Fourier Transform and Homology

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Technion – Israel Institute of Technology

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Plan

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Sum Complexes and Hypertrees
Joint work with N. Linial and M. Rosenthal

- ▶ Sum Complexes and their homology
- ▶ Hypertrees and Chebotarëv's Theorem

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Uncertainty Numbers and Homology

- ▶ Discrete Uncertainty Principles
- ▶ Uncertainty Numbers and Sum Complexes

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Uncertainty Numbers and Homology

- Discrete Uncertainty Principles
- Uncertainty Numbers and Sum Complexes

Balanced Complexes

- Fourier Transform of Coboundaries
- Application to Musiker-Reiner Complexes

Minimal Rank of Circulants

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The minimal rank of $M(x)$ may depend on the field:

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This has something to do with $\mathbb{RP}^2 \ldots$

Topological Terminology

$\Delta_{n-1} = (n-1)$-simplex.

For a simplicial complex X:

$X^{(k)} = k$-dimensional skeleton of X.

$f_k(X) =$ number of k-dimensional faces of X.

$H_k(X; \mathbb{F}) = k$-th Homology of X with $\mathbb{F}$ coefficients.

$h_k(X; \mathbb{F}) = \dim H_k(X; \mathbb{F})$.

An Arithmetical Construction of Complexes [LMR]

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The Sum Complex

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Remark

If $k+1$ is coprime to n then $f_k(X_{A,k}) = \frac{|A|}{k+1} \binom{n-1}{k}$. Hence:

- $|A| > k+1 \Rightarrow f_k(X_{A,k}) > \binom{n-1}{k} \Rightarrow H_k(X_{A,k}) \neq 0$.
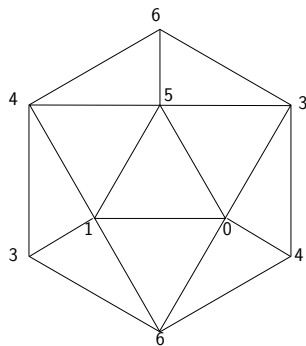- $|A| = k+1 \Rightarrow f_k(X_{A,k}) = \binom{n-1}{k}$ and $X_{A,k}$ is pure.

From Fano Plane to $\mathbb{RP}^2$

$$A = \{0, 1, 3\} \subset G = \mathbb{Z}_7$$

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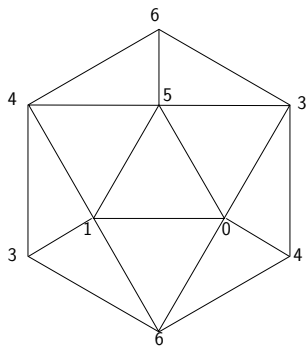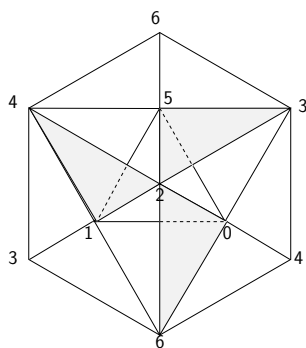$$A = \{0, 1, 3\} \subset G = \mathbb{Z}_7$$

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$X_{A,2} \simeq \mathbb{RP}^2$

The Finite Fourier Transform

G —   *Finite abelian group with exponent ℓ.*

$\mathbb{F}$ —   *Field that contains a primitive $\ell - th$ root of 1.*

$\widehat{G}$ — $\mathbb{F} -$ *valued characters of G.*

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The Fourier Transform $\mathcal{F} : \mathbb{F}[G] \to \mathbb{F}[\widehat{G}]$:

$$\mathcal{F}(f)(\chi) = \widehat{f}(\chi) = \sum_{x \in G} \chi(-x) f(x) \quad .$$

The Homology of $X_{A,k}$

$A = \{a_1, \ldots, a_m\}$, $\widehat{G} = \{1 = \chi_0, \chi_1, \ldots, \chi_{n-1}\}$
$\mathcal{B}_{n,k} = \{B \subset \{1, \ldots, n-1\} : |B| = k\}$
For $B = \{i_1 < \cdots < i_k\} \in \mathcal{B}_{n,k}$ let

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M_B = \begin{bmatrix}
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Theorem [LMR]: If $k + 1$ is coprime to n then:

$$h_k(X_{A,k}; \mathbb{F}) = \frac{m}{k+1}\binom{n-1}{k} - \frac{1}{k+1}\sum_{B \in \mathcal{B}_{n,k}} \text{rank } M_B \ .$$

Simplicial Cohomology

m-Cochains:

$$C^m(G) = \text{skew-symmetric } \mathbb{F}\text{-valued functions on } G^{m+1}$$

Coboundary operator:

$$d_m : C^m(G) \to C^{m+1}(G)$$

$$d_m \phi(x_0, \ldots, x_{m+1}) = \sum_{i=0}^{m+1} (-1)^i \phi(x_0, \ldots, \widehat{x_i}, \ldots, x_{m+1})$$

$H^{k-1}(X_{A,k})$ via Fourier Transform

Let T be the automorphism of $\widehat{G}^k$ given by

$$T(\chi_1, \ldots, \chi_k) = (\chi_2\chi_1^{-1}, \ldots, \chi_k\chi_1^{-1}, \chi_1^{-1})$$

Claim 1: $g \in \mathcal{F}(B^{k-1}(X_{A,k}))$ iff $g \in C^{k-1}(\widehat{G})$ and

$$g(\chi_1, \ldots, \chi_k) = 0$$

whenever $\chi_1, \ldots, \chi_k \neq \mathbf{1}$.

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Claim 2: $g \in \mathcal{F}(Z^{k-1}(X_A, k))$ iff $g \in C^{k-1}(\widehat{G})$ and

$$g(\chi) + \sum_{i=1}^{k} (-1)^{ki} \chi_i(-a) g(T^i \chi) = 0$$

for all $\chi = (\chi_1, \ldots, \chi_k) \in \widehat{G}^k$ and $a \in A$.

The Cohomology of $X_{A,k}$

Theorem [LMR]: If $k+1$ is coprime to n then:

$$h_k(X_{A,k}; \mathbb{F}) = \frac{m}{k+1} \binom{n-1}{k} - \frac{1}{k+1} \sum_{B \in \mathcal{B}_{n,k}} rank\, M_B \quad .$$

Outline of Proof: Using lemmas 1 & 2 we explicitly compute

$$\mathcal{F}(Z^{k-1}(X_A, k)) / \mathcal{F}(B^{k-1}(X_{A,k}))$$

obtaining in particular that

$$h_{k-1}(X_{A,k}; \mathbb{F}) = \binom{n-1}{k} - \frac{1}{k+1} \sum_{B \in \mathcal{B}_{n,k}} rank\, M_B \quad .$$

$h_k(X_{A,k}; \mathbb{F})$ is then determined using the Euler-Poincaré relation.

Higher Dimensional Trees

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A k-Hypertree on n vertices is a simplicial complex X such that:

- $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$
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Remark: A k-Hypertree is always $\mathbb{Q}$-acyclic. However, it may (and usually will) have nontrivial homology over other fields.
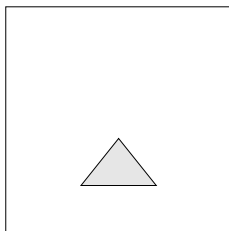
Examples of k-Hypertrees

A Recursive Construction

$\mathcal{C}(n, k)$ = the family of all k-hypertrees on $[n]$.

$$X \in \mathcal{C}(n-1, k) \ , \quad Y \in \mathcal{C}(n-1, k-1) \quad \Rightarrow \quad X \cup Y * n \in \mathcal{C}(n, k).$$

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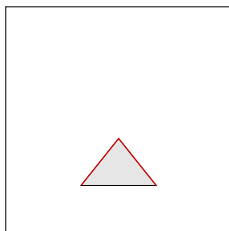
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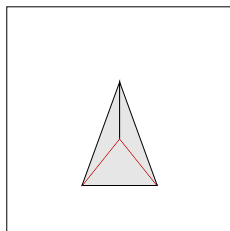
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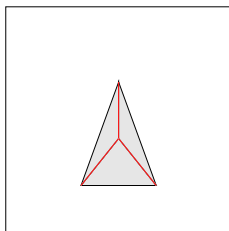
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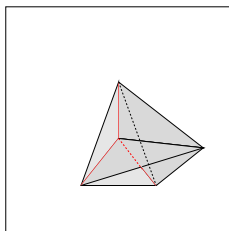
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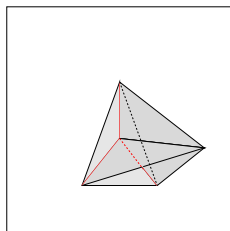
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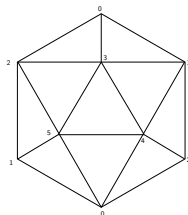$\mathcal{C}(n, k) =$ the family of all k-hypertrees on $[n]$.

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A collapsible 2-Hypertree

$\mathbb{RP}^2$: a Non-Acyclic 2-Hypertree

Enumerative Aspects

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Conjecture:

Most k-hypertrees are not $\mathbb{Z}$-acyclic.

Chebotarëv's Theorem

$\omega = \omega_n = \exp(2\pi i/n)$ a primitive n-th root of unity.

Fourier Matrix:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{bmatrix}$$

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Various other proofs by Resetnyak (1955), Dieudonné (1970), Newman (1975), Evans and Stark (1977), Frenkel (2003), ...

Chebotarëv's Approach

The p-adic Valuation

Let $x = p^k \frac{a}{b}$ with a, b integers coprime to p.

p-adic order of x: $\qquad$ $ord_p(x) = k$.

p-adic valuation of x: $\qquad$ $|x|_p = p^{-k}$.

$\mathbb{Q}_p =$ Completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

The p-adic valuation extends to $\mathbb{Q}_p(\omega_p)$, e.g.

$$ord_p(1 - \omega_p) = \frac{1}{p - 1} .$$

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Chebotarëv's Theorem:

Let M be a $k \times k$ submatrix of the Fourier matrix of order p.

Then:

$$ord_p(\det M) = \frac{k(k-1)}{2(p-1)} \quad .$$

A New Family of Hypertrees

Let p be a prime, $A \subset G = \mathbb{Z}_p$, $|A| = k + 1$.

The k-th homology of $X_{A,k}$ is

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Corollary [LMR]: $X_{A,k}$ is a k-hypertree.

Collapsibility and Acyclicity of Sum Complexes

Theorem [LMR]:

Let $A \subset G = \mathbb{Z}_p$, $|A| = k + 1$. Then:

$X_{A,k}$ is k-collapsible $\Leftrightarrow$ A is an arithmetic progression.

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Example: 2-trees from $A = \{0, 1, 3\} \subset \mathbb{Z}_n$

Assume: p prime and $n = p^m - 1$ is coprime to 3. Then:

$$
h_1(X_{A,2}; \mathbb{F}_p) = \begin{cases} \frac{n-1}{6} & p = 2 \\ \frac{n-2}{6} & p = 3 \\ \frac{n-4}{6} & p > 3. \end{cases}
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Uncertainty Principles

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Classical Uncertainty Inequality:

If $||f||_2 = 1$ then

$$\int_x |x|^2 |f(x)|^2 dx \cdot \int_\xi |\xi|^2 |\widehat{f}(\xi)|^2 d\xi \geq \frac{1}{16\pi^2} \quad .$$

The Discrete Uncertainty Principle

Theorem [Donoho and Stark]: For any $0 \neq f \in \mathbb{F}[G]$

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Case of Equality:

$f = 1_H$ where H is subgroup of G.

The Discrete Uncertainty Principle

Theorem [Donoho and Stark]: For any $0 \neq f \in \mathbb{F}[G]$

$$|supp(f)| \cdot |supp(\widehat{f})| \geq |G| \quad .$$

Case of Equality:

$f = 1_H$ where H is subgroup of G.

A stronger statement holds for $\mathbb{F} = \mathbb{C}$ and $G = \mathbb{Z}_p$.

Theorem [Tao and others]: For a prime p and $0 \neq f \in \mathbb{C}[\mathbb{Z}_p]$:

$$|supp(f)| + |supp(\widehat{f})| \geq p + 1 \quad .$$

A Common Extension

Theorem [M]: Let $f \in \mathbb{C}[G]$ and let $d_1 < d_2$ be two consecutive divisors of $|G| = n$ such that $d_1 \leq k = |supp(f)| \leq d_2$. Then
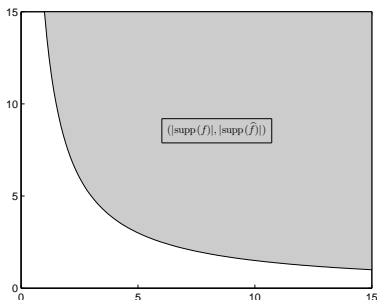
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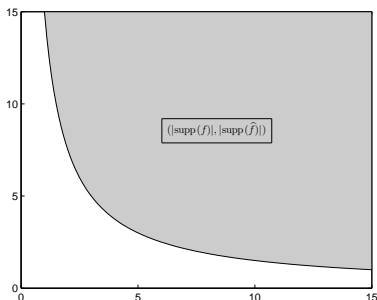$$|supp(\widehat{f})| \geq \frac{n}{d_1 d_2}(d_1 + d_2 - k) \ .$$

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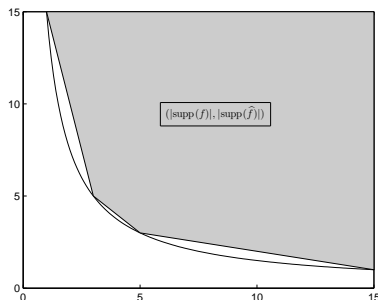
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Uncertainty Numbers

For a subset $A \subset G$ let:

$$u(G, A; \mathbb{F}) = \min\{ \, |supp(\widehat{f})| \; : \; 0 \neq f \in \mathbb{F}[G] \, , \; supp(f) \subset A \, \}.$$

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Matrix Formulation

For $x = (x_a : a \in A)$ let $M(x)$ be the $G \times G$ matrix

$$M(x)(g, h) = \left\{ \begin{array}{ll} x_{h-g} & h - g \in A \\ 0 & h - g \notin A. \end{array} \right.$$

Then

$$u(G, A; \mathbb{F}) = \min\{ \, rank \, M(x) \; : \; 0 \neq x \in \mathbb{F}^A \, \}.$$

Uncertainty and Homology

Theorem [M]:

$$\tilde{H}_k(X_{A,k}; \mathbb{F}) \neq 0 \quad \Rightarrow \quad u(G, A; \mathbb{F}) \leq n - k - 1.$$

If $gcd(k + 1, n) = 1$ then the converse holds:

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Example: $G = \mathbb{Z}_7$, $A = \{0, 1, 3\}$, $X_{A,2} \simeq \mathbb{RP}^2$

$$H_2(\mathbb{RP}^2; \mathbb{C}) = 0 \quad \Rightarrow \quad u(G, A; \mathbb{C}) = 5.$$
$$H_2(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2 \quad \Rightarrow \quad u(G, A; \overline{\mathbb{F}_2}) = 4.$$

Complete Balanced Complexes

$G_0, \ldots, G_k$ finite abelian groups with discrete topology.
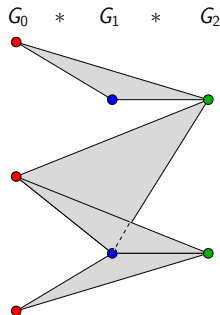$N = \prod_{i=0}^{k}(|G_i| - 1)$.

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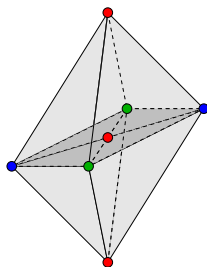
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Fourier Transform of Couboundaries

$G = G_0 \times \cdots \times G_k =$ oriented k-simplices of Y.
The Balanced Complex of $A \subset G$ is $X(A) = Y^{(k-1)} \cup A$.

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Let $\mathbf{1}_i$ be the trivial character of G_i and let

$$\widehat{G}^+ = (\widehat{G}_0 - \{\mathbf{1}_0\}) \times \cdots \times (\widehat{G}_k - \{\mathbf{1}_k\}).$$

Proposition [M]: The k-Coboundaries of $X(A)$ are:

$$B^k(X(A); \mathbb{Z}) = \{f_{|A} : f \in \mathbb{Z}[G] \ \text{such that} \ supp(\widehat{f}) \subset \widehat{G} - \widehat{G}^+\}.$$

Balanced Complexes from Arithmetic

Notation

$p_0, \ldots, p_k$ distinct primes, $n = p_0 \cdots p_k$.

$G = \mathbb{Z}_{p_0} \times \cdots \times \mathbb{Z}_{p_k} = \mathbb{Z}_n$.

$\mathbb{Z}_n^\times = \{m \in \mathbb{Z}_n : gcd(m, n) = 1\}$.

$|\mathbb{Z}_n^\times| = \varphi(n) = \prod_{i=0}^{k}(p_i - 1)$.

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Musiker-Reiner Complexes

Let $A_0 = \{ \varphi(n) + 1, \ldots, n - 1 \}$.

For $A \subset \{ 0, \ldots, \varphi(n) \}$ let

$$K_A = X(A \cup A_0).$$

Let $\omega = \exp(\frac{2\pi i}{n})$ be a primitive n-th root of unity.
The n-th cyclotomic polynomial:

$$\Phi_n(z) = \prod_{j \in \mathbb{Z}_n^{\times}} (z - \omega^j) = \sum_{j=0}^{\varphi(n)} c_j z^j \in \mathbb{Z}[z].$$

Homology of $K_{\{j\}}$

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Theorem [Musiker and Reiner]:
For $j \in \{0, \ldots, \varphi(n)\}$

$$\tilde{H}_i(K_{\{j\}}; \mathbb{Z}) \cong \left\{ \begin{array}{ll} \mathbb{Z}/c_j\mathbb{Z} & i = k-1 \\ \mathbb{Z} & i = k \text{ and } c_j = 0 \\ 0 & otherwise. \end{array} \right.$$

Homology of K_A

For $A \subset \{0, \ldots, \varphi(n)\}$ let $c_A = (c_j : j \in A) \in \mathbb{Z}^A$ and

$$d_A = \begin{cases} gcd(c_A) & c_A \neq 0 \\ 0 & c_A = 0. \end{cases}$$

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Sketch of Proof

The k-Coboundaries of K_A

$$B^k(K_A; \mathbb{Z}) = \{f_{|A \cup A_0} : f \in \mathbb{Z}[\mathbb{Z}_n] \ , \ supp(\widehat{f}) \subset \mathbb{Z}_n - \mathbb{Z}_n^\times\} =$$

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The k-Cohomology of K_A

$$H^k(K_A; \mathbb{Z}) = \frac{\mathbb{Z}[A]}{\mathcal{B}(A)} \stackrel{def}{=} \mathcal{H}(A).$$

For $j \in A \cup A_0$ define $g_j \in \mathbb{Z}[A \cup A_0]$ by $g_j(i) = \delta_{ij}$.
$[g_j]$ = image of g_j in $\mathcal{H}(A)$.

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Claim:

(i) $\mathcal{H}(A)$ is generated by $\{[g_j] \ : \ j \in A\}$.

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$$\sum_{j \in A} c_j[g_j] = 0.$$

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Corollary:

$$H^k(K_A) = \mathcal{H}(A) = \mathbb{Z}[A]/\mathbb{Z}c_A \cong \mathbb{Z}^{|A|-1} \oplus \mathbb{Z}/d_A\mathbb{Z} \ .$$

Proof of (i)

Let $t \in A_0$. There exist $u_0, \ldots, u_{\varphi(n)-1} \in \mathbb{Z}$ such that

$$\sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0.$$

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Hence

$$\widehat{f}(1) = \sum_{\ell=0}^{\varphi(n)-1} u_\ell \omega^\ell + \omega^t = 0 \quad \Rightarrow$$

$$\sum_{j \in A} u_j g_j + g_t = f_{|A \cup A_0} \in \mathcal{B}(A) \quad \Rightarrow \quad [g_t] = -\sum_{j \in A} u_j [g_j].$$

Proof of (ii)

Define $f \in \mathbb{Z}[\mathbb{Z}_n]$ by

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The minimality of this relation follows from the minimality of $\Phi(z)$.