Towards the understanding of homological persistence of maps

Research in progress - joint work with. H. Edelsbrunner

Workshop on Computational Topology
The Fields Institute, Toronto, 10th November 2011

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Goal

- $X \subset \mathbb{R}^d$ - a space
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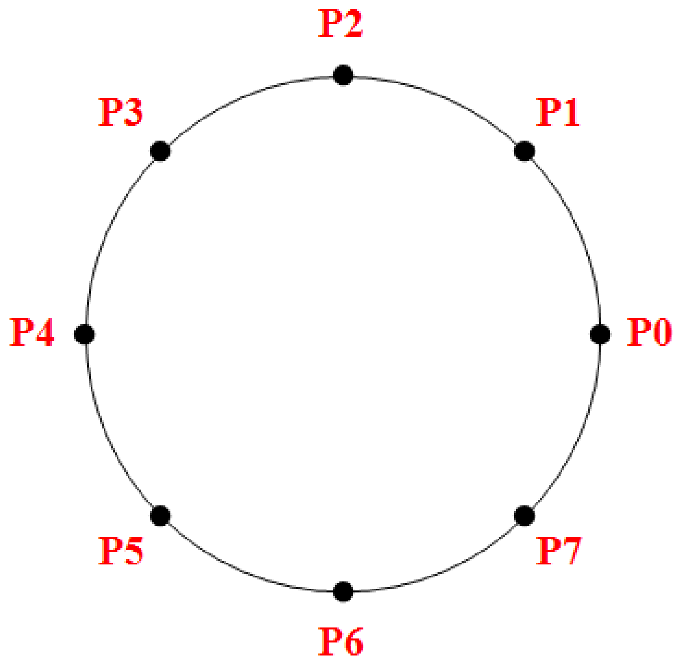
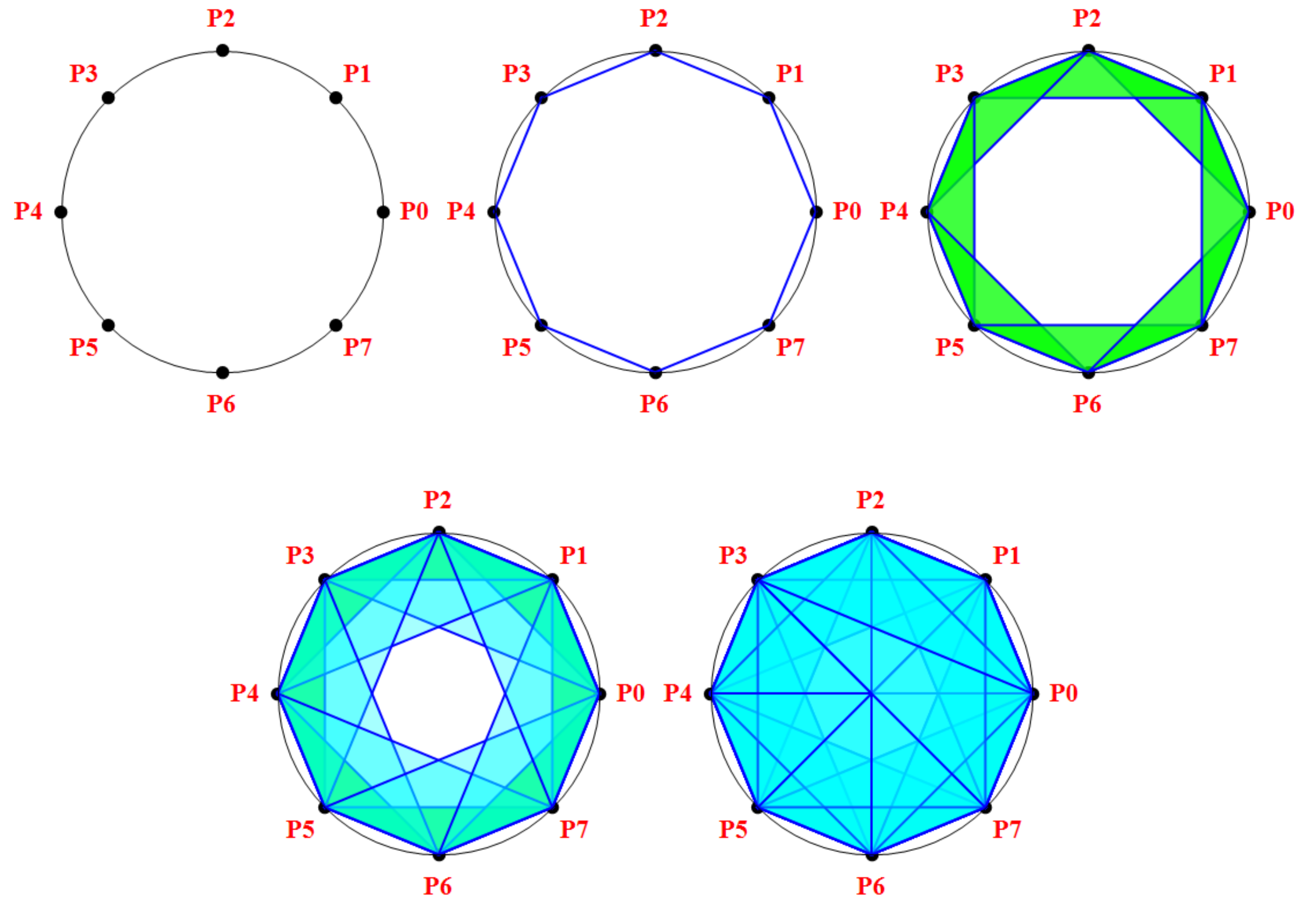
- for today: understanding the persistence of the map induced by f in homology
- ultimate: retrieving the information about the dynamics of f from A and α in the spirit of homological persistence
- in particular: studying the dynamics of a time series governed by a physical process

Example 1 [3]



- Take $S^1 \subset \mathbb{C}$ as the space X.
- The map
$$f : S^1 \ni z \mapsto z^2 \in S^1$$
- Point cloud
$$A_K := \{\, P_k := e^{2k\pi i/K} \mid k = 0, 1, \ldots K-1 \,\}$$
- Point cloud map
$$\alpha : A_K \ni P_k \mapsto P_{2k \,\mathsf{mod}\, K} \in A_K$$

Example 1 - Vietoris-Rips filtration



$$\epsilon_1 \approx 0.765367, \epsilon_2 \approx 1.41421, \epsilon_3 \approx 1.84776, \epsilon_4 = 2.0$$

- $\{P_0\}$ - $[0, \epsilon_4]$
- $\{P_0P_1, P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6, P_6P_7, P_7P_0\}$ - $[\epsilon_1, \epsilon_4)$

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A good starting point is the persistence of eigenspaces.

Eigenvalues and eigenspaces

- $\lambda \in F$ is an eigenvalue of L if the λ-eigenspace of L

$$E(\lambda, L) := \{\, v \in V \mid Lv = \lambda v \,\} \neq \{0\}.$$

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If two automorphisms $L_i : V_i \to V_i$ are conjugate, then $\sigma(L_1) = \sigma(L_2)$ and the respective eigenspaces are isomorphic.

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- Is the point cloud map α simplicial?
- How do we define persistence of eigenspaces?

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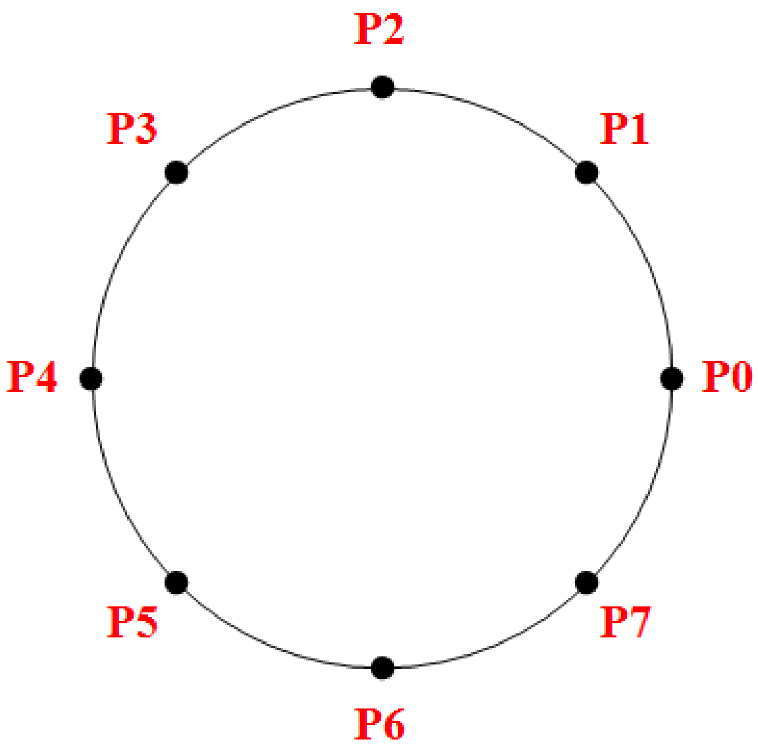

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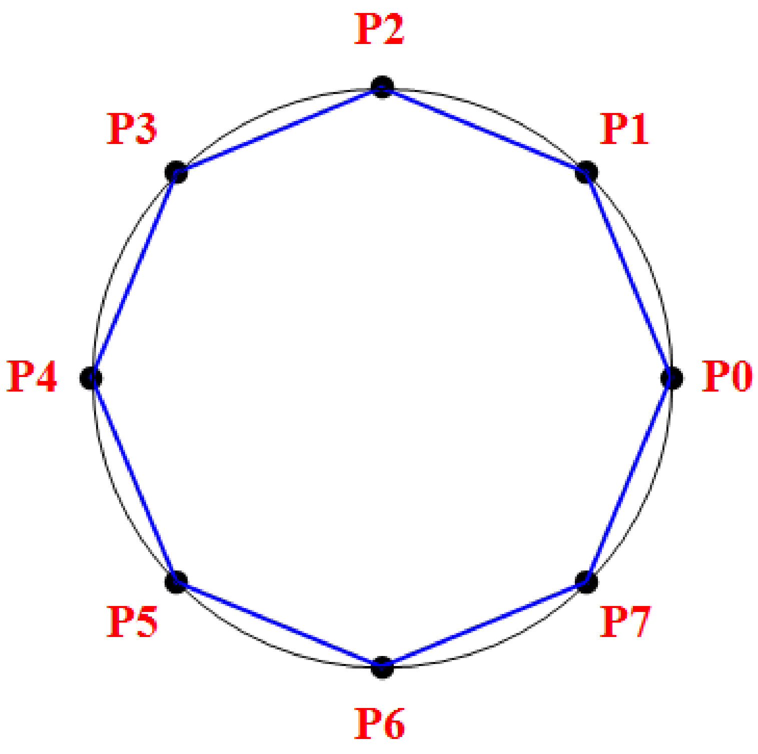

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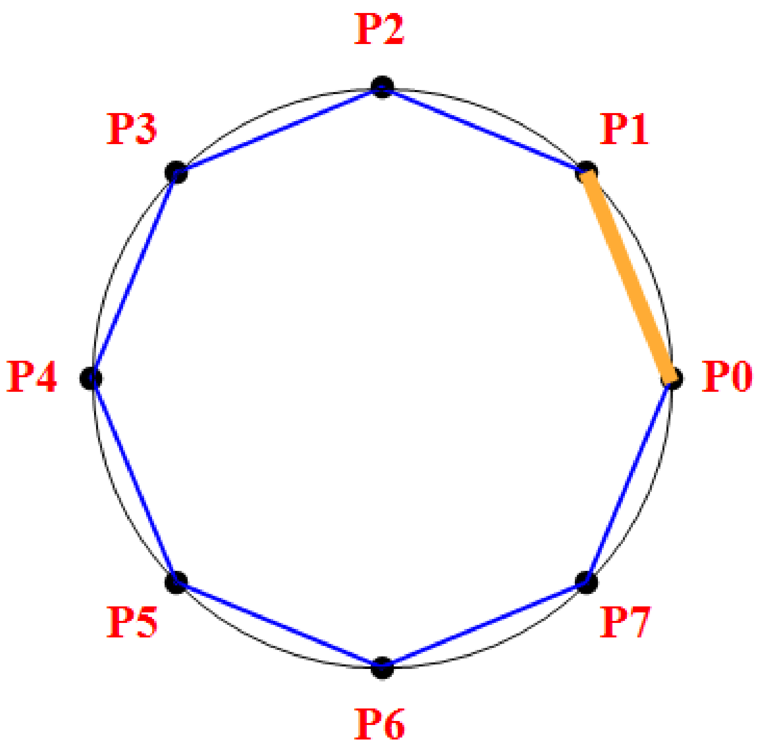

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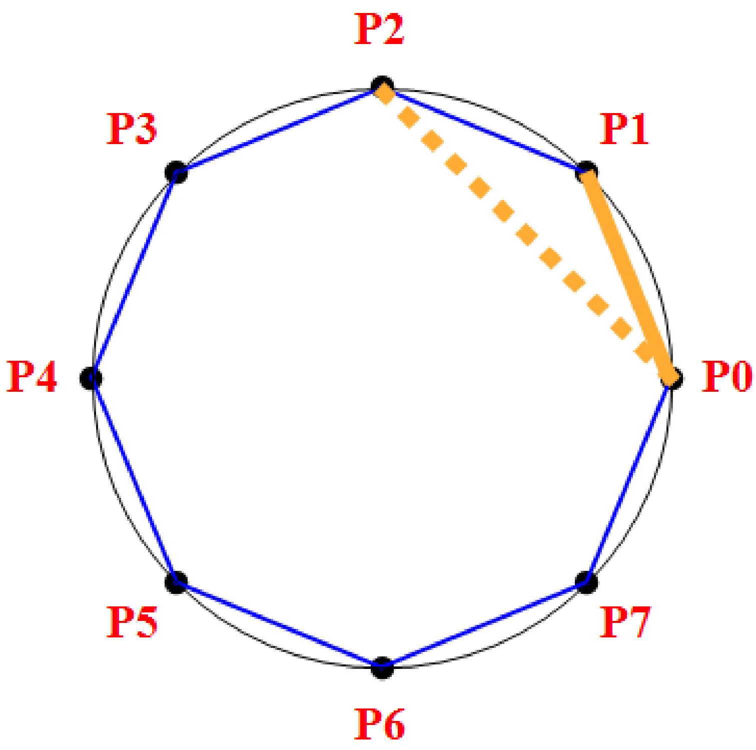

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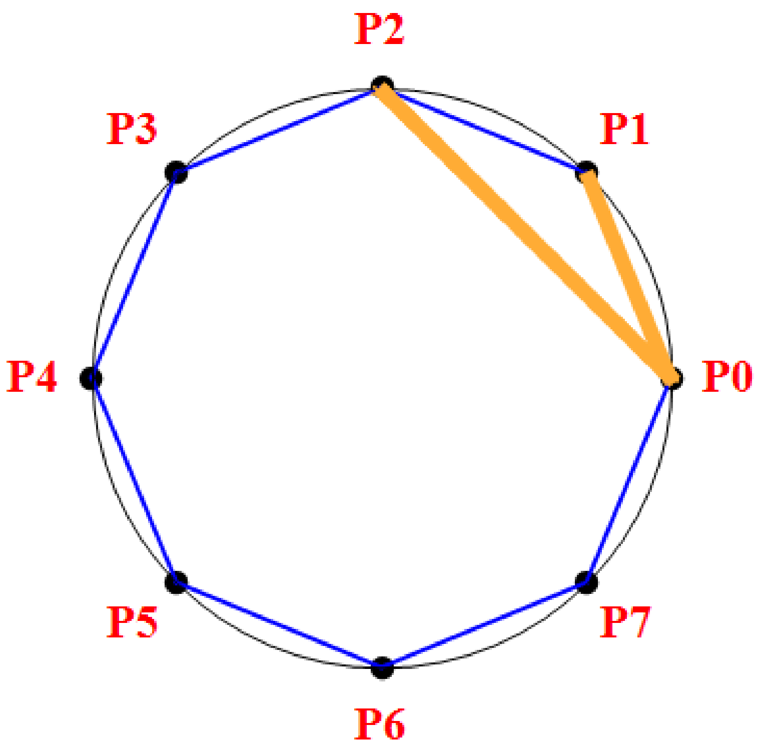

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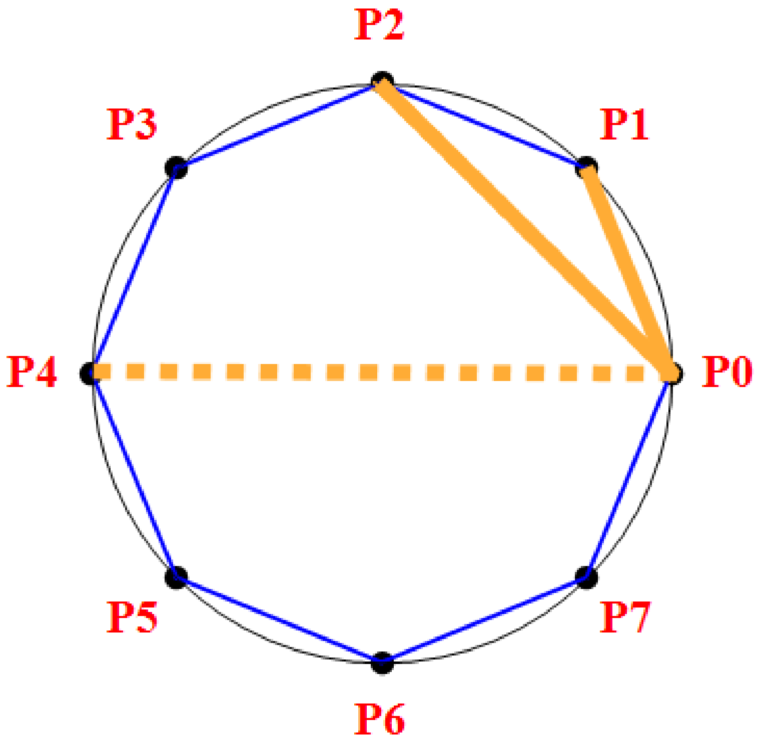

- $f : X \to X$ - a continuous map
- Then the diagram

$$\begin{tikzcd} & G(f) \arrow[dl, "p_{|G(f)}"'] \arrow[dr, "q_{|G(f)}"] & \\ X \arrow[rr, "f"] & & X \end{tikzcd}$$

  commutes.
- Since $p_{|G(f)}$ is a homeomorphism, we have

$$f = q_{|G(f)} \circ p_{|G(f)}^{-1}.$$

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Proposition. Projections

$$p_{|G(\alpha)}, q_{|G(\alpha)} : G(\alpha) \to A$$

are simplicial with respect to Vietoris-Rips complexes $\mathcal{R}_\epsilon(G(\alpha))$ and $\mathcal{R}_\epsilon(A)$.

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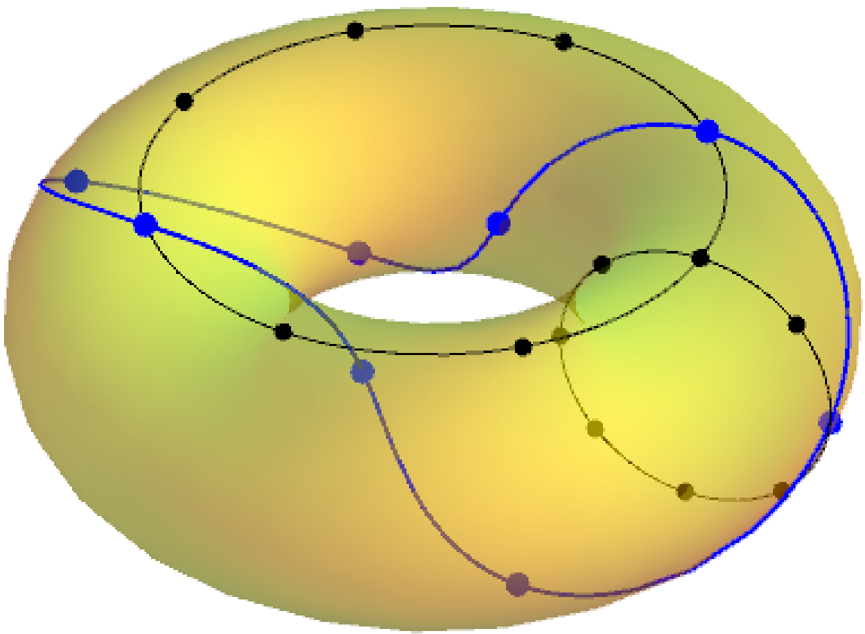

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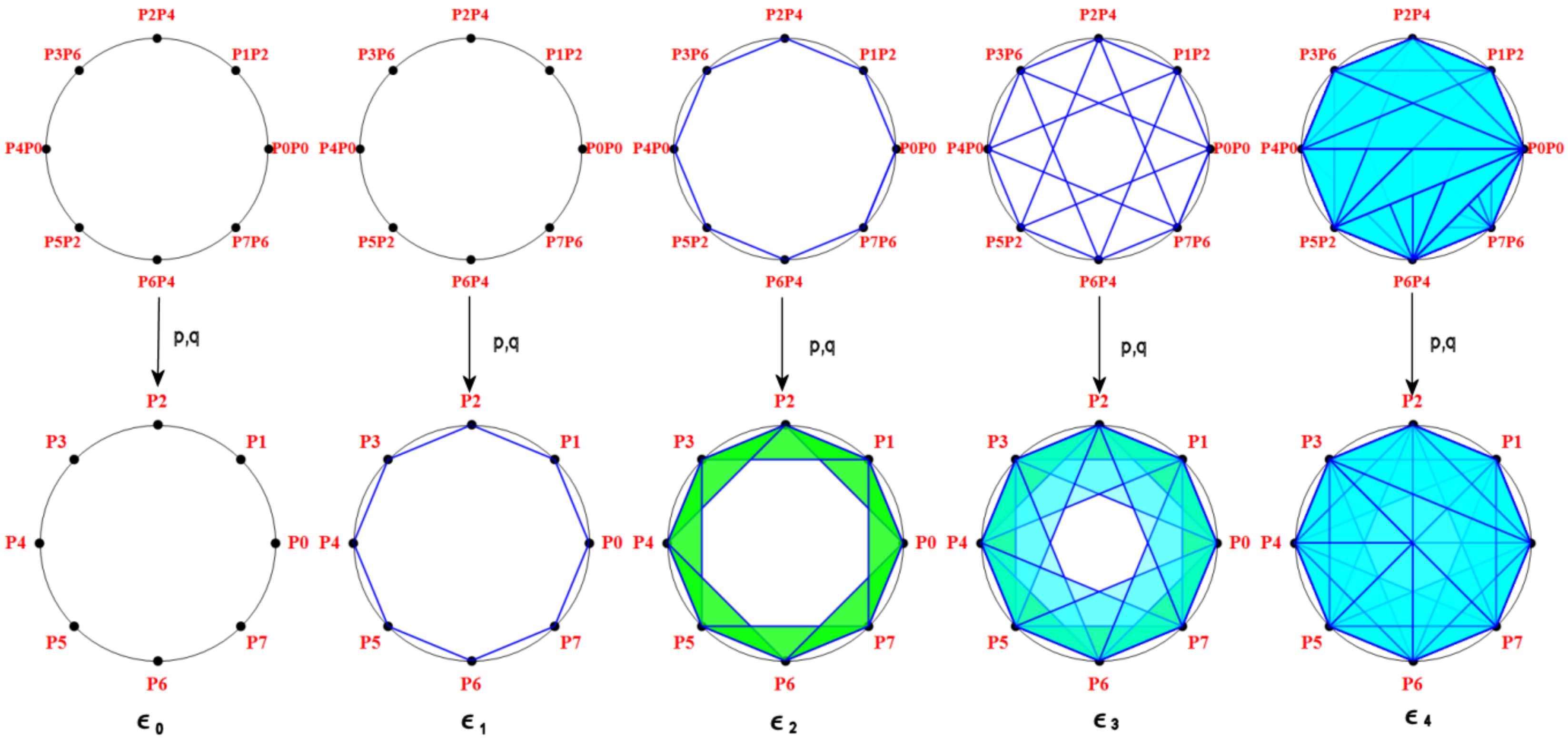

- p_* is not invertible at ϵ_1 and ϵ_3
- $q_* p_*^{-1}$ has one 1-eigenvector in dimension zero at ϵ_0, ϵ_2 and ϵ_4
- $q_* p_*^{-1}$ has one 2-eigenvector in dimension one at ϵ_2

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- For $\lambda \in F$ set
$$\bar{E}(\lambda, p, q) := \{\, w \in \Gamma \mid qw = \lambda pw \,\}.$$
and define the λ-eigenspace of (p, q) by
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- λ is an eigenvalue of the pair (p, q) if $E(\lambda, p, q) \neq \{0\}$.

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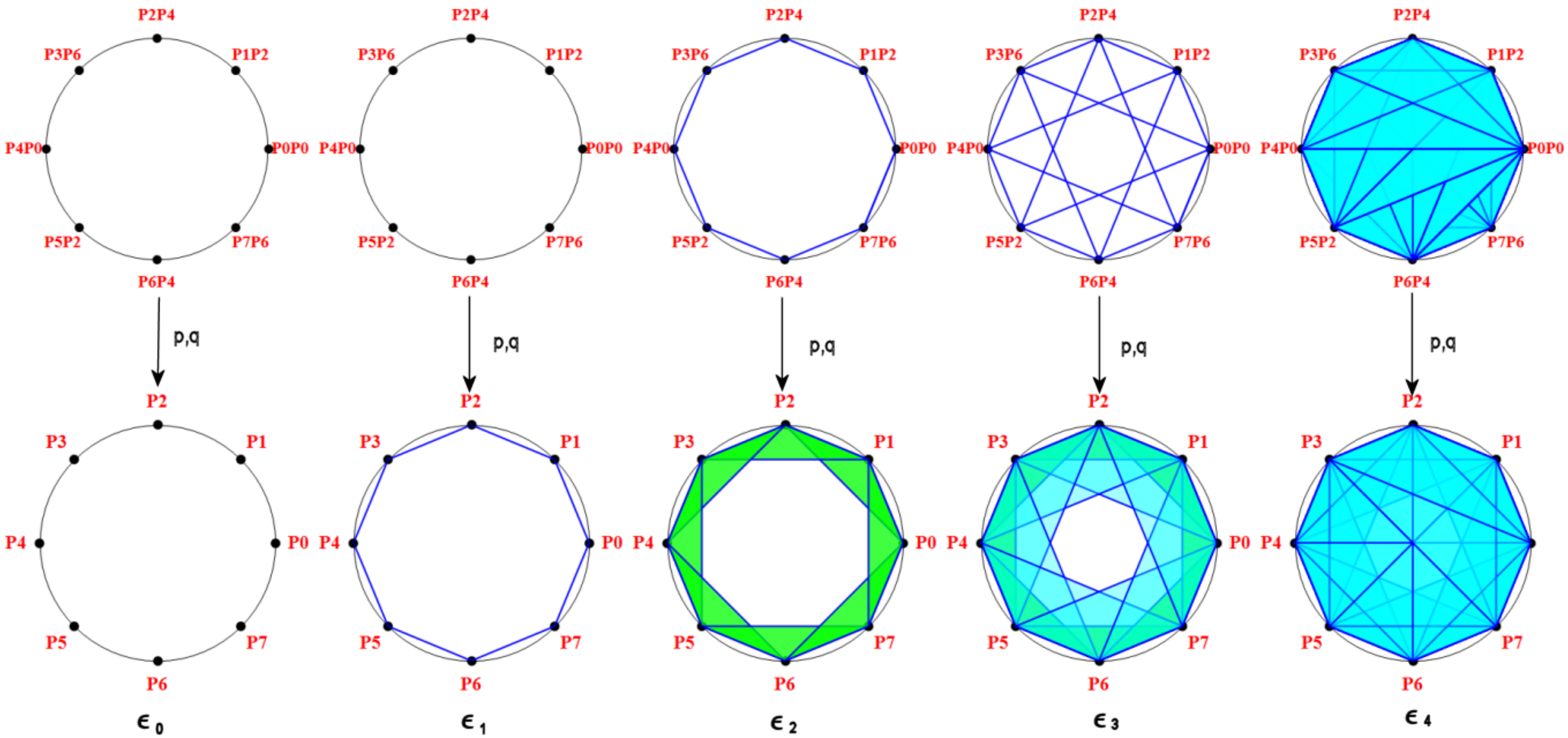

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- $\varphi : (X^i, \xi^i) \to (\bar{X}^i, \bar{\xi}^i)$ - a morphism of towers if $\varphi^i : X^i \to \bar{X}^i$ is linear and

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- $X^{i,p} := \operatorname{im} \xi^{i,p}$ - (i,p)-persistent vector space of $\mathcal{X} := (X^i, \xi^i)_{i \in \mathbb{N}_0}$.
- $\beta^{i,p}(\mathcal{X}) := \dim X^{i,p}$ - the (i,p)-persistent Betti number of $\mathcal{X}$.

Persistent bases 21

- $\mathcal{X} = (X^i, \xi^i)$ - tower of vector spaces over F.
- (E^i, K^i) is a persistent basis of $\mathcal{X}$ if

  (i) $E^i \cup K^i$ is a basis of X^i,

  (ii) ξ^i is zero on K^i and nonzero on E^i,

  (iii) ξ^i maps injectively E^i into $E^{i+1} \cup K^{i+1}$.

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Theorem. Any finitely generated tower of vector spaces admits a persistent basis.

Persistent intervals

- $(x^j)_{j=i,i+1,\dots,i+p}$ - an (i,p)-persistent tower of vectors if $x^j \in X^j$, $\xi^j(x^j) = x^{j+1}$, $x^j \neq 0$ for $j = i, i+1, \dots, i+p-1$, $x^i \notin \operatorname{im} \xi^{i-1}$ and $x^{i+p} = 0$.
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Proposition. Let (E^i, K^i) be a persistent basis of $\mathcal{X}$. Let $L^{i,p}$ denote the cardinality of the set of (i,p)-persistent towers of vectors whose ith nonzero elements belong to $E_i \cup K_i$. Then

$$L^{i,p} = \beta^{i,p-1} - \beta^{i-1,p} - \beta^{i,p} + \beta^{i-1,p+1}.$$

Derived towers

- $\mathcal{X} = (X^i, \xi^i)$ and $\mathcal{Y} = (Y^i, \upsilon^i)$ - towers
- $\mathcal{Y}$ is a subtower of $\mathcal{X}$ if $Y^i \subset X^i$ and $\upsilon^i = \xi^i_{Y^i}$.
- Quotient tower:

$$\mathcal{X}/\mathcal{Y} := (X_i/Y_i, \bar{\xi^i})$$

- $\varphi : \mathcal{X} \to \mathcal{Y}$ - a morphism of towers
- $(\ker \varphi^i, \xi^i_{\ker \varphi^i})$ - tower of kernels
- $(\operatorname{im} \varphi^i, \upsilon^i_{\operatorname{im} \varphi^i})$ - tower of images
- towers of cokernels, coimages, generalized kernels, generalized images ...

Proposition. Let $\varphi : \mathcal{X} \to \mathcal{X}$ be a morphism of towers. Then for any $\lambda \in F$

$$\xi^i(E(\lambda, \varphi^i)) \subset E(\lambda, \varphi^{i+1}).$$

In particular, $(E(\lambda, \varphi^i), \xi^i_{|E(\lambda,\varphi^i)})$ is a tower.

Proposition. Let $\mathcal{X}, \mathcal{Y}$ be two towers of modules and $\varphi, \psi : \mathcal{X} \to \mathcal{Y}$ be morphisms of towers. Then for any $\lambda \in F$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(1, 4, 2, \infty, 6) \qquad (1, 2, 4, 6, \infty)$$

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Step 2b may be skipped if only persistence intervals but not the corresponding towers of vectors are needed.

Graph complexes and graph filtrations.

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- For $U \subset A$ set

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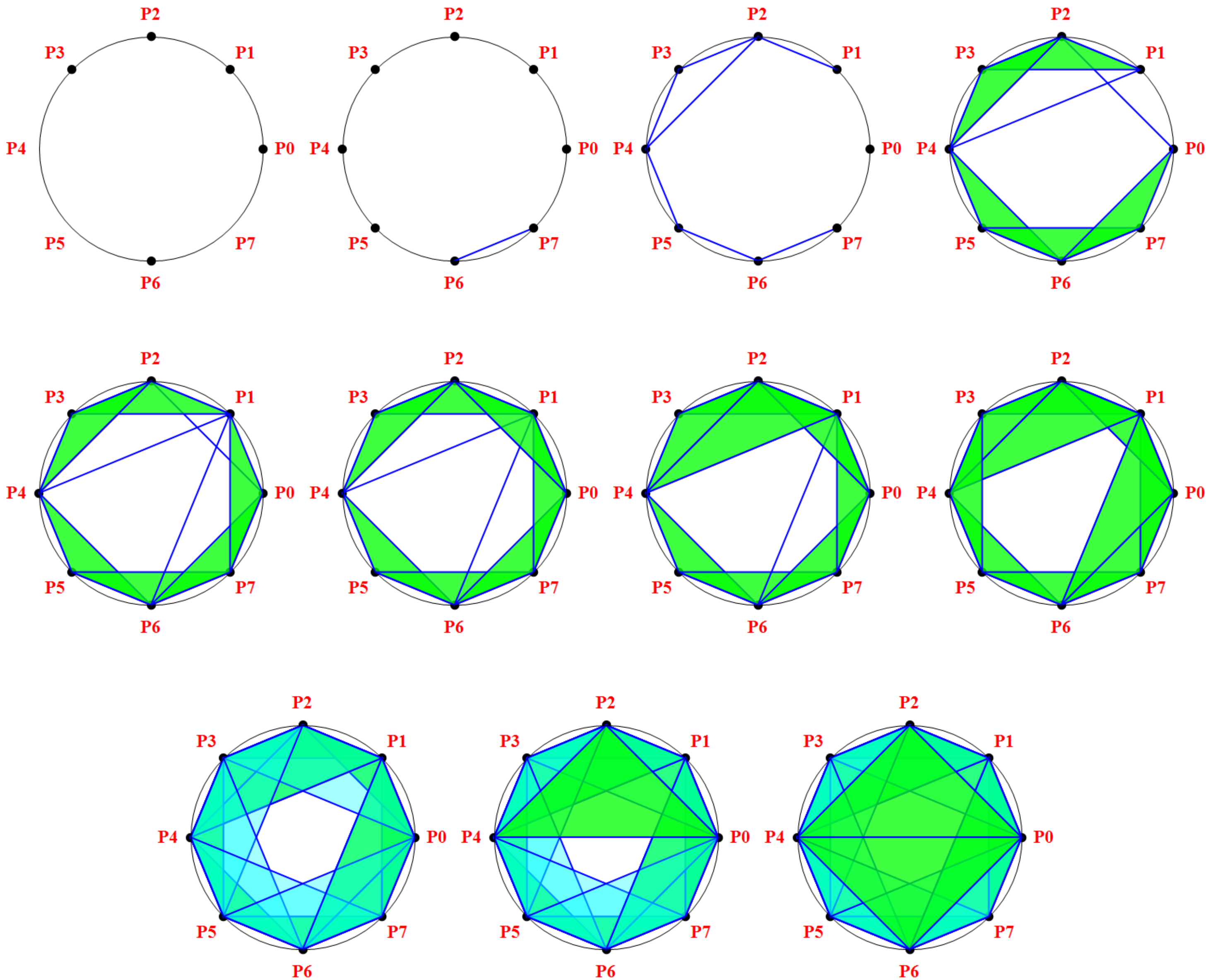
Proposition.

- The family $\mathcal{X}^\star := \{ \sigma^\star \mid \sigma \in \mathcal{X} \text{ and } \alpha(\sigma) \in \mathcal{X} \}$ is a simplicial complex with vertices in $A \times A$ (graph complex).
- The maps

$$p : \mathcal{X}_0^\star \ni (x, \alpha(x)) \mapsto x \in \mathcal{X}_0,$$
$$q : \mathcal{X}_0^\star \ni (x, \alpha(x)) \mapsto \alpha(x) \in \mathcal{X}_0,$$

are simplicial.
- If $\mathcal{X} = (\mathcal{X}^1, \mathcal{X}^2, \ldots, \mathcal{X}^n)$ is a filtration of simplicial complexes in A, then $\mathcal{X}^\star := (\mathcal{X}^{1\star}, \mathcal{X}^{2\star}, \ldots, \mathcal{X}^{n\star})$ is a filtration of simplicial complexes with vertices in $A \times A$ (graph filtration).

Example 2 - a filtration



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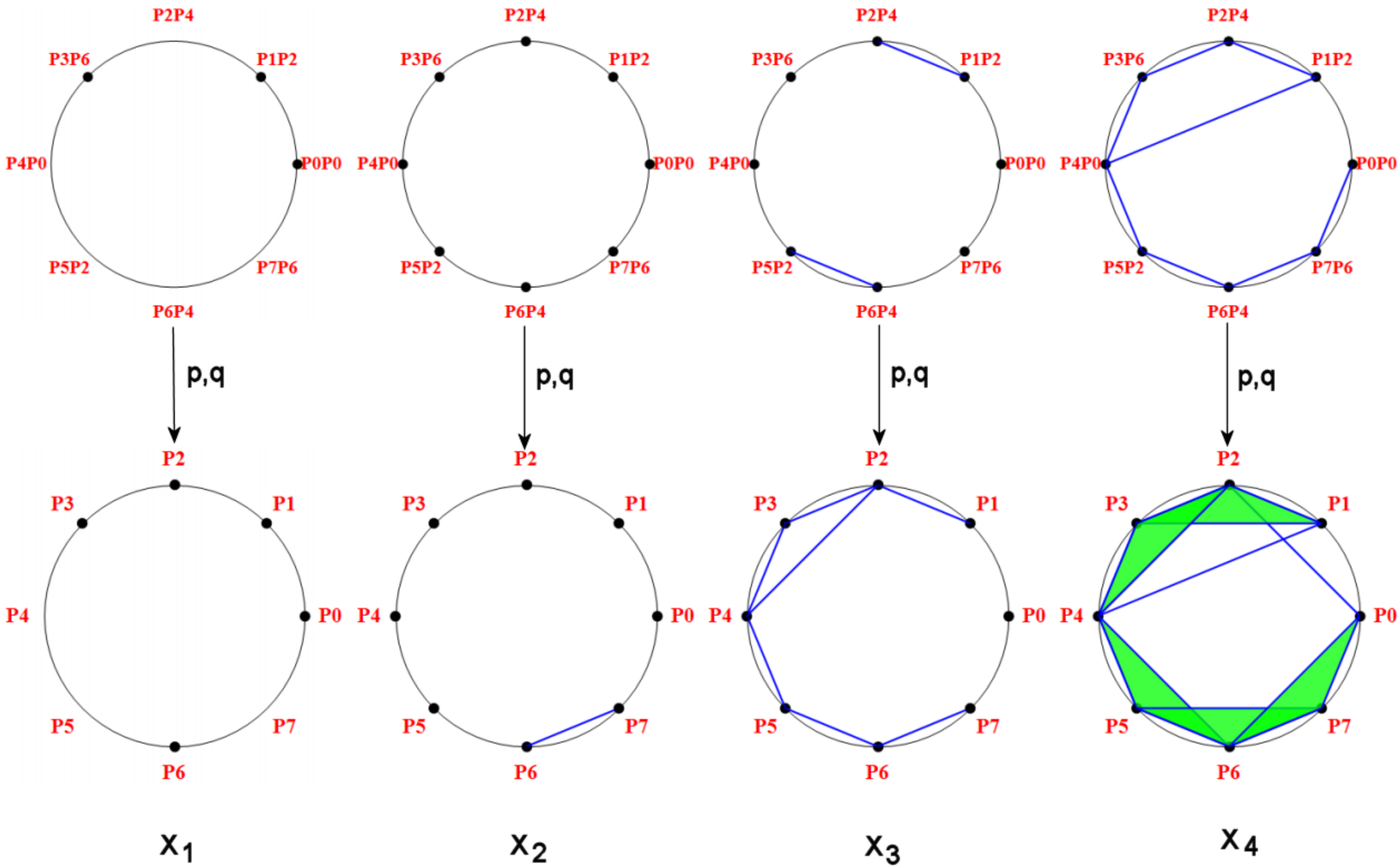

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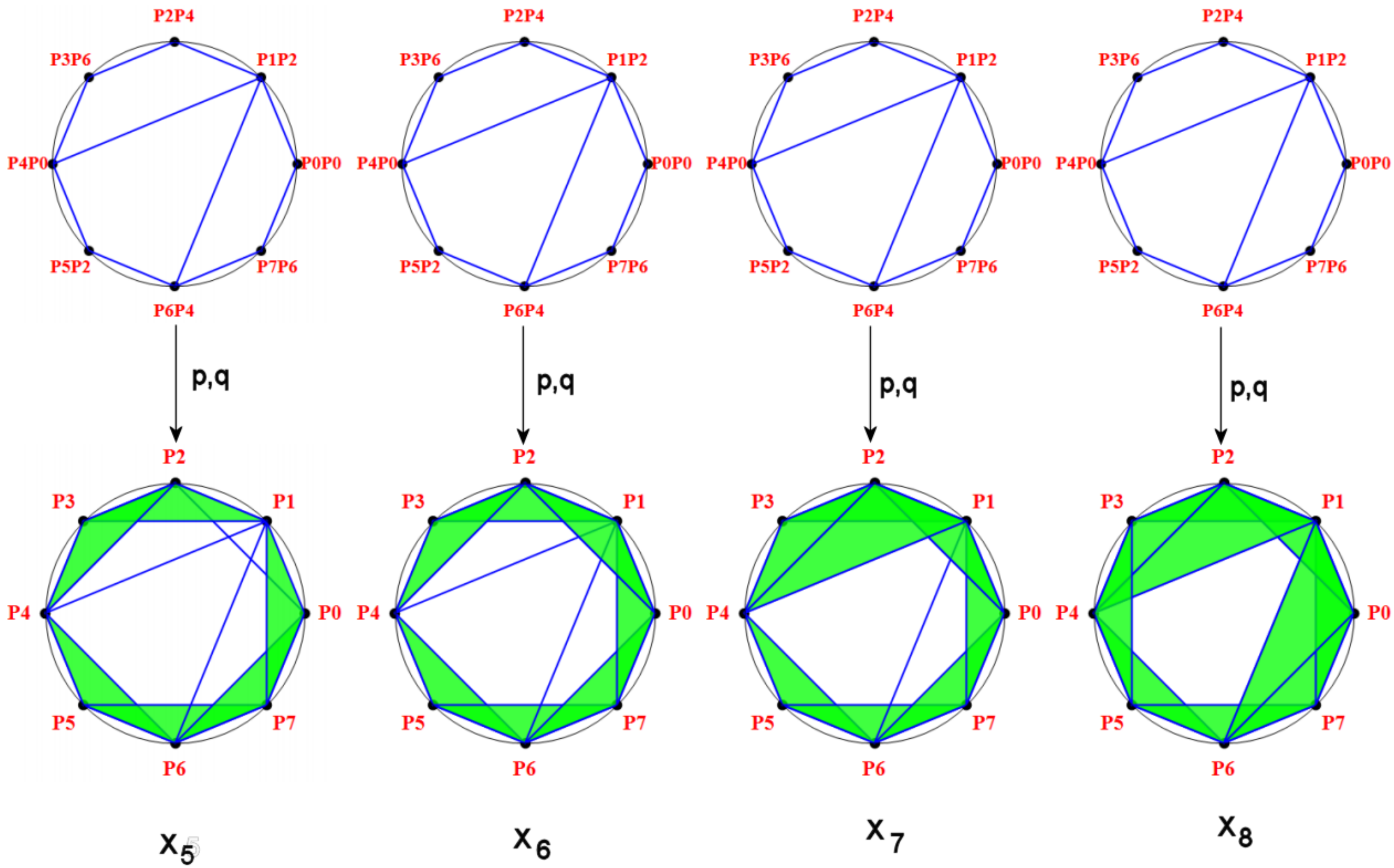

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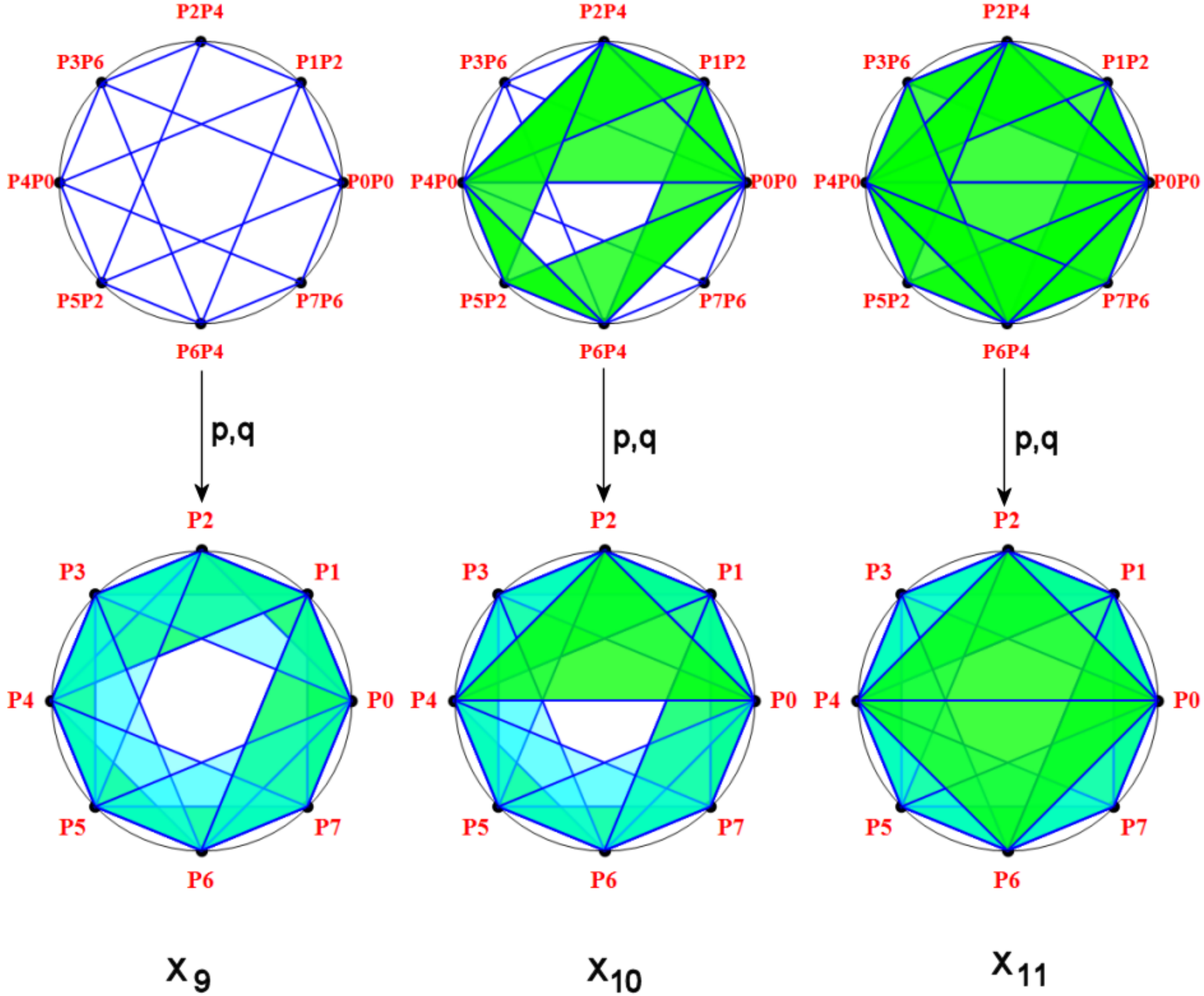


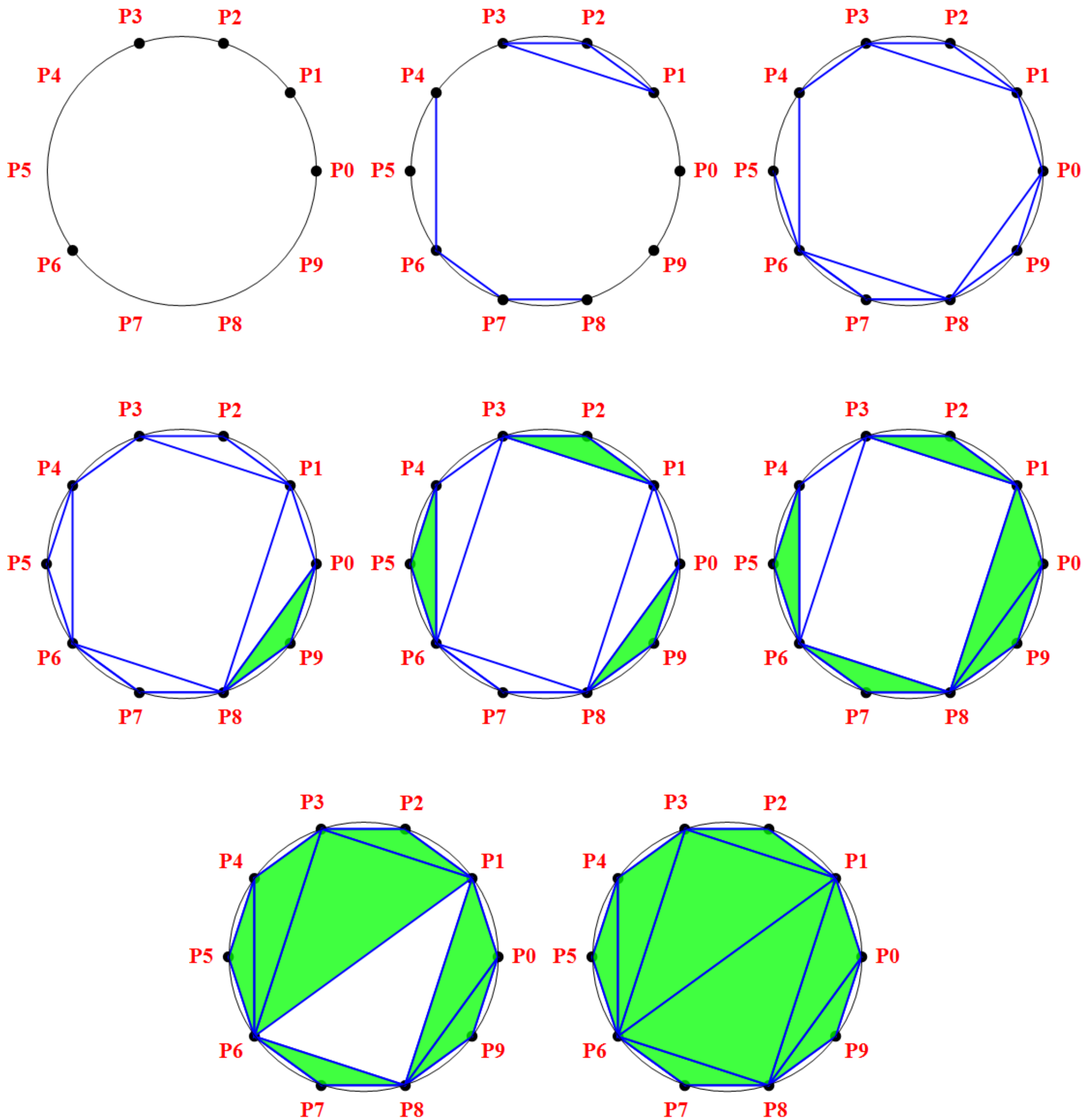
Example 2 - persistence of eigenvalues

- $\lambda = 1$
  - $\{P_0^\star\}$ - $[1, 11)$
  - $\{P_1^\star\}$ - $[2, 3)$
  - $\{P_3^\star\}$ - $[2, 3)$
  - $\{P_5^\star\}$ - $[2, 3)$
  - $\{P_7^\star\}$ - $[2, 3)$
- $\lambda = 2$
  - $\{P_0P_1^\star, P_1P_2^\star, P_2P_3^\star, P_3P_4^\star, P_4P_5^\star, P_5P_6^\star, P_6P_7^\star, P_7P_0^\star\}$ - $[6, 11)$

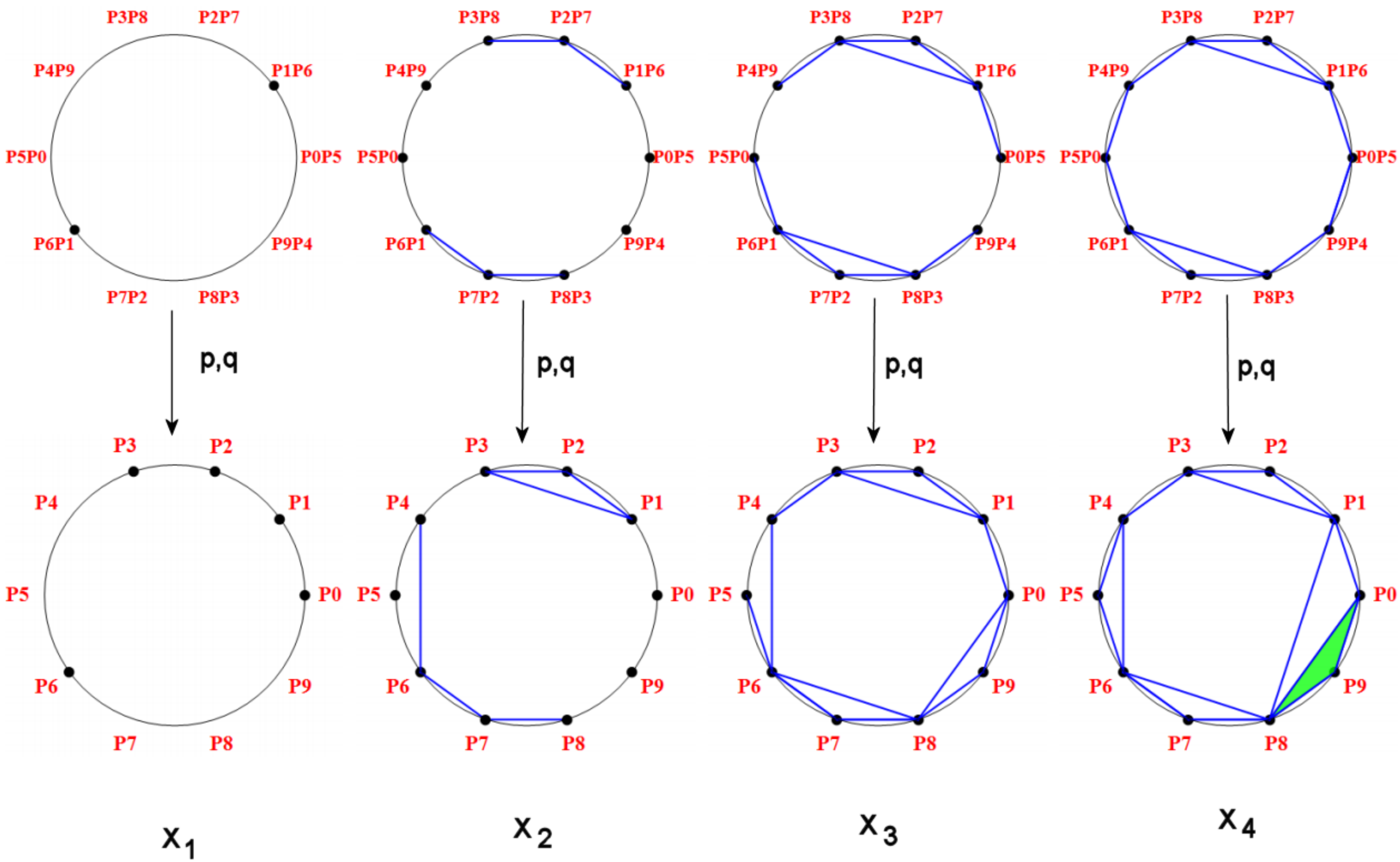
Example 3

- Take the map
$$S^1 \ni z \mapsto -z \in S^1.$$
- For K even we get the map
$$\alpha : A_K \ni P_i \mapsto P_{i+K/2} \in A_K.$$

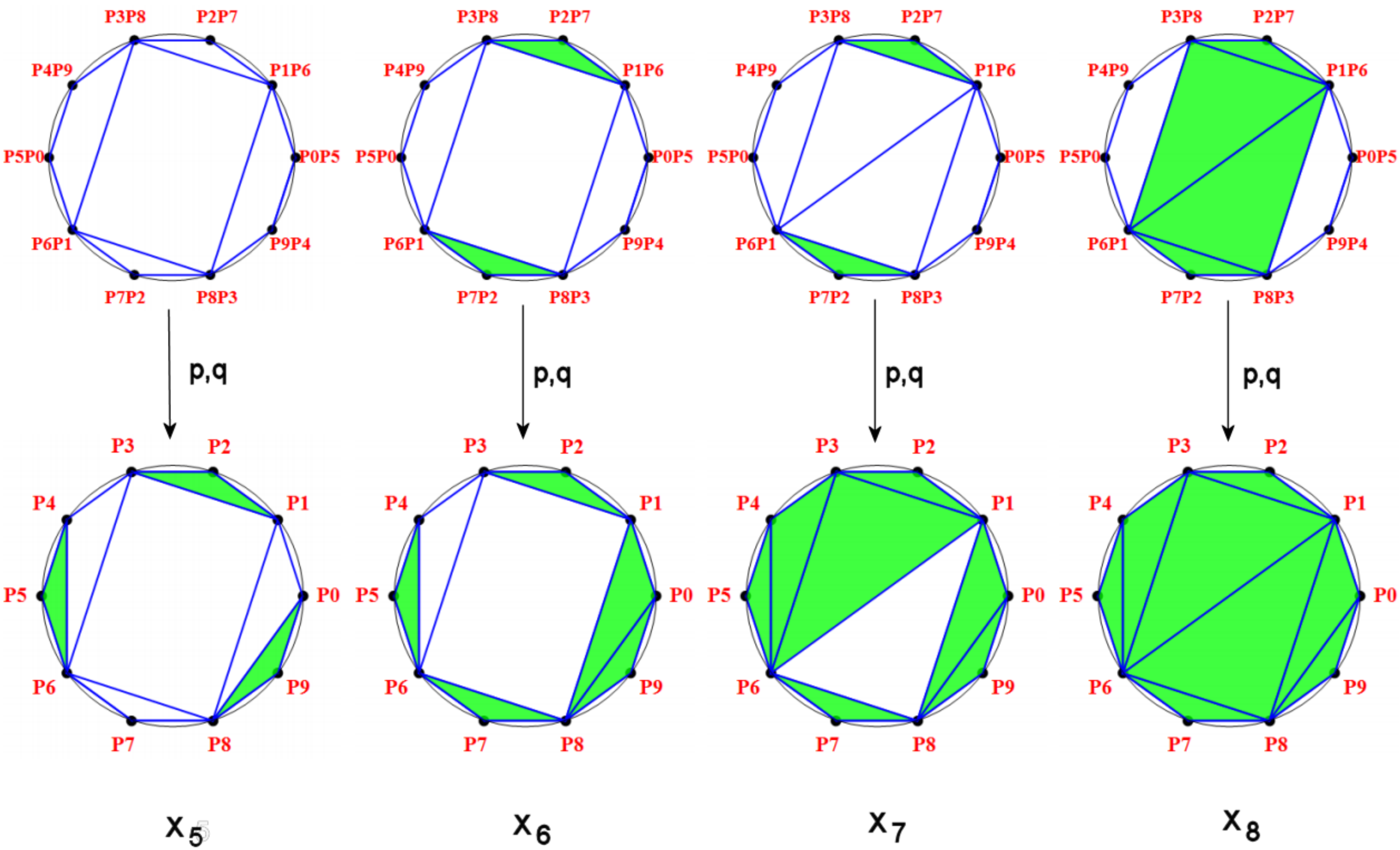
Example 3 - filtration



Example 3 - graph filtration with projections



Example 3 - graph filtration with projections



Example 3 - persistence intervals 37

- $\lambda = 1$
  - $[P_1^\star + P_6^\star]$ – interval $[1, 8)$
  - $[P_4^\star + P_9^\star]$ – interval $[2, 3)$
  - $[P_0^\star + P_5^\star]$ – interval $[2, 3)$
  - $[P_1 P_2^\star + P_2 P_3^\star + P_3 P_1^\star + P_6 P_7^\star + P_7 P_8^\star + P_8 P_6^\star]$ – interval $[3, 6)$
  - $[\Sigma_{i=0}^{9} P_i P_{i+1}^\star]$ – interval $[4, 8)$
  - $[P_1 P_3^\star + P_3 P_6^\star + P_6 P_8^\star + P_8 P_1^\star]$ – interval $[5, 8)$
  - $[\Sigma_{i=3}^{6} P_i P_{i+1}^\star + \Sigma_{i=8}^{11} P_i P_{i+1}^\star]$ – interval $[5, 7)$

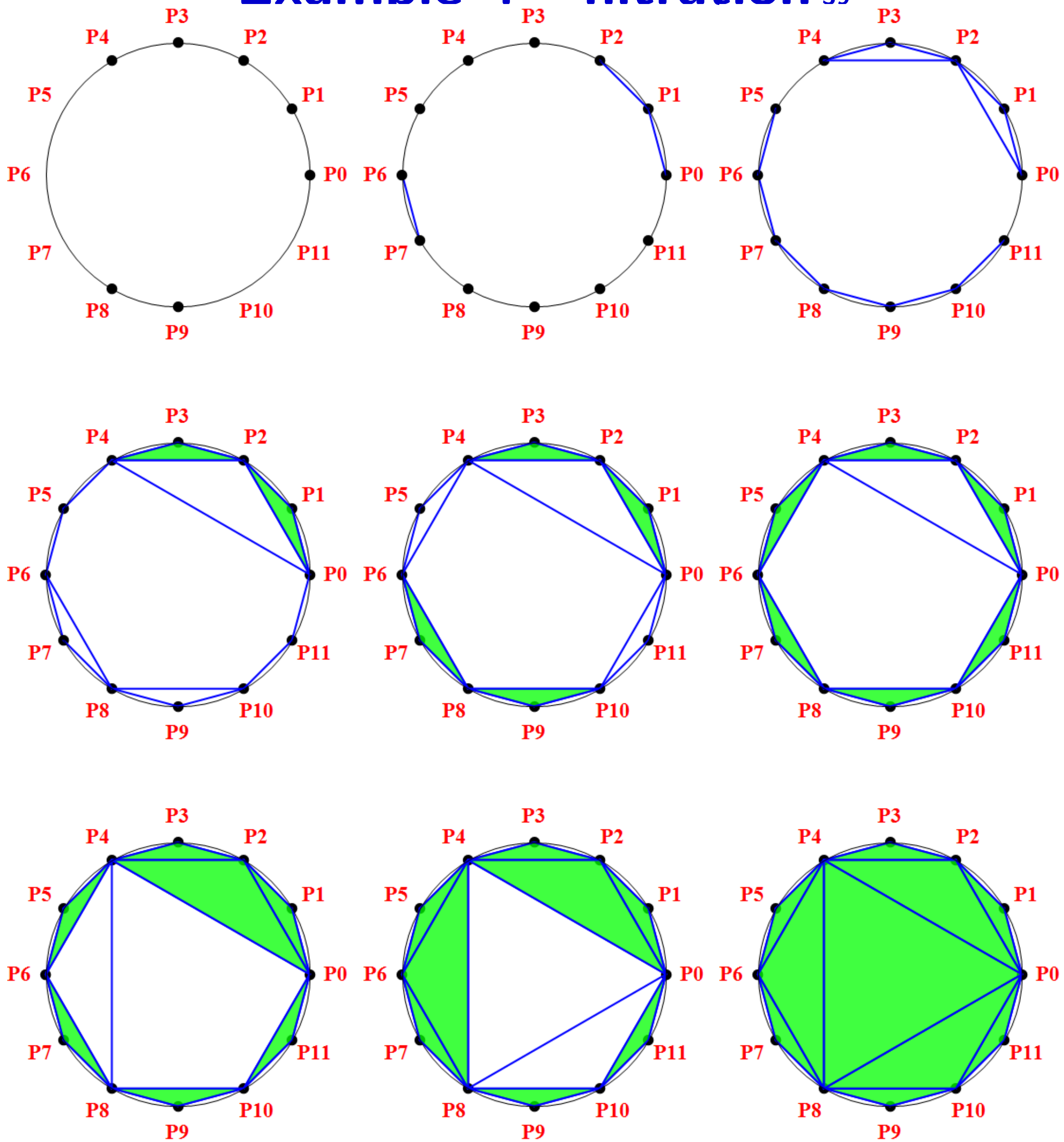
Example 4

- Take again the map

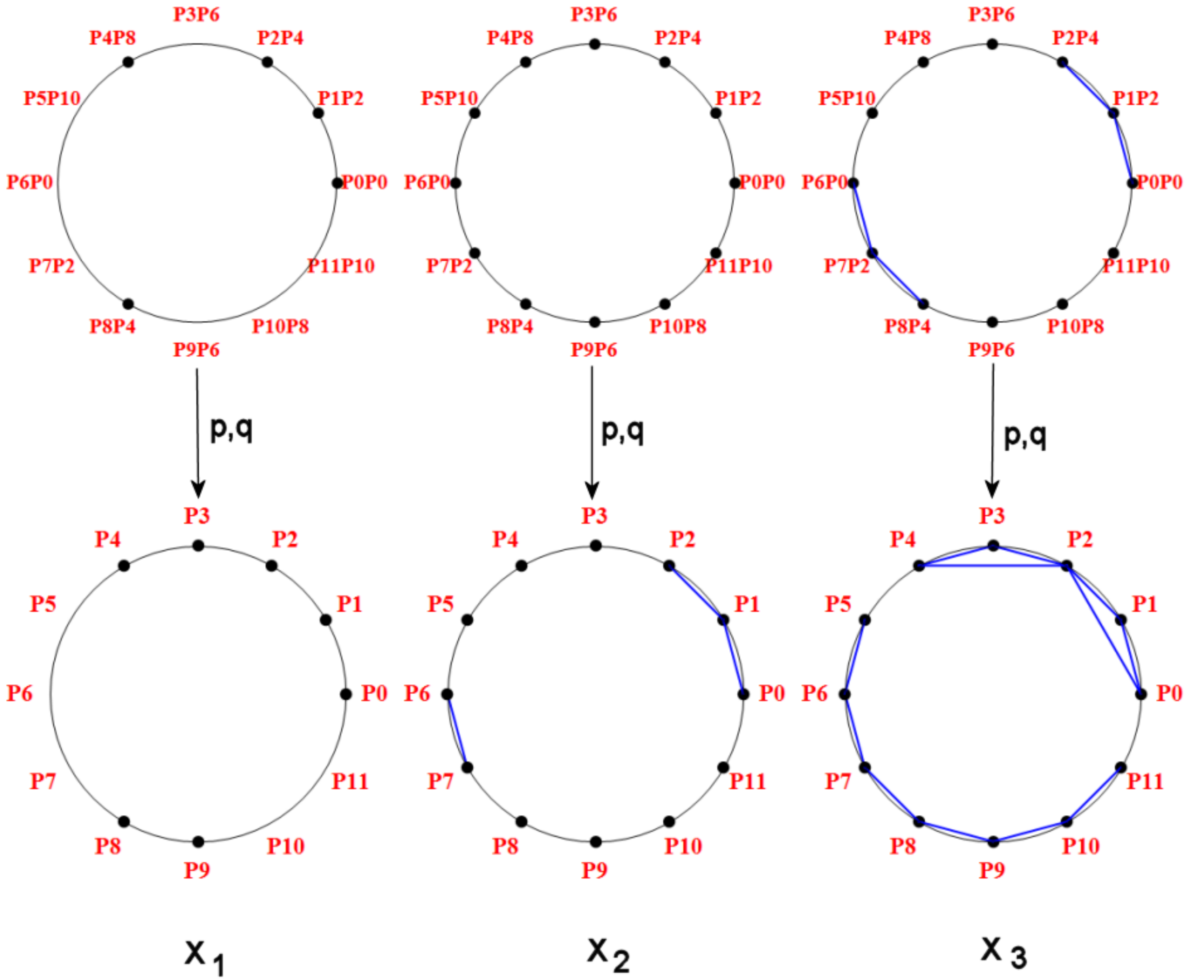
$$S^1 \ni z \mapsto z^2 \in S^1.$$

  and $K := 12$.

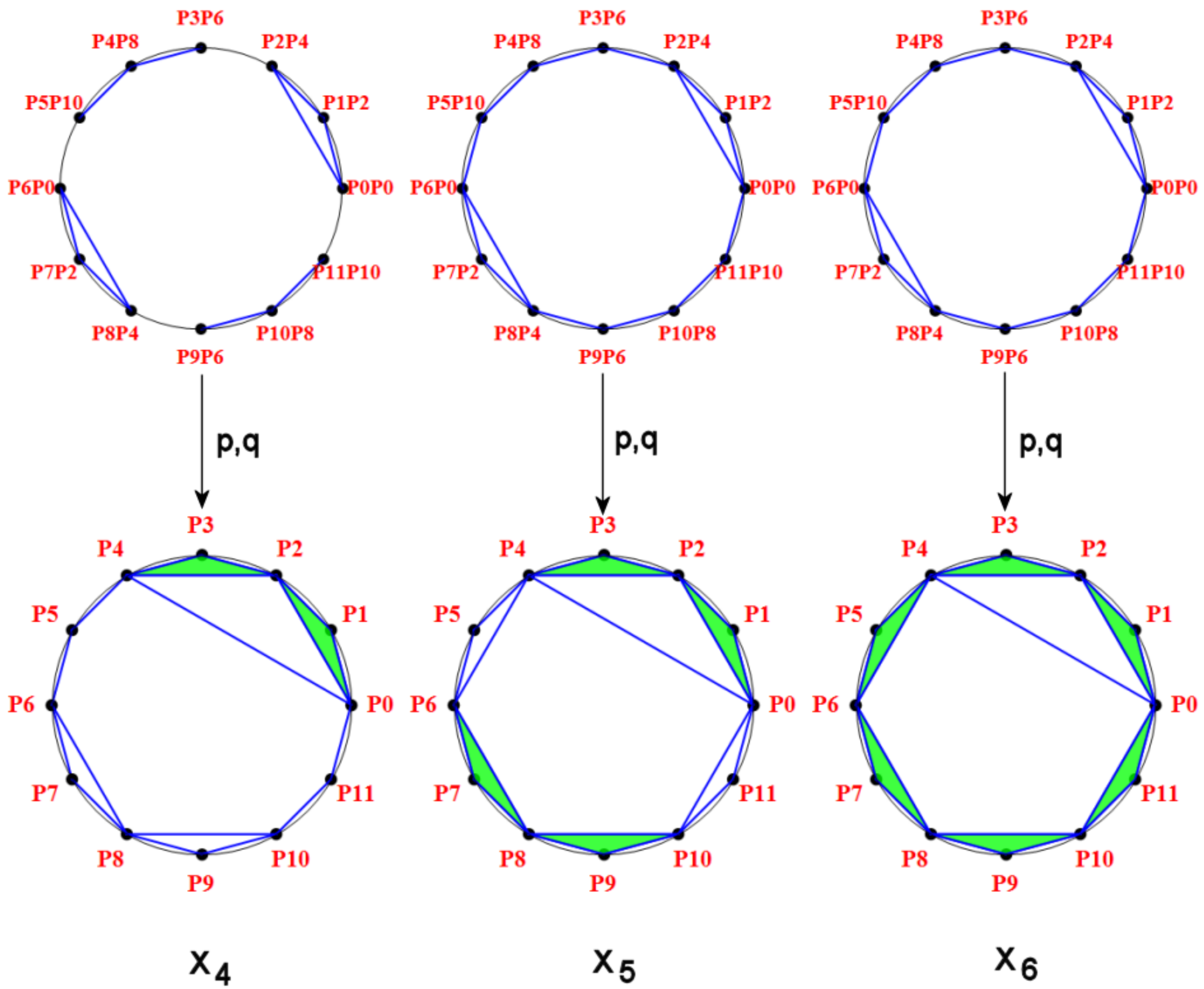
Example 4 - filtration



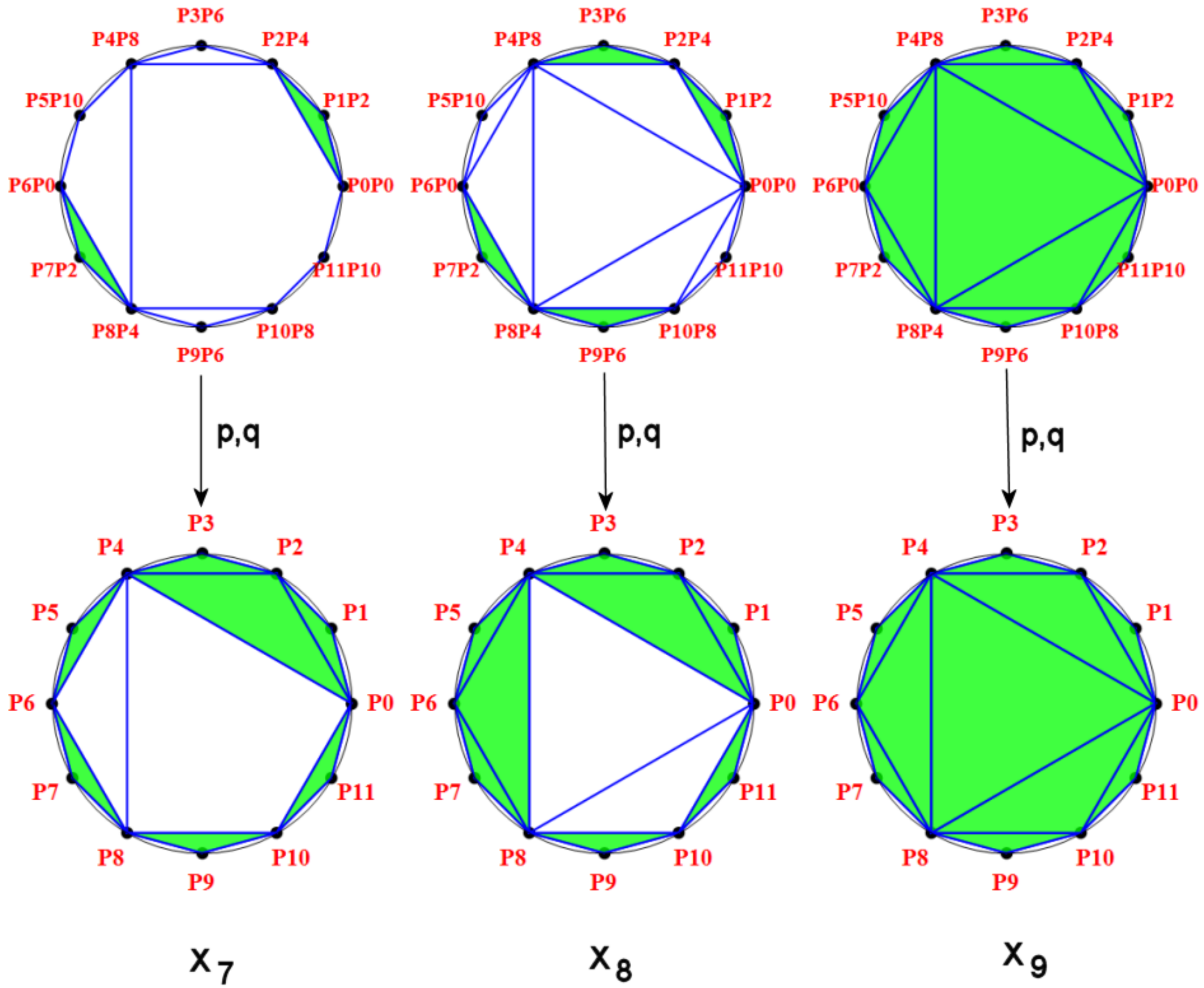
Example 4 - graph filtration with projections



Example 4 - graph filtration with projections



Example 4 - graph filtration with projections



Example 4 - persistence intervals

- $\lambda = 1$
  - $[P_0^\star]$ – interval $[1, 9)$
  - $[P_4^\star + P_8^\star]$ – interval $[4, 5)$
  - $[P_1^\star]$ – interval $[2, 3)$
  - $[P_5^\star]$ – interval $[3, 4)$
  - $[P_{11}^\star]$ – interval $[3, 4)$
  - $[P_9^\star]$ – interval $[3, 4)$
  - $[P_{10}^\star]$ – interval $[3, 5)$
  - $[P_8^\star]$ – interval $[4, 5)$
  - $[P_{10}P_{11}^\star + P_{11}P_0^\star + \Sigma_{i=0}^{8} P_i P_{i+2}^\star]$ – interval $[7, 9)$
- $\lambda = 2$
  - $[\Sigma_{i=0}^{11} P_i P_{i+1}^\star]$ – interval $[6, 9)$
- $\lambda = -1$
  - $[\Sigma_{i=0}^{8} P_i P_{i+4}^\star]$ – interval $[8, 9)$

Future plans

- Implementing the algorithm and experimenting with large data
- Persistence of Jordan form
- A convergence theorem?
- Persistence of topological invariants of dynamical systems: Conley index, fixed point index, connection matrices ?
- Applications to time series dynamics

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Thank you for your attention!