

Computing Cohomology Ring

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joint work with

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Background from *Cubical Homology*

A *cubical set* $X \subset \mathbb{R}^d$ is a finite union of *elementary cubes*

$$Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d,$$

I_i of the form $[k, k + 1]$ or $\{k\}$.

$\mathcal{K}(\mathbb{R}^d)$ all such Q

$\mathcal{K}_k(\mathbb{R}^d)$ those of dimension k

$\mathcal{K}_k(X)$ those in X

$C_k(X)$ free abelian group generated by

$\mathcal{K}_k(X)$, its canonical basis

$$\partial_k(Q) = \sum \pm(k-1)\text{-dim faces of } Q$$

cubical boundary operator

Alternation defined in [CH] by induction on $d = \text{emb } X$

Cubical cochain complex

$\mathcal{C}(X)$ *cubical chain complex*

Note: All is valid for $R(\mathcal{K}(X)) := \mathcal{C}(X; R)$ graded module with coefficients in a ring with unity R .

$(\mathcal{C}^*(X), \delta)$ the *dual complex* of $(\mathcal{C}_*(X), \partial)$

$\delta^k : \mathcal{C}^k(X) \rightarrow \mathcal{C}^{k+1}(X)$ the *coboundary operator*

$$\langle \delta^k c^k, c_{k+1} \rangle := \langle c^k, \partial_{k+1} c_{k+1} \rangle$$

$\mathcal{K}^k(X) := \{Q^* | Q \in \mathcal{K}_k(X)\}$ the dual canonical basis of $\mathcal{C}^k(\mathbb{R}^d)$

$$H^k(X) := Z^k(X)/B^k(X) = \ker \delta^k / \operatorname{im} \delta^{k-1}$$

the k 'th *cohomology group* of X

Cubical cross product

Recall from *CH*: the *cubical product*

$$\times : C_p(\mathbb{R}^n) \times C_q(\mathbb{R}^m) \rightarrow C_{p+q}(\mathbb{R}^{n+m})$$

is defined on $P \in \mathcal{K}_p^n$ and $Q \in \mathcal{K}_q^m$ as the cartesian product $P \times Q$ and extended to chains (c, c') by bilinearity.

The *cubical cross product* of cochains

$c^p \in C^p(X)$ and $c^q \in C^q(Y)$

is a cochain in $C^{p+q}(X \times Y)$

defined on $R \times S \in \mathcal{K}_{p+q}(X \times Y)$ by

$$\langle c^p \times c^q, Q \rangle := \begin{cases} \langle c^p, R \rangle \cdot \langle c^q, S \rangle & \text{if } \dim R = p \text{ and } \dim S = q, \\ 0 & \text{otherwise.} \end{cases}$$

Cup product – general definition

The *cubical cup product*

$$\smile : C^p(X) \times C^q(X) \rightarrow C^{p+q}(X)$$

of cochains c^p and c^q is defined on $Q \in \mathcal{K}_{p+q}(X)$ by:

$$\langle (c^p \smile c^q), Q \rangle := \langle \text{diag}^{p+q}(c^p \times c^q), Q \rangle = \langle c^p \times c^q, \text{diag}_{p+q}(Q) \rangle$$

where diag_{p+q} is the *homology chain map* induced by $\text{diag}(x) = (x, x)$.

The *cup product* $\smile : H^p(X) \times H^q(X) \rightarrow H^{p+q}(X)$ is defined by

$$[z^p] \smile [z^q] := [z^p \smile z^q].$$

Our goal: Obtain an explicit formula suitable for computations.

Chain map of the diagonal map

Theorem (Chain selector CH)

Given $f : X \rightarrow Y$, let $F : X \rightrightarrows Y$,

$$F(x) := \text{ch}(f(\text{ch}(x))),$$

where $\text{ch}(A)$ is the smallest cubical set containing A .

Suppose F acyclic-valued. Then \exists a chain map

$\varphi : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ such that

(a) $|\varphi_k(Q)| \subset (F(\overset{\circ}{Q})) \forall_{Q \in \mathcal{K}_k(X)}$;

(b) $\varphi_0(V) \in \mathcal{K}_0 \forall_{Q \in \mathcal{K}_0(X)}$.

For any chain map satisfying (a,b), we have

$$\varphi_* = H_*(f).$$

Lemma (1)

Let X_1, X_2, Y_1, Y_2 be cubical sets,
 $f : X_1 \rightarrow Y_1, g : X_2 \rightarrow Y_2$ maps which admit acyclic-valued
representations F, G, φ and ψ their chain selectors.

- ① The set-valued map $F \times G : X_1 \times X_2 \rightrightarrows Y_1 \times Y_2$ given by

$$(F \times G)(x, y) := F(x) \times G(y)$$

is an acyclic-valued representation of $f \times g$.

- ② The chain map $\varphi \otimes \psi : \mathcal{C}(X_1 \times X_2) \rightarrow \mathcal{C}(Y_1 \times Y_2)$ given on
generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X_1 \times X_2)$ by

$$(\varphi \otimes \psi)_k(Q) := \varphi_p(Q_1) \times \psi_q(Q_2)$$

is a chain selector of $F \times G$.

Theorem (2)

Let X, Y be cubical sets and let $\lambda : X \times Y \rightarrow Y \times X$ be the transpose given by $\lambda(x, y) := (y, x)$.

- ① The map $\Lambda : X \times Y \rightrightarrows Y \times X$ given by

$$\Lambda(x, y) := Q_2 \times Q_1, \quad Q_1 := \text{ch}(x), Q_2 := \text{ch}(y),$$

is an acyclic-valued representation of λ ;

- ② Let $\lambda_{\#} : \mathcal{C}(X \times Y) \rightarrow \mathcal{C}(Y \times X)$ be defined on generators $Q = Q_1 \times Q_2 \in \mathcal{K}_k(X \times Y)$ by

$$\lambda_k(Q) := (-1)^{\dim Q_1 \dim Q_2} Q_2 \times Q_1.$$

Then $\lambda_{\#}$ is a chain selector of Λ .

Constructing $\varphi = \text{diag}_\#$

Proposition (3)

The map $\text{diag} : X \rightarrow X \times X$ admits an acyclic-valued representation $\text{Diag} : X \rightrightarrows (X \times X)$ given by $\text{Diag}(x) := Q \times Q$ where $Q = \text{ch}(x) \in \mathcal{K}(X)$.

We go by induction on $d = \text{emb}(X)$.

Case $d = 1$:

$$Q = [v] \Rightarrow \text{diag}_0([v]) := [v] \times [v]$$

$$Q = [v_0, v_1] \Rightarrow \text{diag}_1([v_0, v_1]) := [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1]$$

Induction step:

$$\begin{aligned}\text{diag}(x_1, \dots, x_d) &= (x_1, \dots, x_d, x_1, x_2, \dots, x_d) \\ &= \tau(x_1, x_1, x_2, \dots, x_d, x_2, \dots, x_d) \\ &= \tau(\text{diag}(x_1), \text{diag}(x_2, \dots, x_d))\end{aligned}$$

We get

$$\text{diag} = \tau \circ (\text{diag}^{X_1} \times \text{diag}^{X'}) \circ j,$$

where τ is the permutation,

$$j: X \hookrightarrow X_1 \times X_2 \subset \mathbb{R} \times \mathbb{R}^{d-1}$$

the inclusion of X to the product of its projections.

Theorem (4)

Let $\text{emb } X > 1$. Define $\text{diag}_{\#} : \mathcal{C}(X) \rightarrow \mathcal{C}(X \times X)$ by

$$\text{diag}_{\#} := \pi \circ \tau_{\#} \circ (\text{diag}_{\#}^{X_1} \otimes \text{diag}_{\#}^{X'}) \circ \iota,$$

where

- 1 $\iota : \mathcal{C}(X) \rightarrow \mathcal{C}(X_1 \times X')$ is the inclusion map;
- 2 $\text{diag}_{\#}^{X_1}$ and $\text{diag}_{\#}^{X'}$ are defined by induction hypothesis;
- 3 $\text{diag}_{\#}^{X_1} \otimes \text{diag}_{\#}^{X'}$ is given by Lemma 1;
- 4 $\tau_{\#}$ is given by Theorem 3;
- 5 $\pi : \mathcal{C}(X_1 \times X' \times X_1 \times X') \rightarrow \mathcal{C}(X \times X)$ is the projection

$$\pi(Q) := \begin{cases} Q & \text{if } Q \in \mathcal{K}(X \times X), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\text{diag}_{\#}$ is a chain selector for Diag .

Explicit cup product formula

Induction on $d = \text{emb}(X)$. **Case $d = 1$:**

$k = p + q = 0$, $R = [v]$. Then

$$\begin{aligned}\langle P^* \smile Q^*, R \rangle &= \langle P^* \times Q^*, [v] \times [v] \rangle \\ &= \begin{cases} 1 & \text{if } P = Q = [v], \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

$k = p + q = 1$, $R = [v_0, v_1]$. Then

$$\begin{aligned}\langle P^* \smile Q^*, R \rangle &= \langle P^* \times Q^*, [v_0] \times [v_0, v_1] + [v_0, v_1] \times [v_1] \rangle \\ &= \begin{cases} 1 & \text{if } P = [v_0] \text{ and } Q = [v_0, v_1], \\ 1 & \text{if } P = [v_0, v_1] \text{ and } Q = [v_1], \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Induction step

Theorem (5)

Let $\text{emb } X = d > 1$, and suppose that the formula for \smile is given for cochains with $\text{emb} = 1, \dots, d-1$.

Let $P = P_1 \times P_2 \in \mathcal{K}_p(X)$ and $Q = Q_1 \times Q_2 \in \mathcal{K}_q(X)$ with $\text{emb } P_1 = \text{emb } Q_1 = 1$ and $\text{emb } P_2 = \text{emb } Q_2 = d-1$.

Let $x = P_1^* \smile Q_1^*$, $y = P_2^* \smile Q_2^*$ be computed using induction. Then

$$P^* \smile Q^* = \begin{cases} (-1)^{\dim P_2 \dim Q_1} x \times y & \text{if } |x \times y| \in \mathcal{K}(X), \\ 0 & \text{otherwise.} \end{cases}$$

Example Let $X = [0, 1]^2$, $P = [0] \times [0, 1]$, and $Q = [0, 1] \times [0]$.

$$P^* \smile Q^* = (-1)^{1 \cdot 1} ([0]^* \smile [0, 1]^*) \times ([0, 1]^* \smile [0]^*) = -[0, 1]^2{}^*.$$

However, if $X = P \cup Q$, we get $P^* \smile Q^* = 0$.

Coordinate-wise formula

Let

$$P = I_1 \times I_2 \times \cdots \times I_d \text{ and } Q = J_1 \times J_2 \times \cdots \times J_d.$$

Let $P'_j := I_{j+1} \times I_{j+2} \times \cdots \times I_d$. Put

$$\operatorname{sgn}(P, Q) := (-1)^{\sum_{j=1}^d \dim P'_j \dim J_j}.$$

Corollary (6)

With the above notation,

$$P^* \smile Q^* = \operatorname{sgn}(P, Q)(I_1^* \smile J_1^*) \times (I_2^* \smile J_2^*) \times \cdots \times (I_d^* \smile J_d^*),$$

*provided the right-hand side is supported in X ,
and $P^* \smile Q^* = 0$ otherwise.*

Cohomology of S-complexes

S-complex defined by Mrozek *et. al.* (S, κ) is a combinatorial setting for a CW-complex with a chosen canonical basis $S = \{S_k\}$ and given *incidence numbers* κ in a ring of coefficients R such that

$\kappa(a, b) \neq 0$, $a \in S_k$, then b is a $(k - 1)$ -face of a .

Particular cases: Any simplicial complex S , $S = \mathcal{K}(X)$, X a cubical set.

This gives rise to the chain complex $R(S)$, its *dual* $R^*(S) := \text{Hom}(R(S), R)$, the *coboundary map* $\delta^\kappa := (\partial^\kappa)^*$,

$$\delta^\kappa(t^*) := \sum_{s \in S} \kappa(s, t) s^* \quad t \in S.$$

and to the *cohomology* $H^*(S)$.

Reduction and coreduction pairs

Smith normal form algorithm is costly! Our goal: We want a low-cost method for removing as many generators as possible before applying it.

An S -subcomplex S' is *closed* if $bd_S S' \subset S'$ and it is *open* if $S \setminus S'$ is closed.

Consider a pair $(a, b) \in S \times S$ with invertible $\kappa(Q, P)$.

It is a *reduction pair* if $cbd_S(b) = \{a\}$.

It is a *coreduction pair* if $bd_S(a) = \{b\}$.

Theorem (7)

A reduction pair (a, b) is open in S and a coreduction pair is closed in S . In both cases $\{a, b\}$ and $\bar{S} := S \setminus \{a, b\}$ are S -subcomplexes of S , and $H(\{a, b\}) = H^(\{a, b\}) = 0$.*

Consequently,

$$H(S) \cong H(\bar{S}) \quad \text{and} \quad H^*(S) \cong H^*(\bar{S}).$$

Homology Models

Attention: To compute the cohomology ring of a cubical set it is not sufficient to have the cohomology generators in a reduced S-complex.

It is necessary to construct the cohomology generators in the original cubical set.

Theorem (8)

Let (a, b) be a reduction or coreduction pair in S . The maps $\psi = \psi^{(a,b)} : R(S) \rightarrow R(\bar{S}), \iota = \iota^{(a,b)} : R(\bar{S}) \rightarrow R(S)$,

$$\psi(c) = c - \frac{\langle c, a \rangle}{\langle \partial b, a \rangle} \partial b - \langle c, b \rangle b, \quad (1)$$

$$\iota(c) = c - \frac{\langle \partial c, a \rangle}{\langle \partial b, a \rangle} b \quad (2)$$

are mutually inverse chain equivalences.

Reduction sequence

is a finite sequence

$$\omega = \{(a_i, b_i)_{i=1,2,\dots,n}\} \in \mathcal{S}$$

such that (a_i, b_i) is a reduction or coreduction pair
in (S^{i-1}, κ^{i-1}) ,

starting at $S^0 = S$ and ending at $S^\omega := S^n$.

A *homology model of S* is S^ω together with the chain
equivalences:

$$\begin{aligned}\iota^\omega &= \iota^{(a_1, b_1)} \circ \dots \circ \iota^{(a_n, b_n)} : R(S^\omega) \rightarrow R(S), \\ \psi^\omega &= \psi^{(a_n, b_n)} \circ \dots \circ \psi^{(a_1, b_1)} : R(S) \rightarrow R(S^\omega).\end{aligned}$$

Dual chain equivalences

Idea: Use $(\iota^\omega)^\star$ to transport cochains on S to those in S^ω and $(\psi^\omega)^\star$ the reverse way.

Proposition (9)

The duals of ψ and ι are given by

$$\begin{aligned}\psi^\star : R^\star(\bar{S}) \ni c &\mapsto c - \frac{\langle b^\star, \delta c \rangle}{\langle \partial b, a \rangle} a^\star \in R^\star(S), \\ \iota^\star : R^\star(S) \ni c &\mapsto c - \frac{\langle b^\star, c \rangle}{\langle \partial b, a \rangle} \delta a^\star \in R^\star(\bar{S}).\end{aligned}$$

Using models for cohomology

Note: There are even more benefits from the homology model for cohomology computation than for homology computation!

Theorem (10)

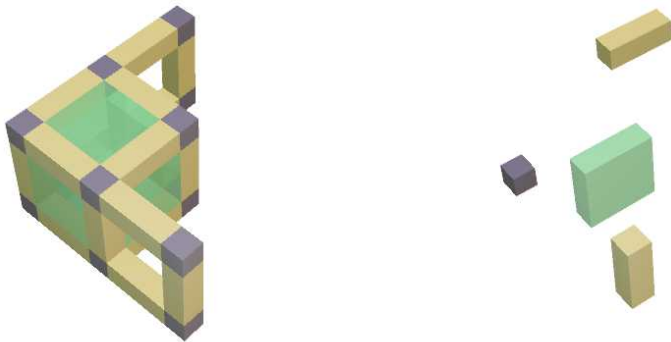
Let ω be a sequence consisting only of elementary coreduction pairs. Then $(\psi^\omega)^$ is an inclusion*

$$R^*(S^\omega) \hookrightarrow R^*(S),$$

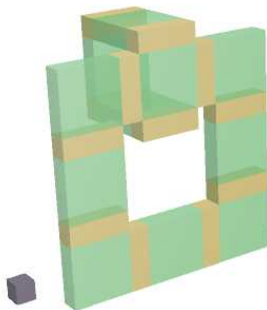
Consequence: For computing the ring structure of a cubical set X , when $S = \mathcal{K}(X)$, the cup product formula may be applied directly to the cohomology generators in the ω -reduction.

Wedge X and torus T

$X = S^2 \vee S^1 \vee S^1$. $H(X) \cong H(T)$, but $H^*(X) \not\cong H^*(T)$ as rings.

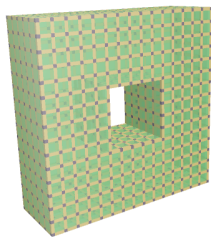
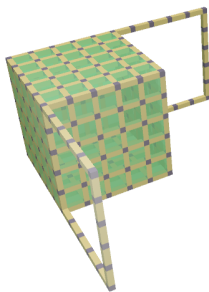


Cubical model for X . **Left:** original, **Right:** coreduced.
Cup product of H^1 -generators computed easily in the coreduced complex is zero.



Cubical model for T . **Left:** original, **Right:** coreduced.
Cup product of H^1 -generators computed easily in the coreduced complex is a H^2 -generator.

Efficiency test on rescaled X and T



Past and future work

Origins of (singular) cubical approach:

J.P. Serre, *Homologie singulière des espaces fibrés*, Annals Math. (1951).

More recent computation oriented work:

R. González-Díaz and P. Real, *Computation of cohomology operations on finite simplicial complexes*, Homology, Homotopy & Appl. (2003).

T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational Homology*, Springer 2004.

M. Mrozek and B. Batko, *Coreduction homology algorithm*, Discrete & Comput. Geom. (2009).

Related program libraries [CHomP](#), [CAPD-RedHom](#).

Future work:

Implementation and experimentation, joint work with P. Dłotko and M. Mrozek.