Comparison of Persistent Homologies for Vector Functions: from continuous to discrete

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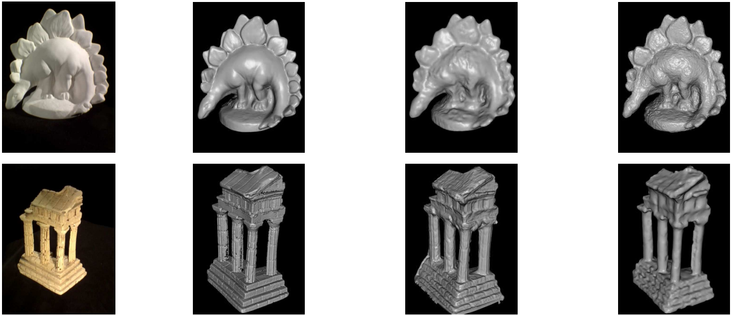
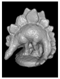
Motivation

Real object | Reconstruction | Reconstruction | Reconstruction



$D\Big($  ,  $\Big) \approx D\Big($  ,  $\Big)$?

Outline

- Multidimensional persistence of a filtration
  - sub-level set filtrations
  - simplicial complex filtrations
- From discrete to continuous filtrations: topological aliasing
- Homological critical values
- Comparison of multidimensional persistence: from continuous to discrete

Persistence of a filtration

Given a space X, a *filtration* is a (finite or infinite) family of nested subspaces:

$$\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X.$$

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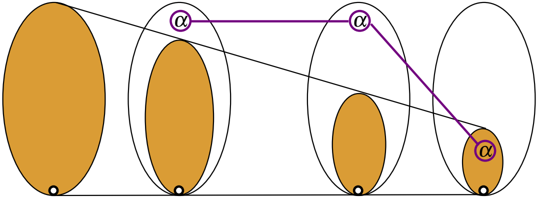
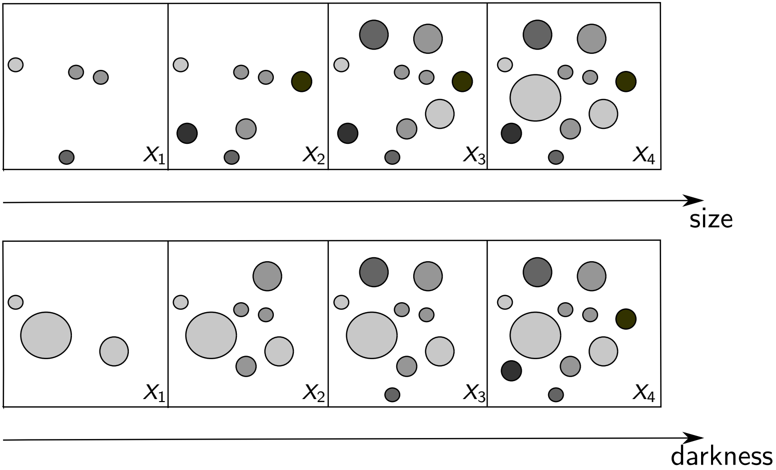
Analyse X by studying the *lifetime* of homology classes



$$\ldots \to H_k(X_{i-1}) \to H_k(X_i) \to \ \ldots \ \to H_k(X_{j-1}) \to H_k(X_j) \to \ldots$$

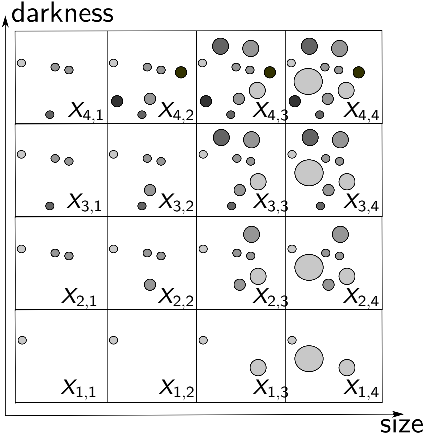
1-D vs. multi-D Persistence

1-D persistence captures the topology of a one-parameter filtration.



1-D vs. multi-D Persistence

Multi-D persistence captures the topology of a family of spaces filtered along multiple geometric dimensions.



Sublevelset filtrations

Any continuous function $f = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ induces sub-level sets:

$$X_\alpha = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, \alpha_i]), \qquad \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k.$$

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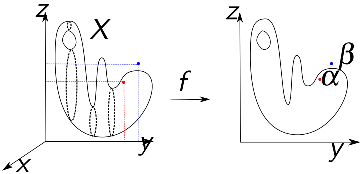
Setting

$$\alpha = (\alpha_i) \preceq \beta = (\beta_i) \text{ iff } \alpha_i \leq \beta_i \text{ for every } i$$

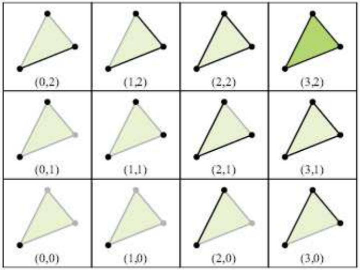
we get a k-parameter filtration of X by sub-level sets:

$$\alpha \preceq \beta \text{ implies } X_\alpha \subseteq X_\beta.$$



Discrete filtrations

Let $\mathscr{K}$ be a simplicial complex and $K = |\mathscr{K}|$ its carrier.
Any family $\{\mathscr{K}_\alpha\}_{\alpha \in \mathbb{R}^k}$ of simplicial sub-complexes of $\mathscr{K}$ with $\mathscr{K}_\alpha \leq \mathscr{K}_\beta$, for $\alpha \preceq \beta$, yields a filtration of K.

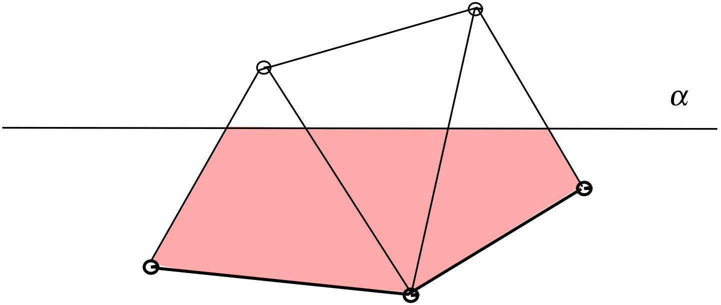


(from [Carlsson and Zomorodian, 2009])

Discrete filtrations

Given a simplicial complex $\mathscr{K}$ and a function $\varphi : \mathscr{V}(K) \to \mathbb{R}^k$, let

$$\mathscr{K}_\alpha = \{\sigma \in \mathscr{K} \,|\, \varphi(v) \preceq \alpha \text{ for all vertices } v \leq \sigma\}.$$

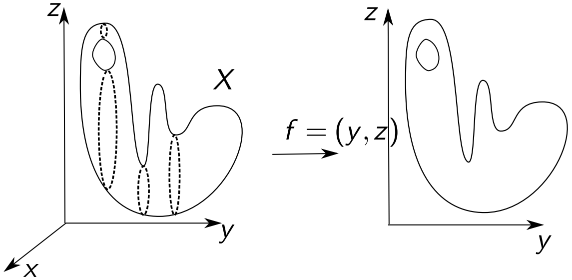


Rank invariant

It is a function that encodes the changes in persistent Betti numbers along the filtration.
For a filtration $\mathscr{F} = \{X_\alpha\}_{\alpha \in \mathbb{R}^k}$,

$$\rho_{\mathscr{F}} : \{(\alpha, \beta) \in \mathbb{R}^k \times \mathbb{R}^k | \alpha \prec \beta\} \to \mathbb{N} \cup \{\infty\},$$

$$\rho_{\mathscr{F}}(\alpha, \beta) = \dim \operatorname{im} H_*(X_\alpha \hookrightarrow X_\beta).$$



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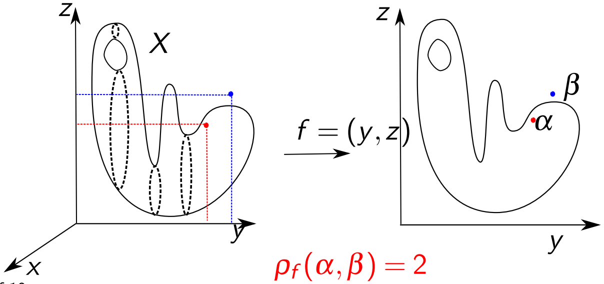

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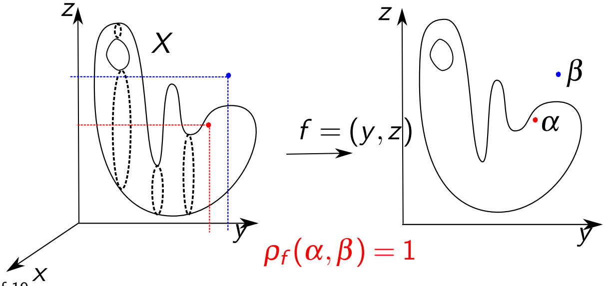

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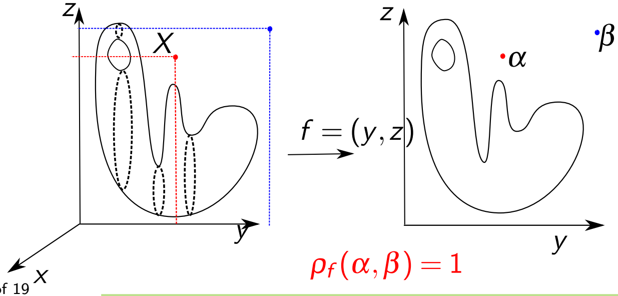

Continuous vs discrete setting

- Sub-level set filtrations are those for which stability results hold: $\forall f, f' : X \to \mathbb{R}^k$ continuous functions, $D(\rho_f, \rho_{f'}) \leq \|f - f'\|_\infty$.

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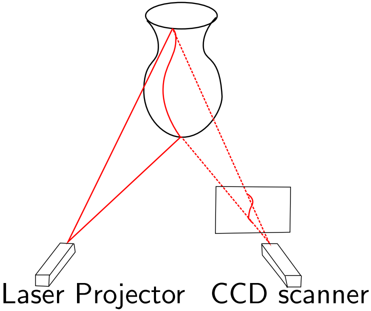
- Sub-level set filtrations are those for which stability results hold: $\forall f, f' : X \to \mathbb{R}^k$ continuous functions, $D(\rho_f, \rho_{f'}) \leq \|f - f'\|_\infty$.
- Discrete filtrations are those actually used in computations:



Stable comparison of rank invariants obtained from discrete data??

From discrete to continuous filtrations

Question: How to extend $\varphi : \mathscr{V}(K) \to \mathbb{R}^k$ to a continuous function $K \to \mathbb{R}^k$ so that its sub-level set filtration coincides with $\{K_\alpha\}_{\alpha \in \mathbb{R}^k}$?

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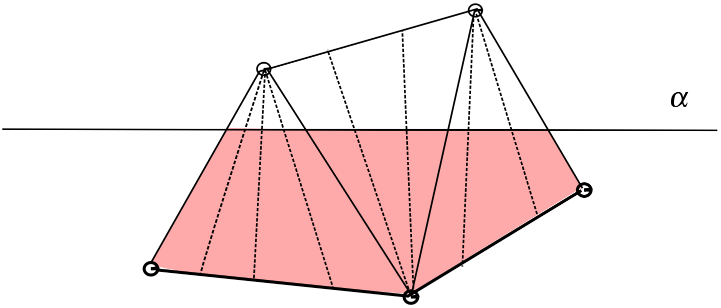
Answer: ☺ 1-D persistence: use linear interpolation [Morozov, 2008]



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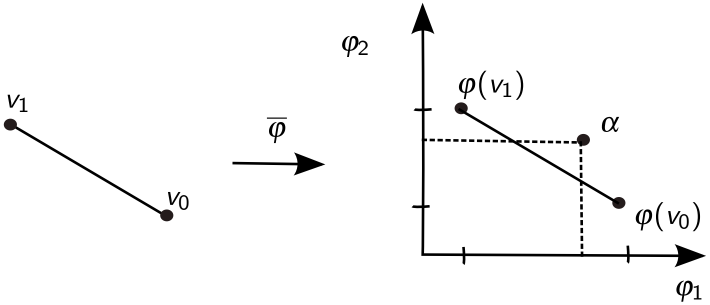
topological aliasing



From discrete to continuous filtrations

Answer: ☺ Multi-D persistence: axis-wise interpolation does the job

- Given any $\sigma \in \mathscr{K}$,

$$\mu(\sigma) = \text{least upper bound of } \{\varphi(v) \,|\, v \text{ is a vertex of } \sigma\}.$$

- Use induction to define $\varphi^{\urcorner} : K \to \mathbb{R}^k$ on σ and a point $w_\sigma \in \sigma$ s.t.
  - For all $x \in \sigma$, $\varphi^{\urcorner}(x) \preceq \varphi^{\urcorner}(w_\sigma) = \mu(\sigma)$;
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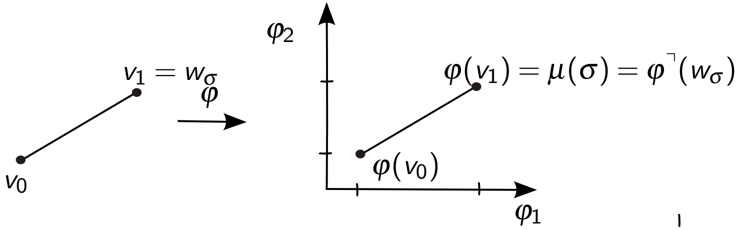

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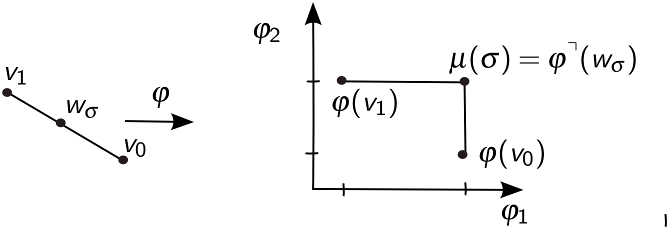

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Theorem

For any $\alpha \in \mathbb{R}^k$, K_α is a strong deformation retract of $K_{\varphi^{\urcorner} \preceq \alpha}$.

Topological Aliasing

 vs. 

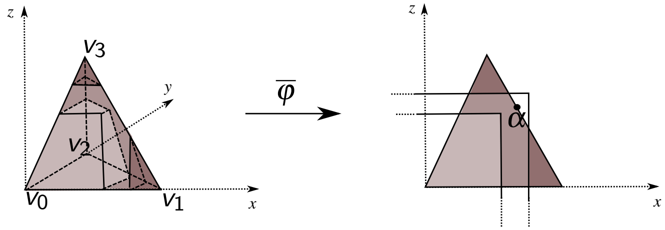
	Nonsub	Linear	Axis-wise	Diff	% Diff
H_1	0.031129	0.031129	0.031129	*0.000000*	*0.000000*
	0.039497	0.039497	0.039497	*0.000000*	*0.000000*
	0.046150	**0.040576**	**0.046150**	**-0.005574**	**-13.737185**
H_0	0.118165	0.118165	0.118165	*0.000000*	*0.000000*
	0.032043	0.032043	0.032043	*0.000000*	*0.000000*
	0.225394	**0.207266**	**0.225394**	**-0.018128**	**-8.746249**

Homological critical values

Definition

Let $\tilde{\varphi} : K \to \mathbb{R}^k$ be a continuous vector function. A value $\alpha \in \mathbb{R}^k$ is a *homological critical value* of $\tilde{\varphi}$ if q exists s.t., for all sufficiently small real values $\varepsilon > 0$, two values $\alpha', \alpha'' \in \mathbb{R}^k$ can be found with $\alpha' \preceq \alpha \preceq \alpha''$, $\|\alpha' - \alpha\| < \varepsilon$, $\|\alpha'' - \alpha\| < \varepsilon$, such that the map $H_q(K_{\tilde{\varphi} \preceq \alpha'} \hookrightarrow K_{\tilde{\varphi} \preceq \alpha''})$ is not an isomorphism.

Case $\tilde{\varphi} = \overline{\varphi}$:



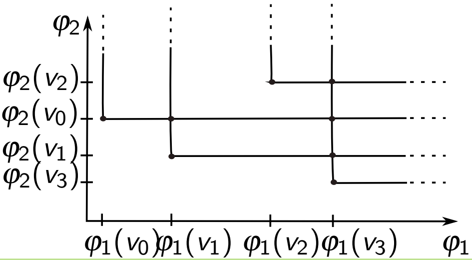
Homological critical values

Case $\tilde{\varphi} = \varphi^{\urcorner}$:

Theorem

The set of homological critical values of $\varphi^{\urcorner}$ is contained in a finite union of cones $C = \bigcup_{j,v} C_j(v)$, $v \in \mathscr{V}(\mathscr{K})$, $j = 1, 2, \dots k$, where

$$C_j(v) := \{\alpha \in \mathbb{R}^k \mid \alpha_j = \varphi^{\urcorner}{}_j(v) \text{ and } \alpha_i \geq \varphi^{\urcorner}{}_i(v) \text{ for all } i = 1, 2, \dots k\}.$$



Homological critical values

Corollary

The set of homological critical values of $\varphi^{\urcorner}$ *is a nowhere dense set in* $\mathbb{R}^k$*. Moreover its k-dimensional Lebesgue measure is zero.*

Proposition

For any $\alpha \in C = \bigcup_{j,v} C_j(v)$*, there exists* λ *in*

$$\Lambda = \{\lambda \in C \mid \forall\, j = 1, 2, \ldots k,\ \exists\, v \in \mathscr{V}(\mathscr{K}) : \lambda_j = \varphi_j(v)\}$$

such that $K_\alpha = K_\lambda$.

From continuous to discrete filtrations: the stability problem

- X and Y homeomorphic triangulable spaces (real objects);
- $f : X \to \mathbb{R}^k$,$g : Y \to \mathbb{R}^k$ continuous functions (real measurements);
- $\mathscr{K}'$ and $\mathscr{L}'$ simplicial complexes with $|\mathscr{K}'| = K$, $|\mathscr{K}'| = L$ (approximated object);
- $\tilde{\varphi} : K \to \mathbb{R}^k$, $\tilde{\psi} : L \to \mathbb{R}^k$ continuous functions (approximated measurements);

Theorem: If two homeomorphisms $\xi : K \to X$, $\zeta : L \to Y$ exist s.t.

$$\|\tilde{\varphi} - f \circ \xi\|_\infty \le \varepsilon/4, \quad \|\tilde{\psi} - g \circ \zeta\|_\infty \le \varepsilon/4$$

then, for any sufficiently fine subdivision $\mathscr{K}$ of $\mathscr{K}'$ and $\mathscr{L}$ of $\mathscr{L}'$,

$$\left|\mathrm{D}(\rho_f, \rho_g) - \mathrm{D}(\rho_\varphi, \rho_\psi)\right| \le \varepsilon,$$

$\varphi : \mathscr{V}(\mathscr{K}) \to \mathbb{R}^k$, $\psi : \mathscr{V}(\mathscr{L}) \to \mathbb{R}^k$ being restrictions of $\tilde{\varphi}$ and $\tilde{\psi}$.

Sketch of the proof

- $\exists \delta > 0$ s.t. $\max\{\mathrm{diam}\,\sigma \mid \sigma \in \mathscr{K}$ or $\sigma \in \mathscr{L}\} < \delta \implies$

$$|\mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) - \mathrm{D}(\rho_{\varphi^{\urcorner}}, \rho_{\psi^{\urcorner}})| < \varepsilon/2.$$

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- $\rho_{\varphi} = \rho_{\varphi^{\urcorner}}$, $\rho_{\psi} = \rho_{\psi^{\urcorner}}$.

$\square$

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$$|\mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) - \mathrm{D}(\rho_{\varphi}, \rho_{\psi})| < \varepsilon/2.$$
- $$\begin{aligned} \mathrm{D}(\rho_f, \rho_g) &\leq \mathrm{D}(\rho_f, \rho_{f\circ\xi}) + \mathrm{D}(\rho_{f\circ\xi}, \rho_{\tilde{\varphi}}) + \mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) \\ &+ \mathrm{D}(\rho_{\tilde{\psi}}, \rho_{g\circ\zeta}) + \mathrm{D}(\rho_{g\circ\zeta}, \rho_g) \\ &\leq \varepsilon \end{aligned}$$
.

$\square$

Conclusions

We have shown that in multidimensional persistence

- stability of rank invariants for continuous filtrations passes to stability for discrete filtrations
- two peculiar phenomena occurr:
  - topological aliasing
  - homological critical values are non-discrete