

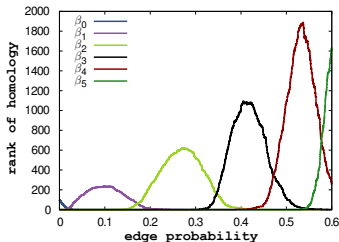
Higher-dimensional expanders

Fields Institute — Computational topology

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Collaborators and acknowledgements

Some of this is joint work with Dominic Dotterer (Toronto), and some of it with Christopher Hoffman and Elliot Paquette (Washington).

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Thanks also to Uli Wagner, Roy Meshulam, and Andrzej Żuk for helpful conversations.

Big picture of expander graphs

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They provide metric spaces that can only be embedded in Euclidean space with large distortion.

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Ramanujan graphs of Lubotzky, Philips, and Sarnak, after Deligne's proof of the Weil conjectures.

Recent work of Breuillard, Green, and Tao on Suzuki groups — finishes proof that “all finite simple groups have expanders.”

Edge expansion

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Here $e(S, \bar{S})$ is the number of edges between a set of vertices S and its complement \bar{S} , $|S|$ is the number of vertices in S , and the min is taken over all nonempty vertex subsets which are less than half the size of the graph.

Spectral expansion

Define the *Laplacian*, a linear operator on functions $f : G \rightarrow \mathbb{R}$, by

$$\mathcal{L}[f] = f - A[f],$$

where $A[f]$ is the averaging operator

$$A[f](v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(u).$$

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Let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of the Laplacian $\mathcal{L}[G]$. Then the *spectral gap* of G is the smallest positive eigenvalue $\lambda_2(G)$.

Expander families

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Inequalities due to Cheeger and Buser in the continuous case, and Tanner, Alon, and Milman in the discrete case, relate $h(G)$ and $\lambda_2(G)$.

Notions of higher-dimensional expanders

Linial and Meshulam's use of “homological expansion” to find a sharp vanishing threshold for $H_1(Y, \mathbb{Z}_2)$, where $Y \in Y(n, p)$ is a random 2-dimensional simplicial complex

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Fox, Gromov, Lafforgue, Naor, and Pach and “geometric overlap” properties of expanders, and Uli Wagner's use of coarse expansion to give non-embeddable theorems.

Gromov and Guth's recent proof that there exist isotopy classes of knots of arbitrarily large distortion, using expander-like properties of arithmetic hyperbolic manifolds.

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Ramanujan complexes — Li; Lubotzky, Samuels, and Vishne

Definitions of higher-dimensional expanders

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$$\begin{aligned} h(G) &= \min_{0 < |S| < |G|/2} \frac{e(S, \bar{S})}{|S|} \\ &= \min_{X \in C^0 - \{0\}} \frac{|\partial X|}{\|X\|} \end{aligned}$$

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Here X is understood to be a 0-cochain (with $(\mathbb{Z}/2)$ -coefficients), $|\cdot|$ denotes the Hamming norm, and $\|\cdot\|$ denotes the quotient norm — i.e. we assume that X is minimal in its coboundary class.

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This definition works equally well with other coefficients and norms. The case \mathbb{R} coefficients and L^2 norm is particularly important, e.g. when $k = 0$, this corresponds to spectral gap of a graph λ_2 .

Results — random polyhedra as expanders

Dotterrer and K. studied the asymptotics of h_k for various types of random $(k + 1)$ -dimensional complexes (including Linial-Meshulam and Meshulam-Wallach). This shows that many random polyhedra are (degree-relative) expander families.

Property (T)

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More precisely, the trivial representation is an isolated point in its unitary dual equipped with the Fell topology.

Theorem

(Žuk) If Δ is a pure 2-dimensional simplicial complex, such that every vertex link is connected and has $\lambda_2 > 1/2$, then $\pi_1(\Delta)$ has property (T).

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(Ballman–Swiatkowski) If Δ is a pure d -dimensional simplicial complex, such that the link of every $(d - 2)$ -face is connected and has $\lambda_2 > (d - 1)/d$, then $H^{d-1}(\Delta, \mathbb{R}) = 0$.

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Theorems

Theorem

(Linial–Meshulam) Let $\omega \rightarrow \infty$ as $n \rightarrow \infty$. If

$$p \leq \frac{2 \log n - \omega}{n}$$

then a.a.s. $H_1(Y, \mathbb{Z}_2) \neq 0$, and if

$$p \geq \frac{2 \log n + \omega}{n}$$

then a.a.s. $H_1(Y, \mathbb{Z}_2) = 0$.

Theorems

Theorem

(Hoffman–K.–Paquette) If

$$p \leq \frac{2 \log n - \omega}{n}$$

then a.a.s. $\pi_1(Y)$ does not have property (T), and if

$$p \geq \frac{2 \log n + \omega \sqrt{\log n \log \log n}}{n}$$

then a.a.s. $\pi_1(Y)$ has property (T) .

We show that every vertex link has a large spectral gap, and apply Żuk's theorem.

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This work requires new estimates and concentration results on the spectral gap of Erdős-Rényi random graphs.

Spectral gap concentration

Theorem

(Hoffman – K. – Paquette) Fix $k \geq 0$ and $\epsilon > 0$, and let $G \in G(n, p)$. Let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of the normalized Laplacian of G . There is a constant $C = C(k)$ so that when

$$p \geq \frac{(k+1) \log n + C\sqrt{\log n \log \log n}}{n}$$

is satisfied, then

$$\lambda_2 > 1 - \epsilon,$$

with probability at least $1 - o(n^{-k})$.

Results — random flag complexes

Theorem

(K.) If

$$p \geq \left(\frac{(1 + k/2) \log n + \omega \sqrt{\log n}}{n} \right)^{1/(k+1)}$$

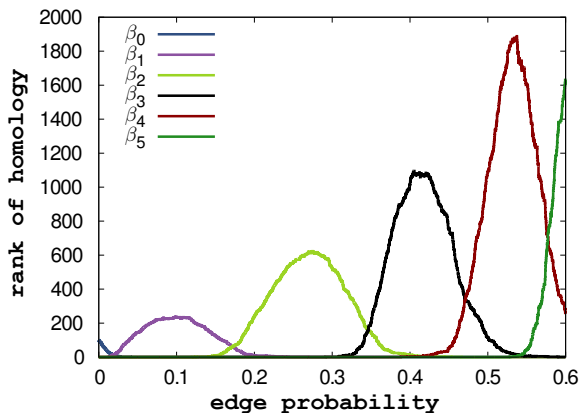
or

$$p = o\left(\frac{1}{n^{1/k}}\right)$$

then a.a.s. $H_k(X, \mathbb{R}) = 0$, and if

$$\left(\frac{\omega}{n}\right)^{1/k} \leq p \leq \left(\frac{(1 + k/2) \log n - \omega \log \log n}{n} \right)^{1/(k+1)}$$

then a.a.s. $H_k(X, \mathbb{R}) \neq 0$.



A random flag complex on $n = 100$ vertices.
(Computation and image courtesy of Afra Zomorodian.)

Open problems

Higher dimensional analogues of Cheeger and Buser inequalities.

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Define higher-dimensional analogues of k -regular graphs.

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Define higher-dimensional analogues of k -regular graphs.

Describe the “evolution” of the random fundamental group $\pi_1(Y)$ from free group to property (T) group.

Proposed applications

Spectral sparsification of simplicial complexes, as in Batson, Spielman, and Srivastava's work on graph sparsification.

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Fast mixing of Markov chains on k -cochains.