Alexander Duality for Functions

The Persistent Behavior of Land and Water and Shore

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Motivation

Let $f : \mathbb{S}^{n+1} \to [0, 1]$ be perfect Morse.

Let $\mathbb{U} \cup \mathbb{V} = \mathbb{S}^{n+1}$, $\mathbb{M} := \mathbb{U} \cap \mathbb{V}$ n-manifold.

Questions:

- Can we obtain $\mathrm{Dgm}(f|_{\mathbb{V}})$ from $\mathrm{Dgm}(f|_{\mathbb{U}})$?
- Can we obtain $\mathrm{Dgm}(f|_{\mathbb{M}})$ from $\mathrm{Dgm}(f|_{\mathbb{U}})$?

Results:

- The Land and Water Theorem:

$$\tilde{\mathrm{D}}\mathrm{gm}(f|_{\mathbb{U}}) = \tilde{\mathrm{D}}\mathrm{gm}(f|_{\mathbb{V}})^T$$

- The Euclidean Shore Theorem:

$$\mathrm{Dgm}(f|_{\mathbb{M}}) = \mathrm{Dgm}(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}(f|_{\mathbb{U}})^T$$

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Outline

- (Reduced) persistence diagrams
- The Land and Water Theorem
- The Shore Theorem

Reduced persistence diagrams

$$0 \to C_n \to \ldots \to C_1 \to C_0 \to 0$$

Let ω denote a new vertex (born at 1) and

$$\mathbb{X}_\omega^t = \mathbb{X} \cup \omega * \mathbb{X}^t.$$

Define the **reduced filtration** as

$$0 \to \tilde{X}_0 \to \ldots \to \tilde{X}_m \to \ldots \to \tilde{X}_{2m} \to 0$$

with

$$\tilde{X}_i = \begin{cases} \tilde{H}(\mathbb{X}_{t_i}) & 0 \leq i \leq m-1 \\ \tilde{H}(\mathbb{X}) & i = m \\ \tilde{H}(\mathbb{X}_\omega^{t_{m-i}}) & m+1 \leq i \leq 2m \end{cases}$$

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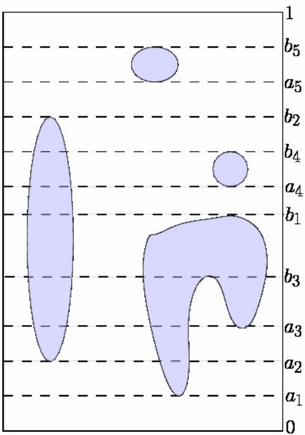
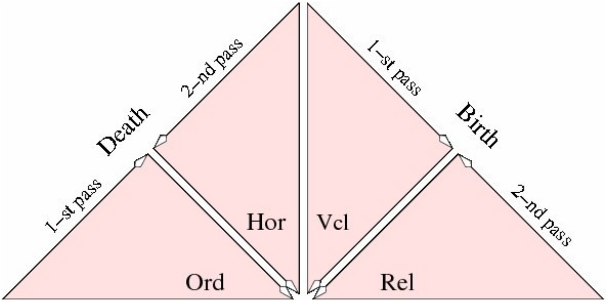
Cascades





A dot (a, b) is *left-extreme*, if there is no dot with birth value smaller a and death value larger b.

Cascade Lemma: $\tilde{\mathrm{D}}\mathrm{gm}(f|_{\mathbb{X}}) = \mathrm{Dgm}(f|_{\mathbb{X}})^C$

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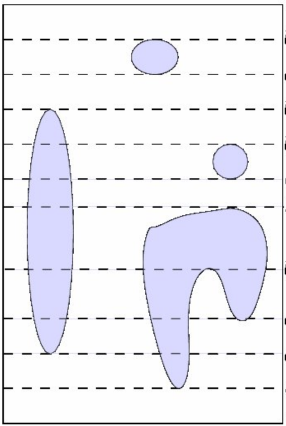

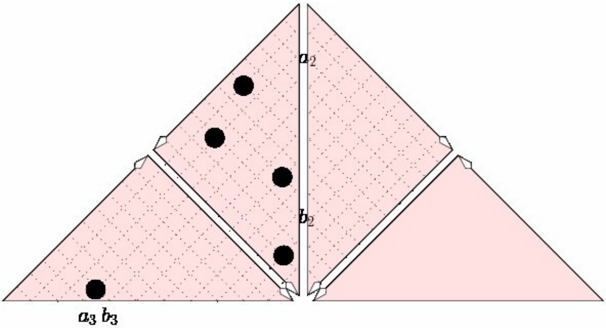

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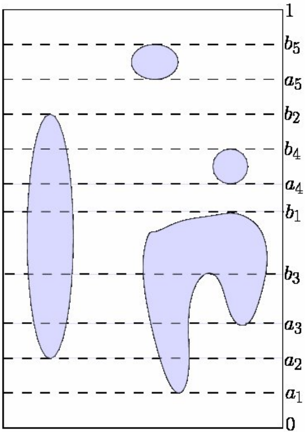

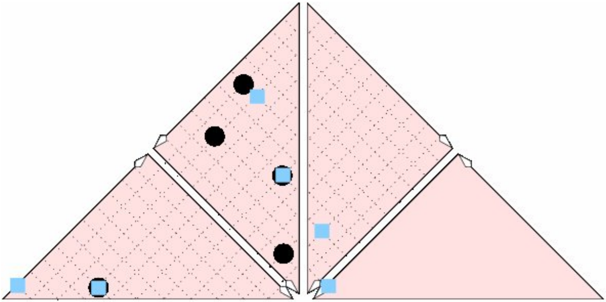

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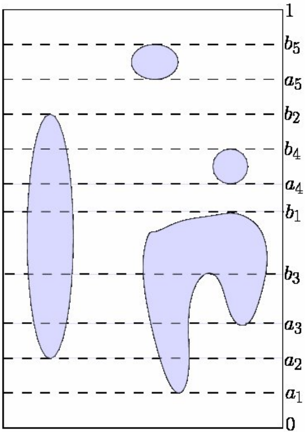

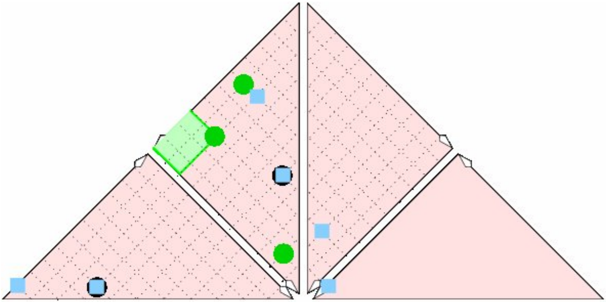

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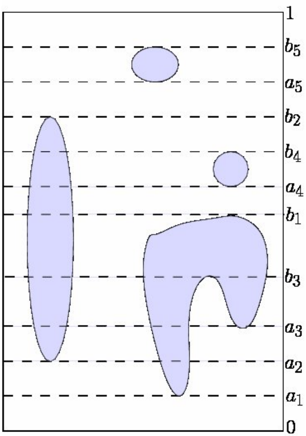

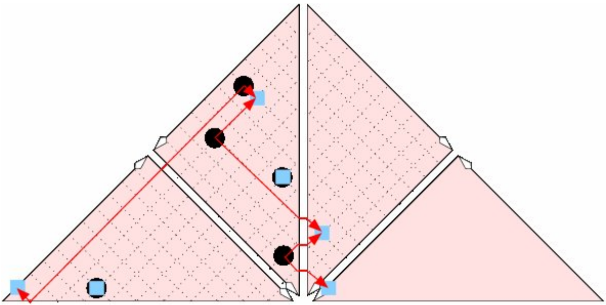

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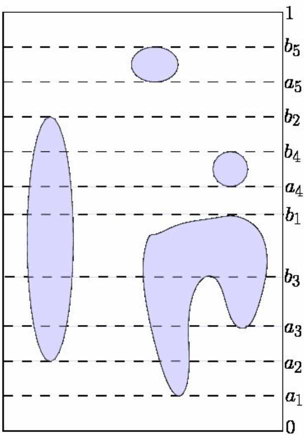

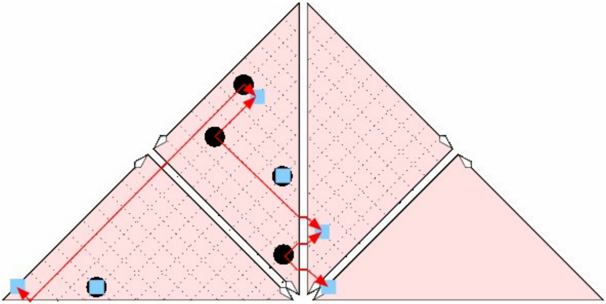

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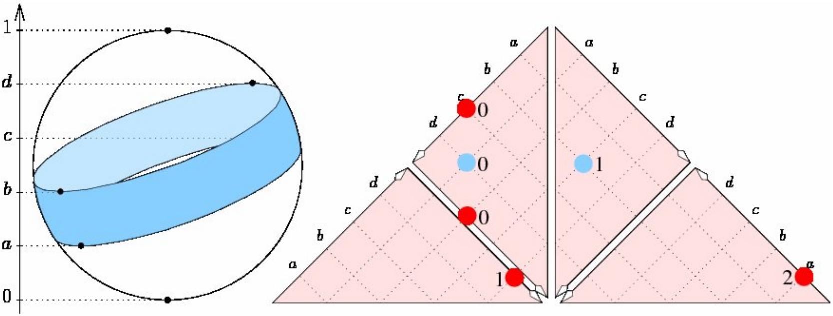

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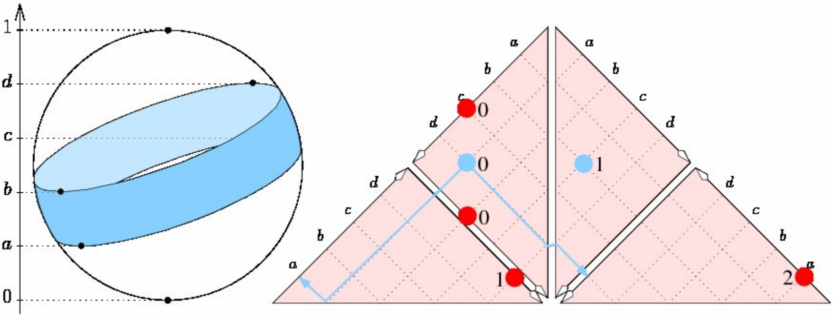

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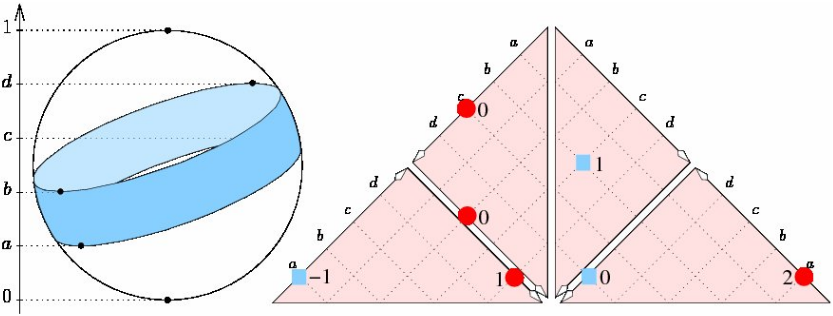

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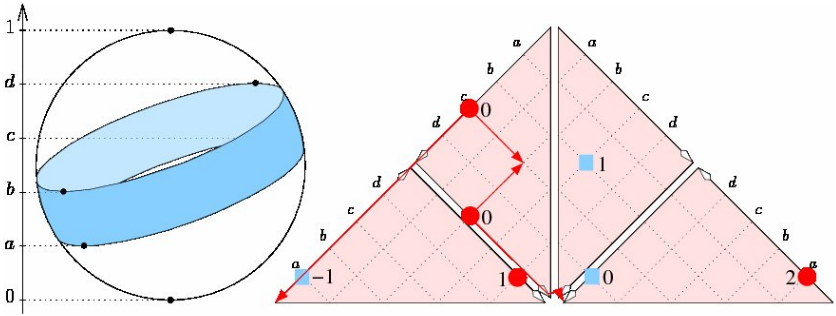

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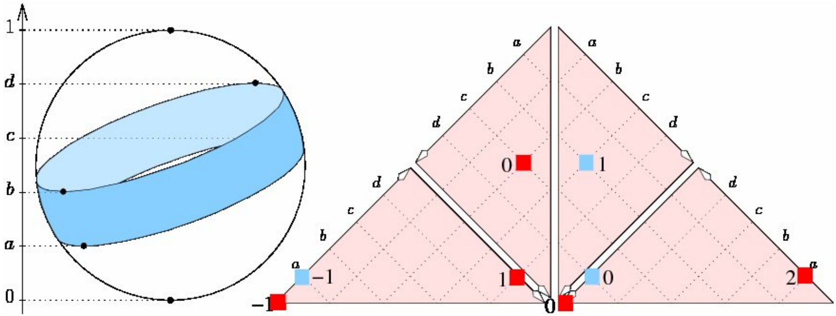

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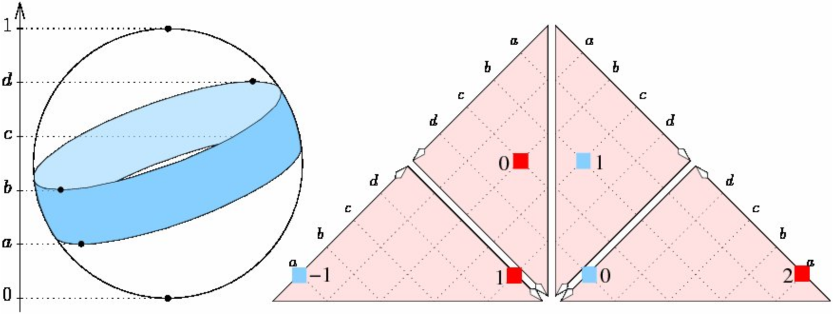

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The Land and Water Theorem – Proof outline

$$
\begin{array}{ccccccccc}
\tilde{U}_0^p & \to & \dots & \to & \tilde{U}_m^p & \to & \dots & \to & \tilde{U}_{2m}^p \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\tilde{V}_{2m}^q & \leftarrow & \dots & \leftarrow & \tilde{V}_m^q & \leftarrow & \dots & \leftarrow & \tilde{V}_0^q,
\end{array}
$$

- Vertical isomorphisms
- Need some *compatibility* condition between isomorphisms
- Implied by compatibility of vertical pairings
- Reduction to (compatible) pairing induced by Lefschetz duality [Cohen-Steiner, E., Harer 09]

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The General Shore Theorem

Let $f : \mathbb{S}^{n+1} \to [0, 1]$ be perfect Morse. Let $\mathbb{U} \cup \mathbb{V} = \mathbb{S}^{n+1}$, $\mathbb{M} := \mathbb{U} \cap \mathbb{V}$ n-manifold. Then

$$\begin{aligned} \mathrm{Dgm}_0(f|_{\mathbb{M}}) &= [\mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_0(f|_{\mathbb{V}})]^C, \\ \mathrm{Dgm}_p(f|_{\mathbb{M}}) &= \mathrm{Dgm}_p(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_p(f|_{\mathbb{V}}), \\ \mathrm{Dgm}_n(f|_{\mathbb{M}}) &= [\mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_0(f|_{\mathbb{V}})]^{CT}, \end{aligned}$$

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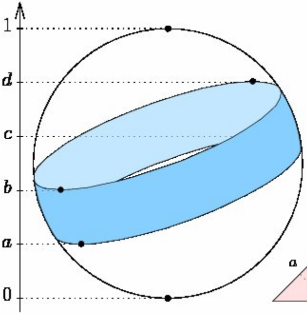

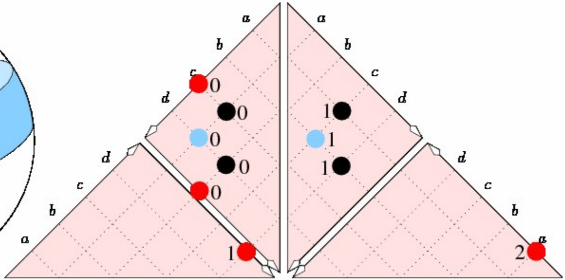

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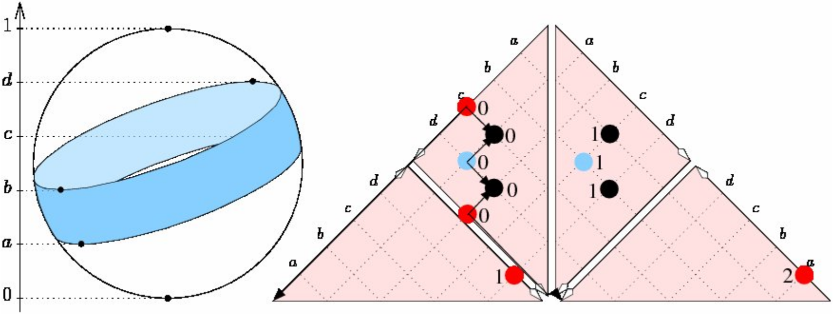
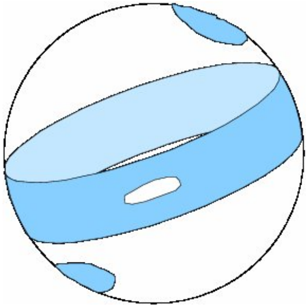


The General Shore Theorem – Proof outline

$$\mathrm{Dgm}_0(f|_{\mathbb{M}}) = [\mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_0(f|_{\mathbb{V}})]^C$$

- Latitudinal n-manifold: Component of $\mathbb{M}$ that separates the poles
- Latitudinal component: Component of $\mathbb{U}$ or of $\mathbb{V}$ neighboring a lat. n-manifold.
- Left-extreme dots in $[\mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_0(f|_{\mathbb{V}})]$ correspond to lat. components
- Show that lat. n-manifolds and lat. components are related by a cascade



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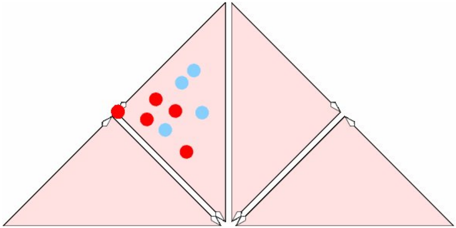

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$$\begin{aligned}
& \mathrm{Dgm}_0(f|_{\mathbb{M}}) \\
= \; & [\mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_0(f|_{\mathbb{V}})]^C \\
= \; & \mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \tilde{\mathrm{D}}\mathrm{gm}_0(f|_{\mathbb{V}}) \\
= \; & \mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \tilde{\mathrm{D}}\mathrm{gm}_n(f|_{\mathbb{U}})^T \\
= \; & \mathrm{Dgm}_0(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}_n(f|_{\mathbb{U}})^T
\end{aligned}$$

Conclusion and Outlook

We have proved relations

- between the (reduced) diagrams of $\mathbb{U}$ and $\mathbb{V}$
- between the diagrams of $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{M}$

Further questions:

- Measure “imperfection” of Morse function?
- Can we always decompose $\mathbb{S} = \mathbb{U} \cup \mathbb{V}$ as before such that

$$\mathrm{Dgm}(f|_{\mathbb{S}}) \sqcup \mathrm{Dgm}(f|_{\mathbb{M}}) \sim \mathrm{Dgm}(f|_{\mathbb{U}}) \sqcup \mathrm{Dgm}(f|_{\mathbb{V}})?$$

- Divide-and-conquer algorithm for computing $\mathrm{Dgm}(f|_{\mathbb{S}})$?
- Generalize to other $(n+1)$-manifolds

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