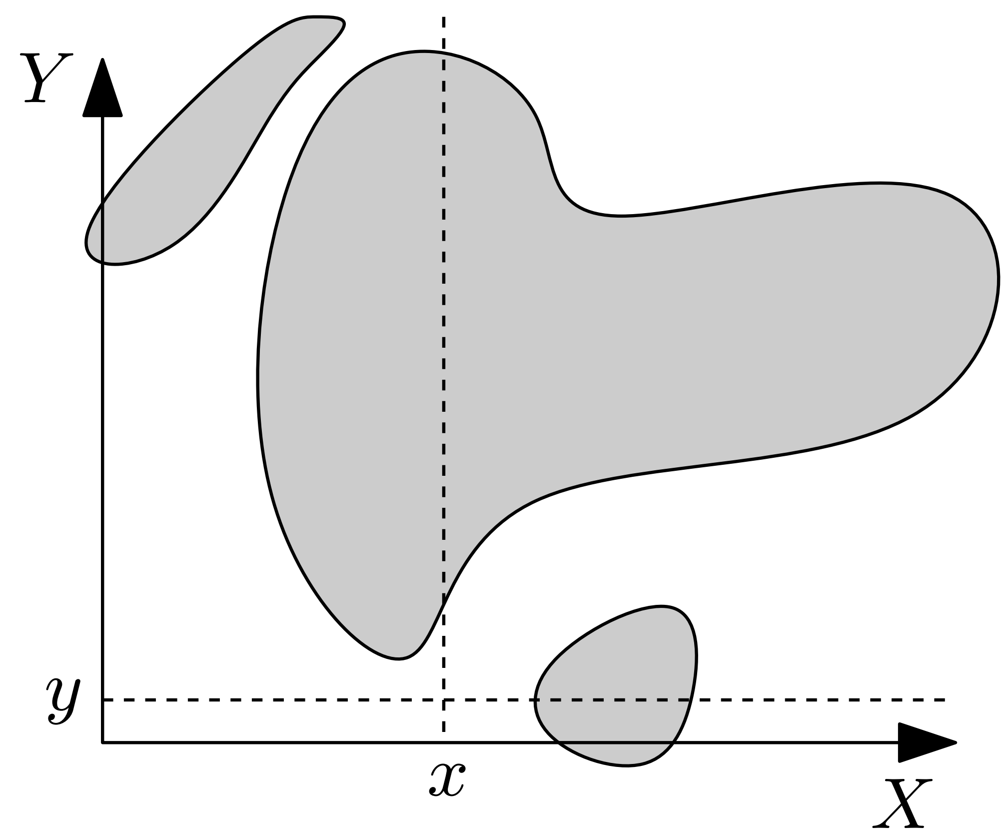

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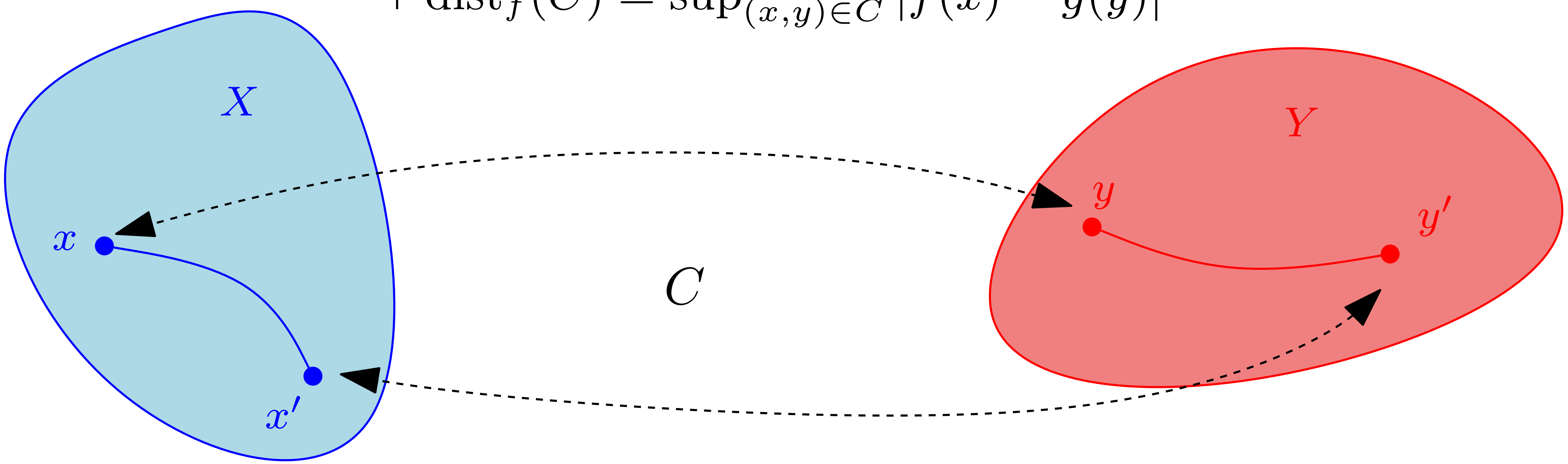
- distortion:

$$\mathrm{dist}_m(C) = \sup_{(x,y),(x',y')\in C} |\mathrm{d}_X(x, x') - \mathrm{d}_Y(y, y')|$$

$$\mathrm{dist}_f(C) = \sup_{(x,y)\in C} |f(x) - g(y)|$$



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- Gromov-Hausdorff distance:

$$\mathrm{d}_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{C \in \mathcal{C}(X,Y)} \mathrm{dist}_m(C)$$

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$\rightarrow$ In our bounds we decouple $\mathrm{dist}_m(C)$ and $\mathrm{dist}_f(C)$

Stability of our Signatures

Desired stability result:

Theorem (Stability): Let (X, d_X) and (Y, d_Y) be two compact metric spaces equipped with c-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)).$$

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Note: this is nothing but a stability theorem for persistence diagrams

- improves over [CEH'05] (functions have different domains)

- improves over [dAFL'08] (domains are in different homeomorphism classes)

- relies on [CCGGO'09] with more explicit conditions

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Note: this is nothing but a stability theorem for persistence diagrams

But it is false in such generality:

- $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) < \infty \Rightarrow (X, \mathrm{d}_X)$ and (Y, d_Y) are homologically equivalent
- $\mathrm{dist}_m(C)$ and $\mathrm{dist}_f(C)$ are finite regardless of homological types of X, Y

$X = \bullet \qquad\qquad Y = \bullet \xleftrightarrow{\ \varepsilon\ } \bullet$

Stability of our Signatures

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$\rightarrow$ Restrict the focus to a class of *sufficiently regular* metric spaces

Stability of our Signatures

Obtained stability result:

length spaces of curvature bounded above

Theorem (Stability): Let (X, d_X) and (Y, d_Y) be two compact ~~metric spaces~~ equipped with c-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X), \varrho(Y)\}$,

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \cancel{\in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C))}.$$

$$\leq 19\, c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$$

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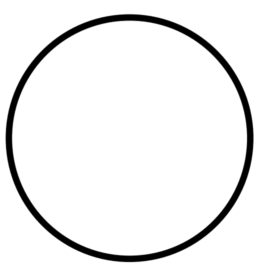
Prerequisite: $\mathrm{d}_{\mathrm{GH}}(X, Y) < \frac{1}{20}\min\{\varrho(X),\ \varrho(Y)\}$

$X =$ 

$Y =$ 

$$\mathrm{d}_{\mathrm{GH}}(X, Y) < \infty = \varrho(Y)$$

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) = \infty$$

Stability of our Signatures

Theorem (Stability): Let (X, d_X) and (Y, d_Y) be two compact length spaces with curvature bounded above, equipped with c-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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Corollary 1: If $\mathrm{d}_{\mathrm{GH}}(X, Y) < \frac{1}{20} \min\{\varrho(X),\ \varrho(Y)\}$,
then $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, \mathrm{ecc}_X,\ \mathrm{Dg}\, \mathrm{ecc}_Y) \leq 40\, \mathrm{d}_{\mathrm{GH}}(X, Y)$.

Stability of our Signatures

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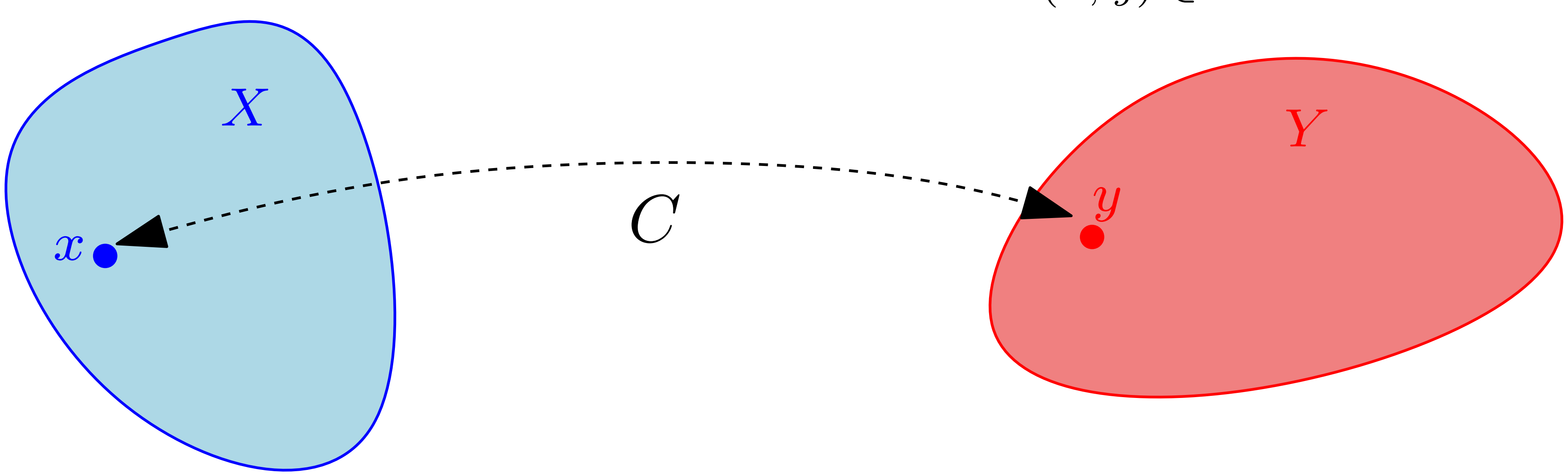
Corollary 2: For any base points $x \in X$ and $y \in Y$,

$$\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, \mathrm{d}_X(x, \cdot),\ \mathrm{Dg}\, \mathrm{d}_Y(y, \cdot)) \leq 20 \inf_{\substack{C \in \mathcal{C}(X, Y) \\ \mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X),\ \varrho(Y)\} \\ (x, y) \in C}} \mathrm{dist}_m(C).$$



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Corollary 3: If $\mathrm{d}_{\mathrm{GH}}(X, Y) < \frac{1}{20} \min\{\varrho(X),\ \varrho(Y)\}$, then given any base point $x \in X$, for any $\varepsilon > 0$ there is a basepoint $y \in Y$ such that $\mathrm{d}_B^\infty(\mathrm{Dg}\,\mathrm{d}_X(x, \cdot),\ \mathrm{Dg}\,\mathrm{d}_Y(y, \cdot)) \leq 40\mathrm{d}_{\mathrm{GH}}(X, Y) + \varepsilon$.

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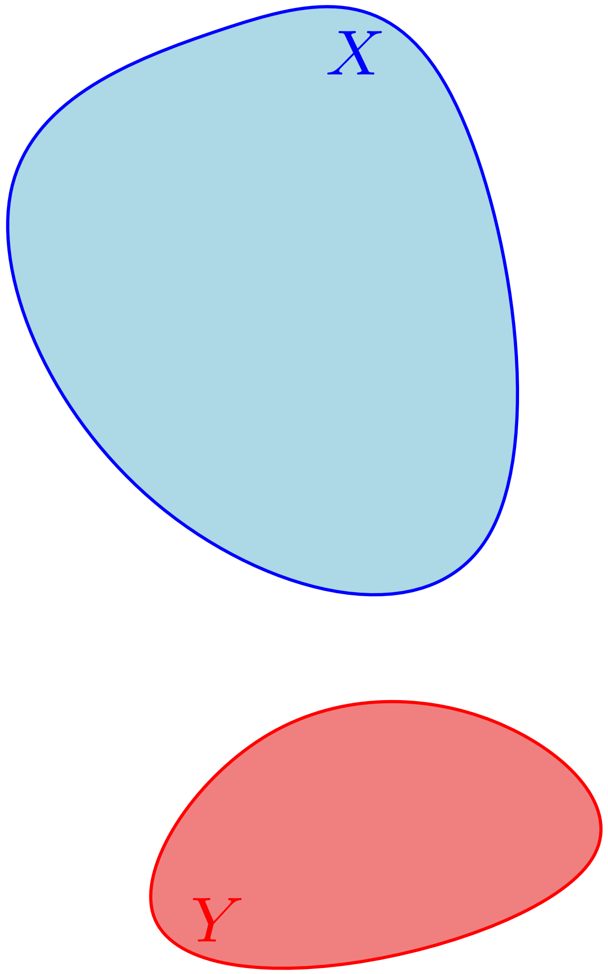
Proof: by reduction to *Scalar Fields Analysis from Point Cloud Data*:

[Chazal, Guibas, O., Skraba 11]



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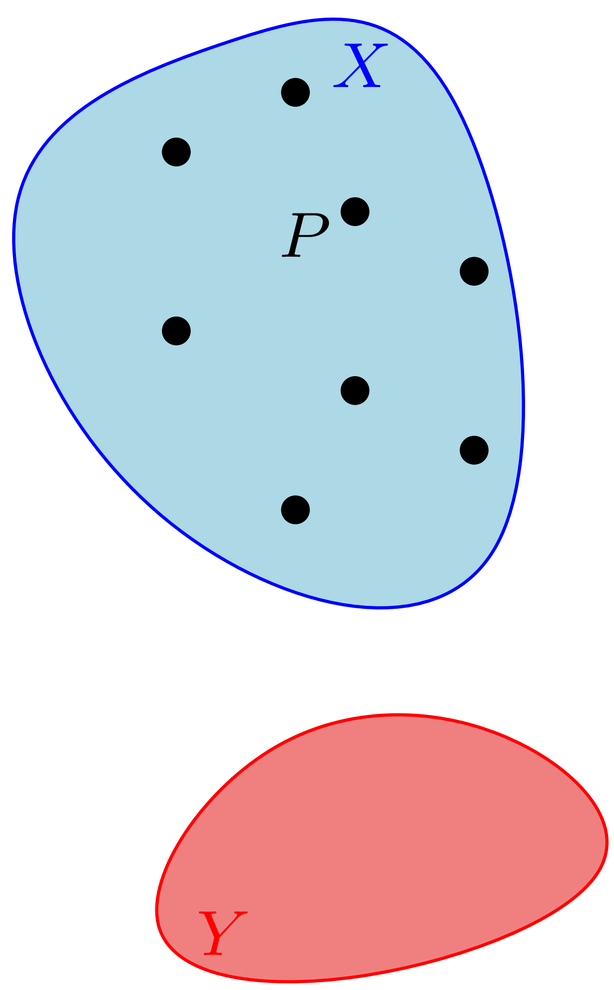

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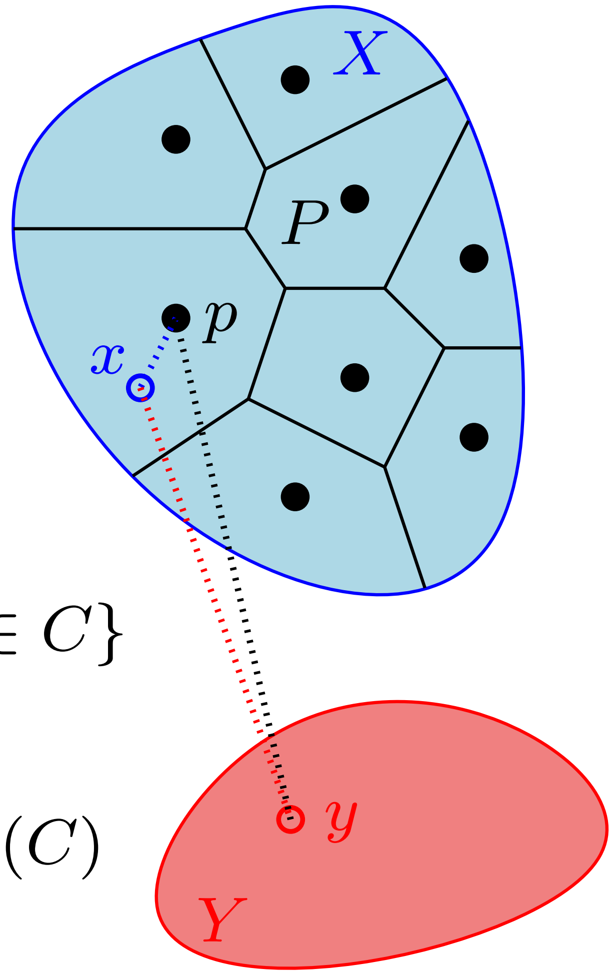

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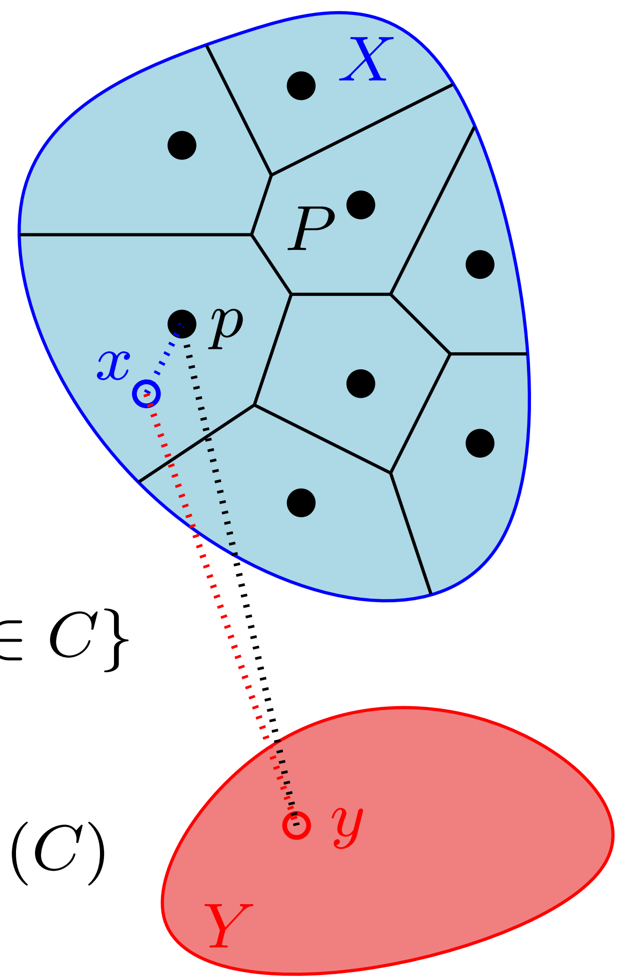

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$\to$ goal: approximate persistence diagram from GH-close finite metric space

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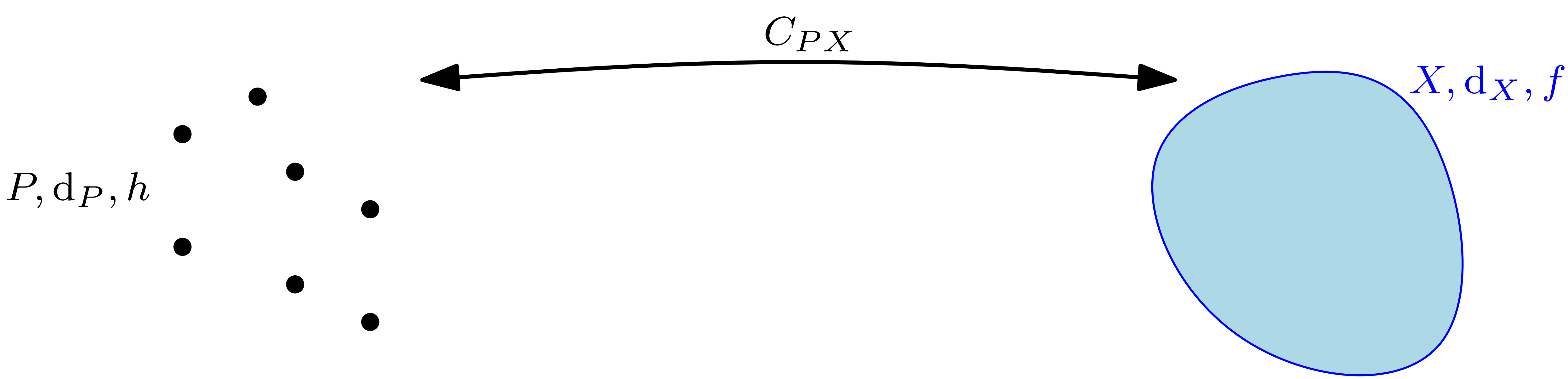

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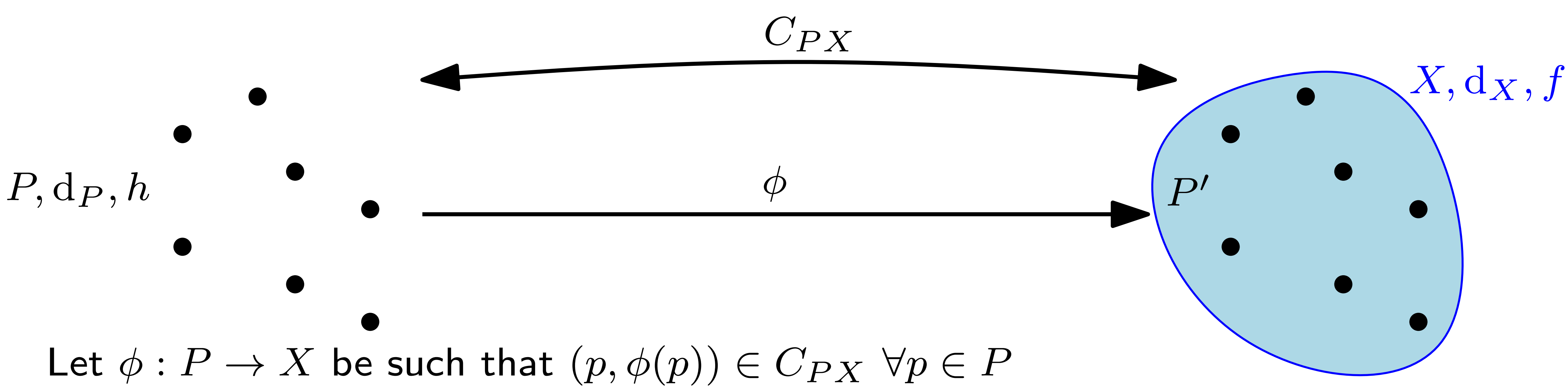

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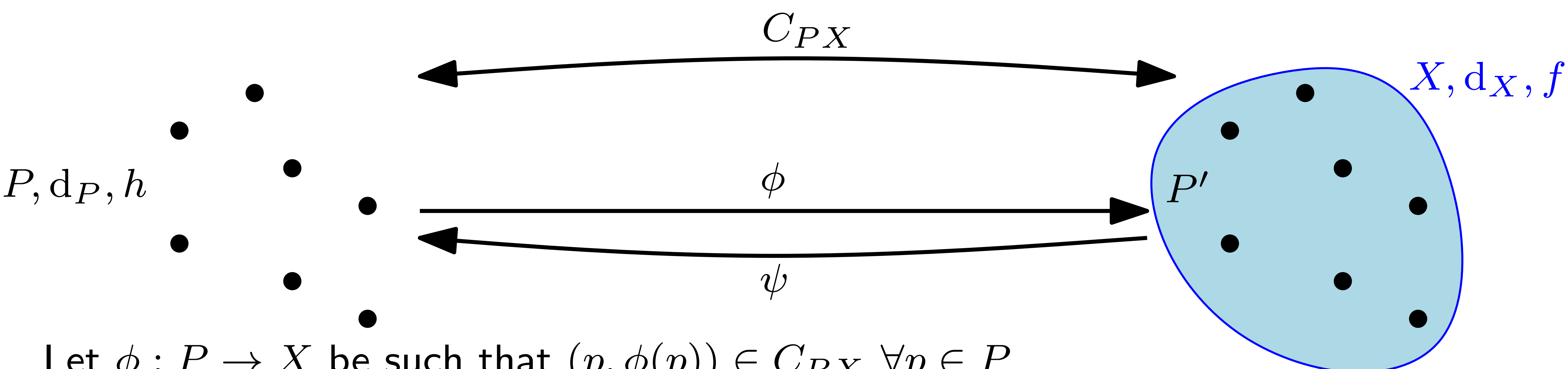

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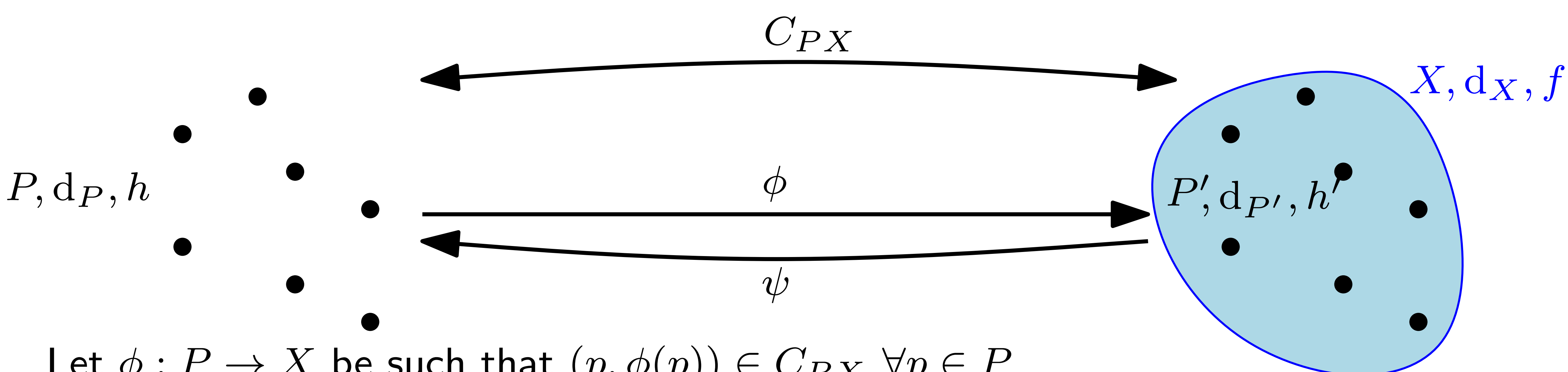

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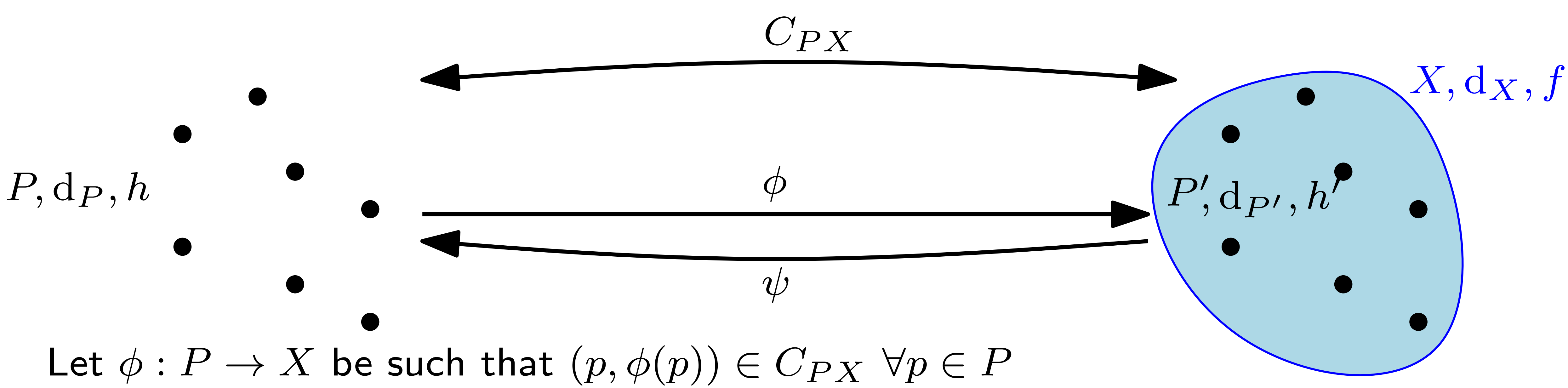

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scenario considered in [CGOS'11]

$\to$ | $\mathrm{d}_{\mathrm{H}}(P', X) \leq \mathrm{dist}_m(C_{PX})$
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In practice, the signatures can only be approximated:

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- when a triangulation of the space X is given:

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- when a finite approximation (P, d_P, g) of (X, d_X, f) is given:
  - choose a neighborhood parameter $\delta > 0$
  - build the filtrations $\{\mathrm{R}_\delta(g^{-1}((-\infty, \alpha]))\}_{\alpha\in\mathbb{R}}$ and $\{\mathrm{R}_{3\delta}(g^{-1}((-\infty, \alpha]))\}_{\alpha\in\mathbb{R}}$
  - compute the PD of the image persistence module induced by inclusions:

$$\{\mathrm{Im}\, H_*(\mathrm{R}_\delta(g^{-1}((-\infty, \alpha])) \to H_*(\mathrm{R}_{3\delta}(g^{-1}((-\infty, \alpha]))\}_{\alpha\in\mathbb{R}}$$

  - bottleneck distance to $\mathrm{Dg}\,f$ is controlled by the results in [CGOS'11]

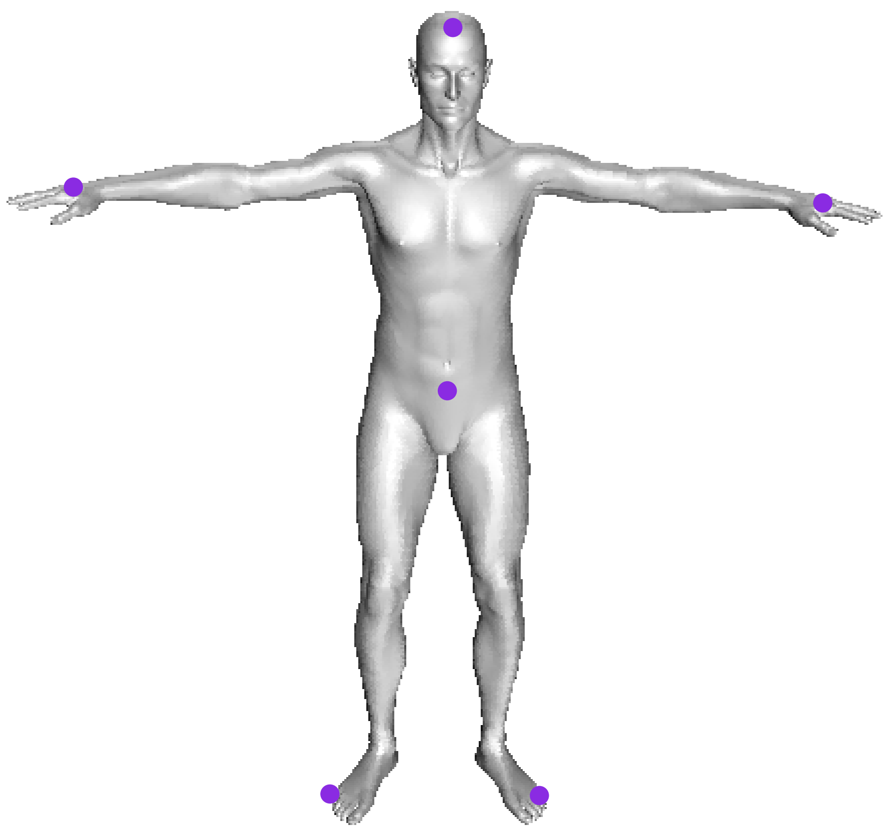
Some Experimental Results

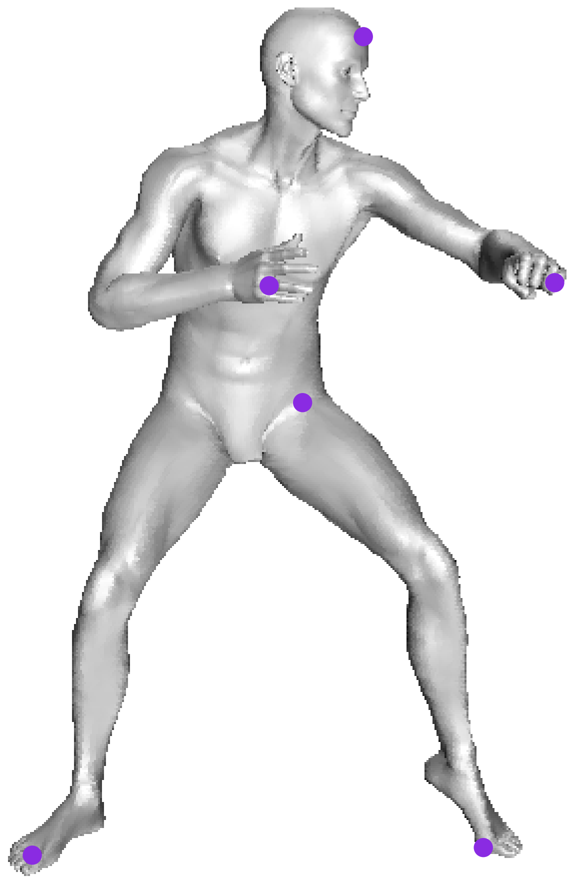
- input: shapes from the TOSCA database, in *mesh* form

- select a few base points by hand on each shape

- approximate geodesic distances to base points using the 1-skeleton graph

- use the PDs of the PL interpolations over the meshes as signatures





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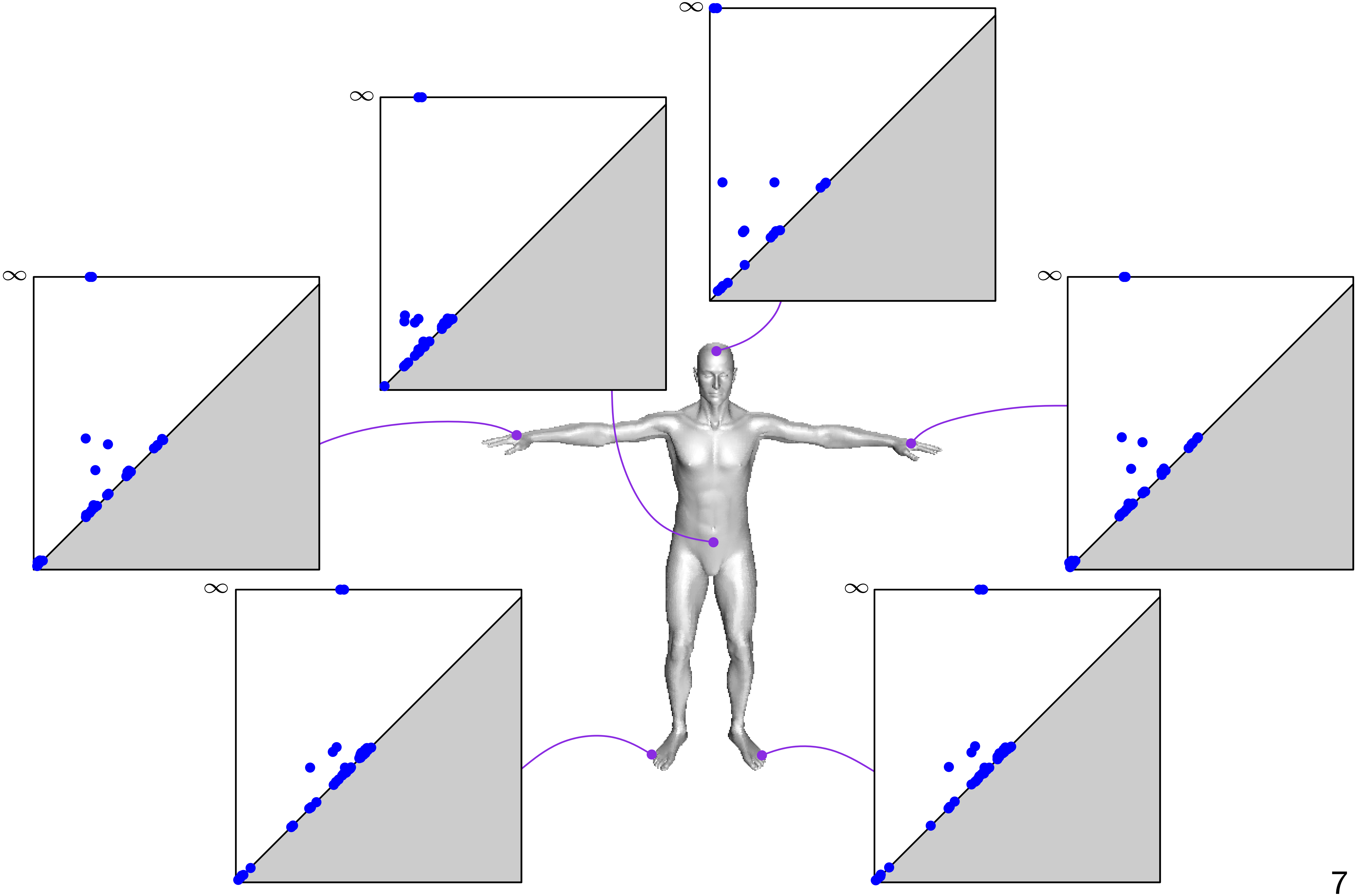

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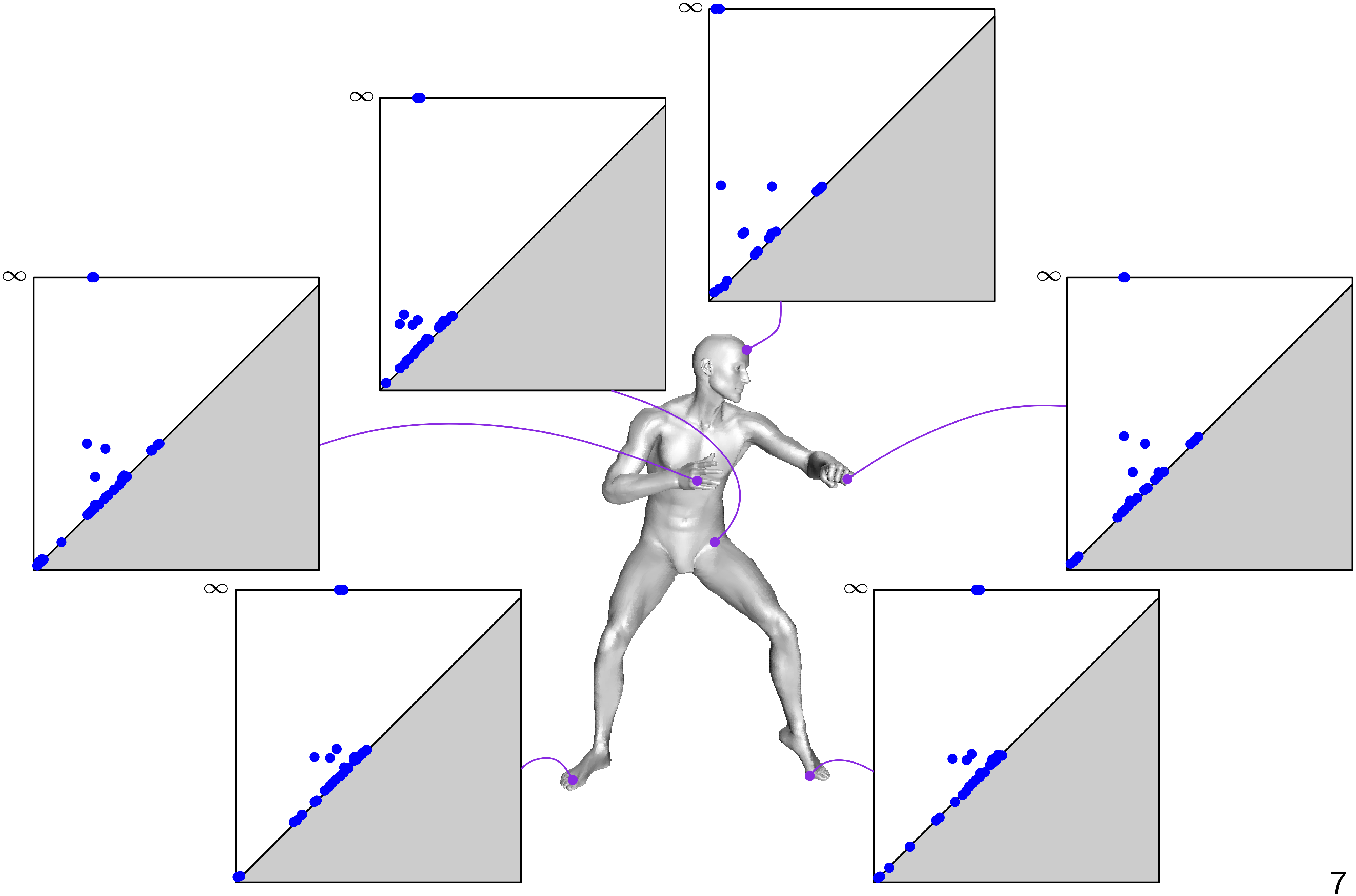

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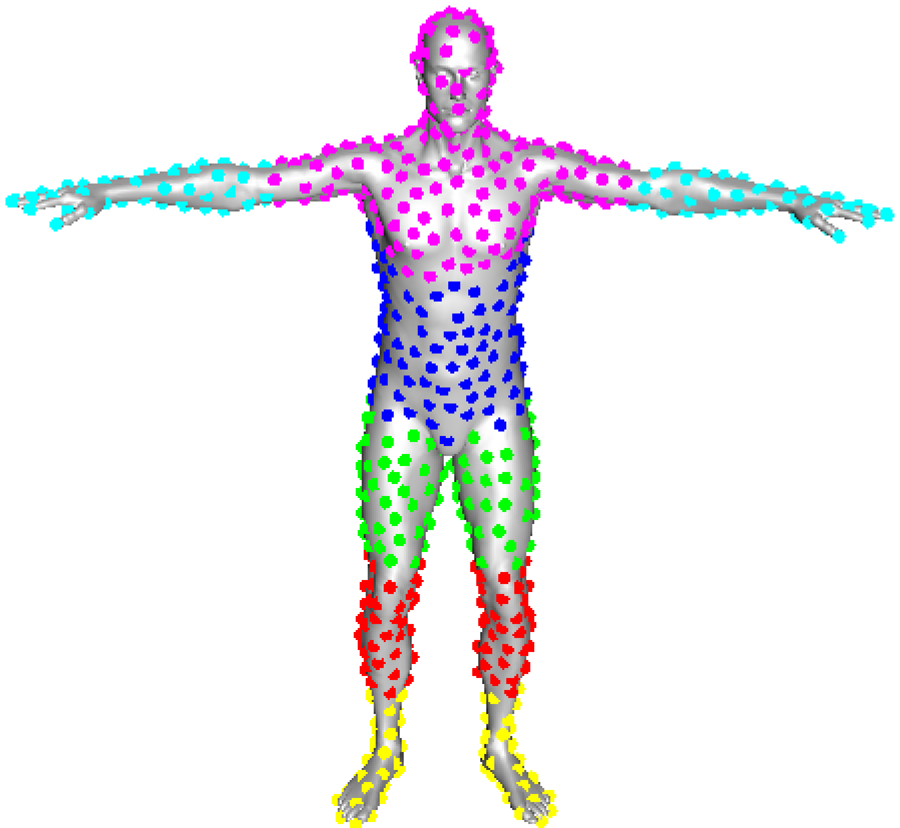

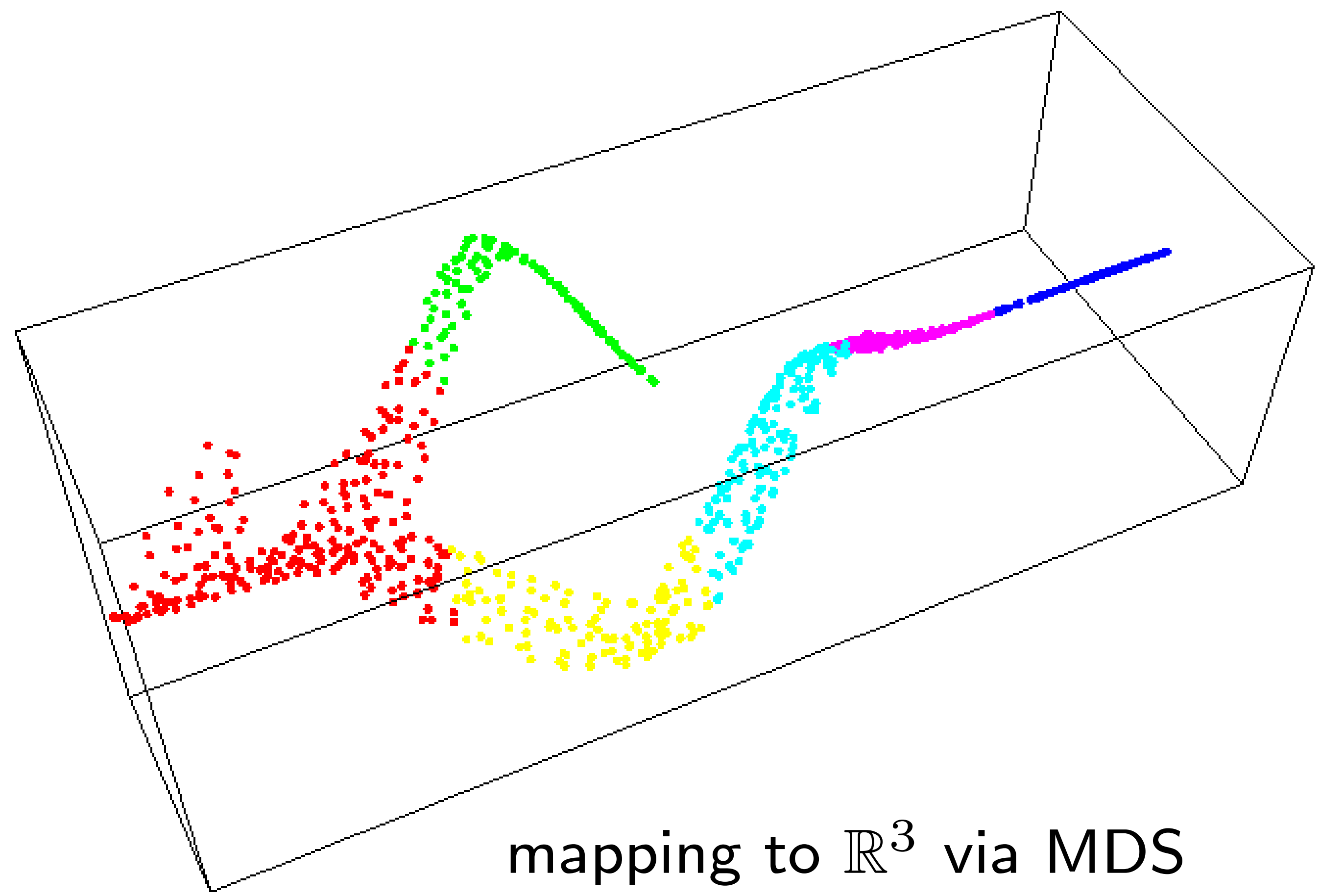

mapping to $\mathbb{R}^3$ via MDS

k-means in $\mathbb{R}^3$

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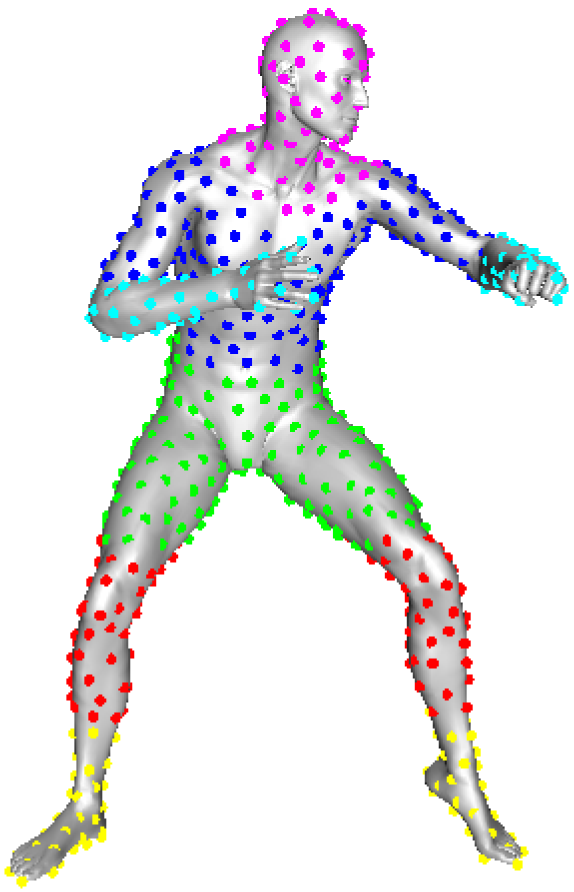

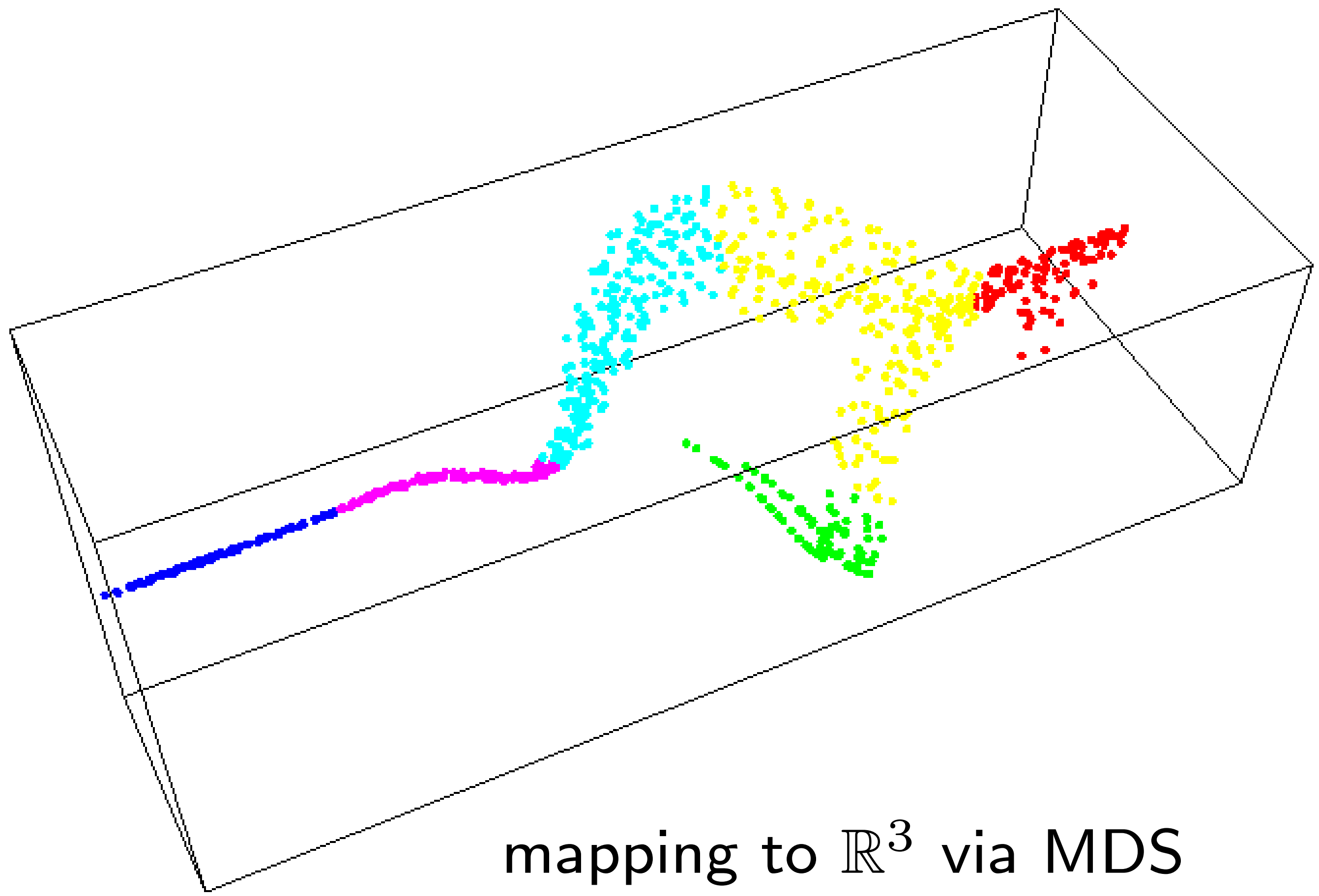

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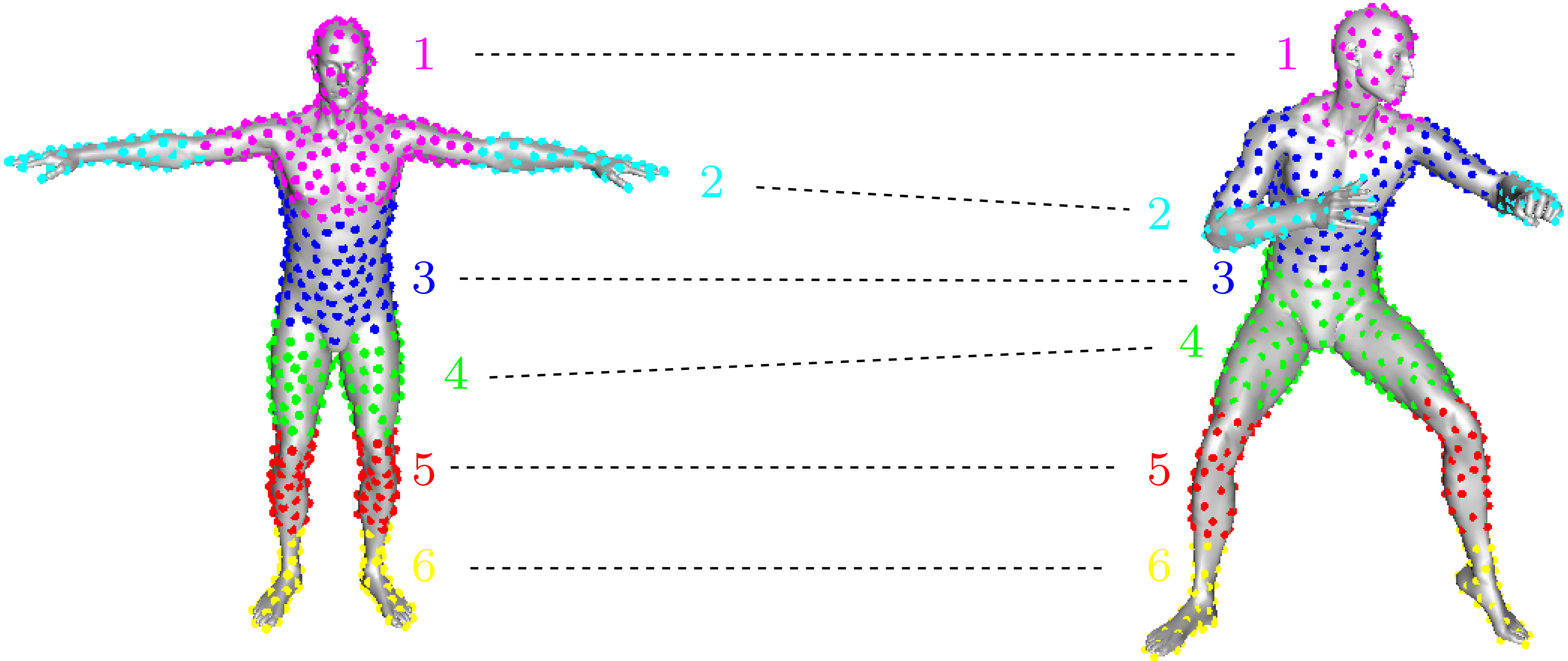
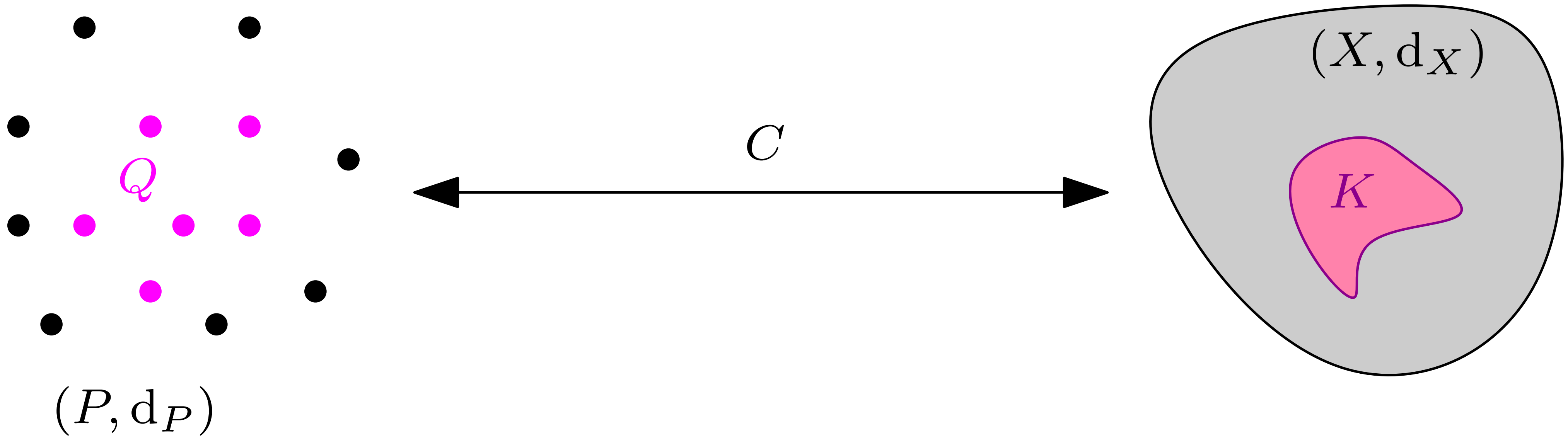
Some Experimental Results

	1	2	3	4	5	6
1	**20.322**	107.945	71.9286	102.34	130.602	189.146
2	85.3632	**21.872**	67.7957	67.7957	43.098	93.697
3	54.843	69.1745	**21.609**	45.2831	73.5451	132.089
4	90.0162	69.1745	29.3636	**15.327**	35.5291	89.044
5	104.753	69.1745	45.1231	26.101	**23.927**	74.307
6	172.427	74.568	110.585	81.213	52.951	**15.161**



Current / Future Directions

- Remote analysis of the distance to a compact set:



Idea: approximate $\mathrm{Dg}\, \mathrm{d}_X(\cdot, K)$ by the diagram of the filtration of $\mathrm{R}_\delta(P, \mathrm{d}_P) \hookrightarrow \mathrm{R}_{3\delta}(P, \mathrm{d}_P)$ defined by $\mathrm{d}_P(\cdot, Q)$.

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- Remote analysis of the distance to a compact set:

- Upper bounds on the Gromov-Hausdorff distance:

  $\rightarrow$ by picking up sufficiently many functions (e.g. distances to all points), can one obtain a constant-factor approximation of the GH-distance?

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- Remote analysis of the distance to a compact set:

- Upper bounds on the Gromov-Hausdorff distance:

- Relaxation of the hypotheses of the stability theorem:

  $\rightarrow$ remove the assumption that $\mathrm{d_{GH}}(X, Y) < \frac{1}{20}\varrho(Y)$

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Thank You