

Approximation algorithm for the multidimensional matching distance

S. Biasotti¹ A. Cerri^{2,3} Patrizio Frosini³ D. Giorgi¹

¹CNR – IMATI, Genova, Italy

²ARCES – University of Bologna, Italy

³PRIP – Vienna University of Technology, Austria



Workshop on Computational Topology
Fields Institute – 11.10.2011

1 Our approach to Shape Comparison

2 Persistence

3 Multidimensional persistent Betti numbers (rank invariant)

- Algorithm
- Experimental results

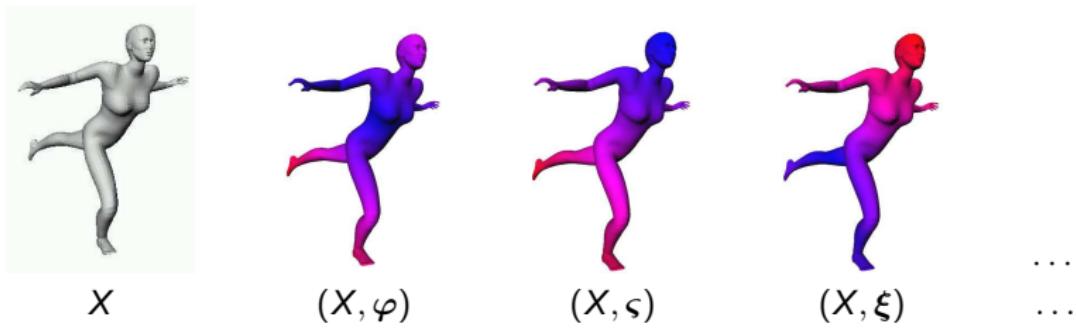
4 Conclusions

Our goal: Define a metric of (dis)similarity on a given database of objects.

- We think that comparing objects shapes should be done with respect to some “**relevant properties**” of objects themselves;

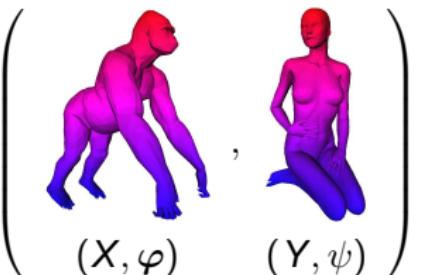
Our goal: Define a metric of (dis)similarity on a given database of objects.

- We think that comparing objects shapes should be done with respect to some “relevant properties” of objects themselves;
- To model a shape we consider pairs (X, φ) s.t.
 - X represents the object;
 - $\varphi : X \rightarrow \mathbb{R}^k$ is a function, called a **filtering (or measuring) function** and describing the **relevant properties**.

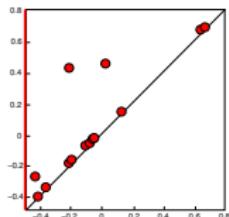
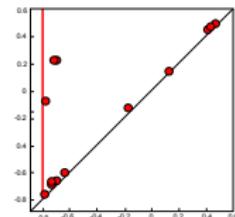


Comparing shapes

- How can we compare two pairs $(X, \varphi), (Y, \psi)$?

$$d \left(\begin{array}{c} (X, \varphi) \\ (Y, \psi) \end{array} \right) = ?$$


- Persistence** allows us to describe such a pair by means of suitable shape descriptors (**persistent Betti Numbers**).



- Instead of comparing shapes, we can compare shape descriptors.

1 Our approach to Shape Comparison

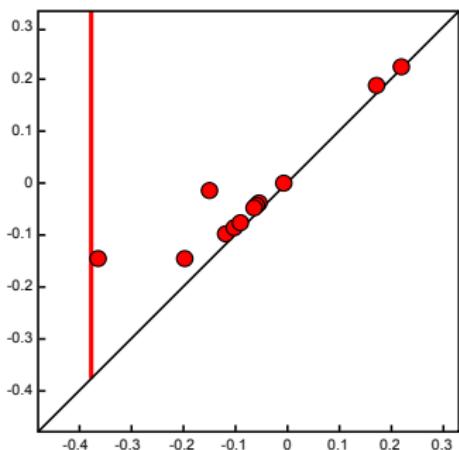
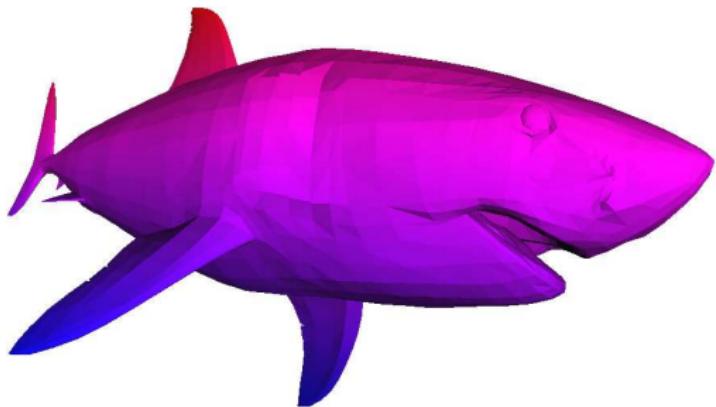
2 Persistence

3 Multidimensional persistent Betti numbers (rank invariant)

- Algorithm
- Experimental results

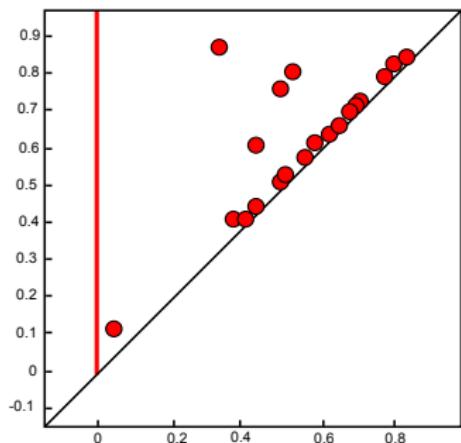
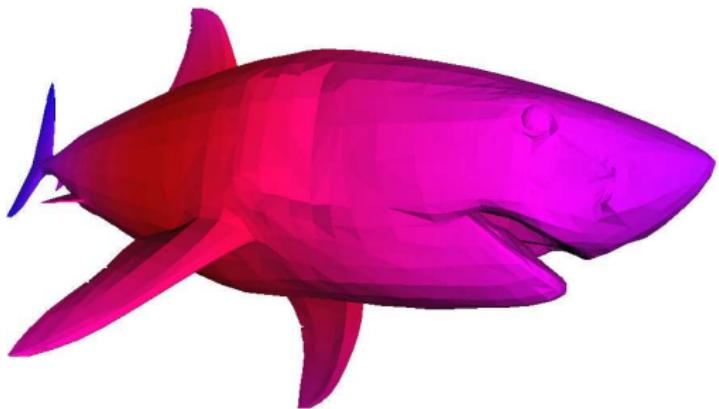
4 Conclusions

Invariance properties inherited from the filtering functions.



Persistence diagrams – Modularity

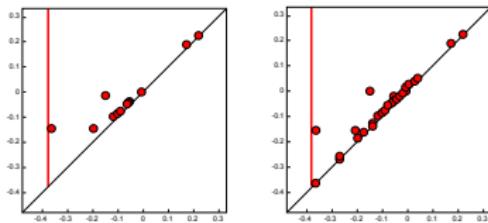
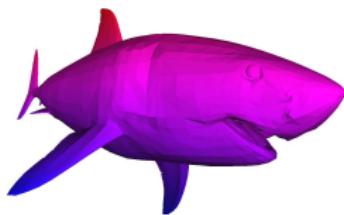
Changing filtering function produces different information.



Persistence diagrams – Stability

Stability with respect to the **bottleneck (or matching) distance**:

$$\text{If } \|f - g\| \leq \varepsilon, \text{ then } d_B(D_*(f), D_*(g)) \leq \varepsilon.^a$$



This implies **resistance to noise**.

^aCohen-Steiner et al. (2005), Chazal et al. (2009), d'Amico et al. (2010)...

1 Our approach to Shape Comparison

2 Persistence

3 Multidimensional persistent Betti numbers (rank invariant)

- Algorithm
- Experimental results

4 Conclusions

Why \mathbb{R}^k -valued filtering function?

- To consider several properties **at the same time**;

Why \mathbb{R}^k -valued filtering function?

- To consider several properties **at the same time**;
- To consider **multidimensional properties** of shapes;

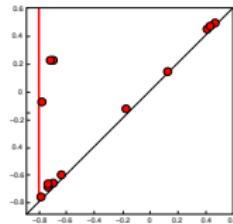
Why \mathbb{R}^k -valued filtering function?

- To consider several properties **at the same time**;
- To consider **multidimensional properties** of shapes;
- k -dimensional persistent Betti numbers (rank invariant¹):

¹Carlsson & Zomorodian (2007)

Why \mathbb{R}^k -valued filtering function?

- To consider several properties **at the same time**;
- To consider **multidimensional properties** of shapes;
- k -dimensional persistent Betti numbers (rank invariant¹):
 - ⌚ no complete and discrete representation is available;



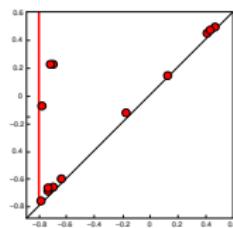
$k > 1$ 



¹Carlsson & Zomorodian (2007)

Why \mathbb{R}^k -valued filtering function?

- To consider several properties **at the same time**;
- To consider **multidimensional properties** of shapes;
- k -dimensional persistent Betti numbers (rank invariant¹):
 - ⌚ no complete and discrete representation is available;



$k > 1$ 

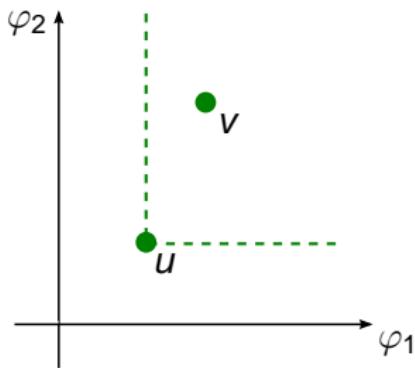


- ⌚ stable w.r.t the k -dimensional matching distance.

¹Carlsson & Zomorodian (2007)

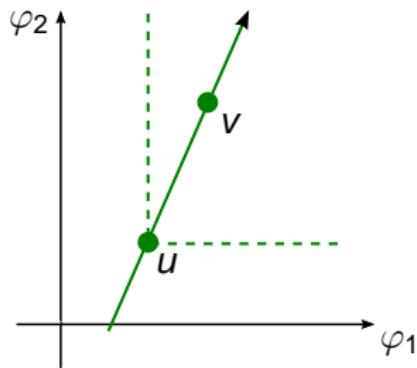
k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



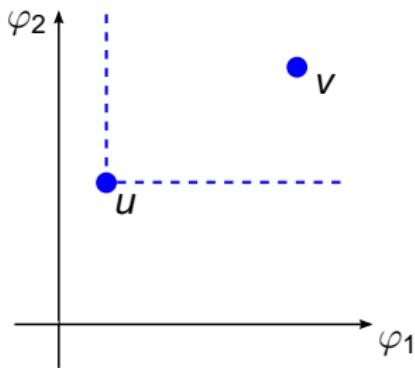
k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



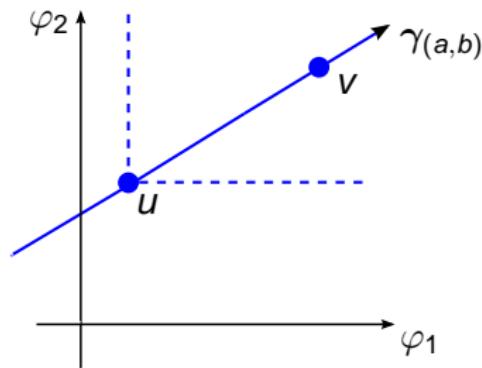
k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



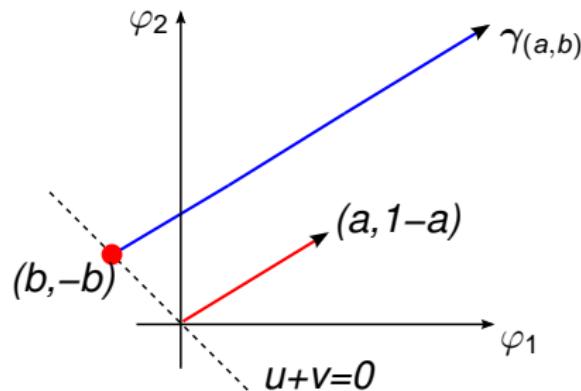
k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



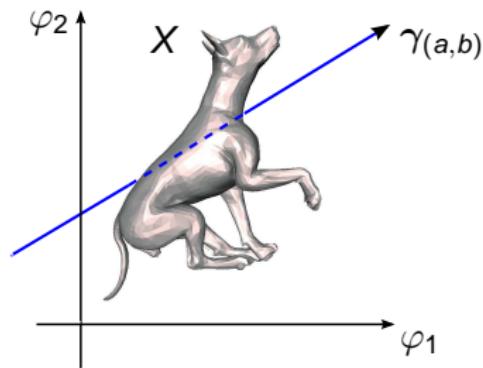
k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. **Example:** $k = 2$.



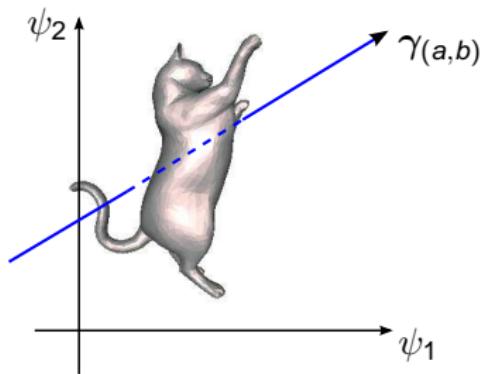
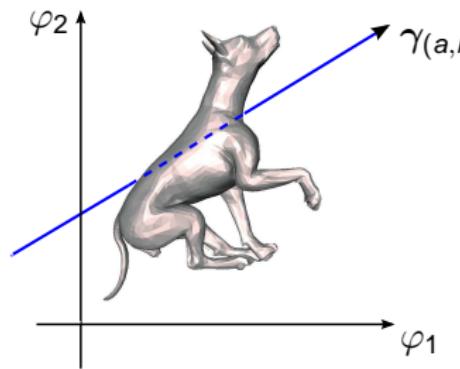
Each line $\gamma_{(a,b)}$ can be associated to a **1-filtration** w. r. t. $\varphi_{(a,b)} : X \rightarrow \mathbb{R}$

Reduction Theorem

The collection $\{D(\varphi_{(a,b)})\}$ completely determines ρ_φ ;

k -dimensional matching distance

$\varphi = (\varphi_1, \varphi_2) : X \rightarrow \mathbb{R}^k$, then ρ_φ lives in $\mathbb{R}^k \times \mathbb{R}^k$. Example: $k = 2$.



Each line $\gamma_{(a,b)}$ can be associated to a **1-filtration** w. r. t. $\varphi_{(a,b)} : X \rightarrow \mathbb{R}$

Reduction Theorem

The collection $\{D(\varphi_{(a,b)})\}$ completely determines ρ_φ ;

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

Properties:

- $d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)}))$ is stable w.r.t. to small changing in (a, b) ;

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

Properties:

- $d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)}))$ is stable w.r.t. to small changing in (a, b) ;

Multidimensional stability theorem

If X is triangulable and $\|\varphi - \psi\|_\infty \leq \varepsilon$, then $D_{\text{match}}(\rho_\varphi, \rho_\psi) \leq \varepsilon$.

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

• Problem:

- Approximations of D_{match} are needed ($\sup \rightarrow \max$);

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

• Problem:

- Approximations of D_{match} are needed ($\sup \rightarrow \max$);
- How can we choose a suitable, finite subset of $\gamma_{(a,b)}$?

Definition

Let be $\varphi, \psi : X \rightarrow \mathbb{R}^k$. The k -dimensional matching distance is

$$D_{\text{match}}(\rho_\varphi, \rho_\psi) = \sup_{\gamma_{(a,b)}} d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})).$$

• Problem:

- Approximations of D_{match} are needed ($\sup \rightarrow \max$);
- How can we choose a suitable, finite subset of $\gamma_{(a,b)}$?

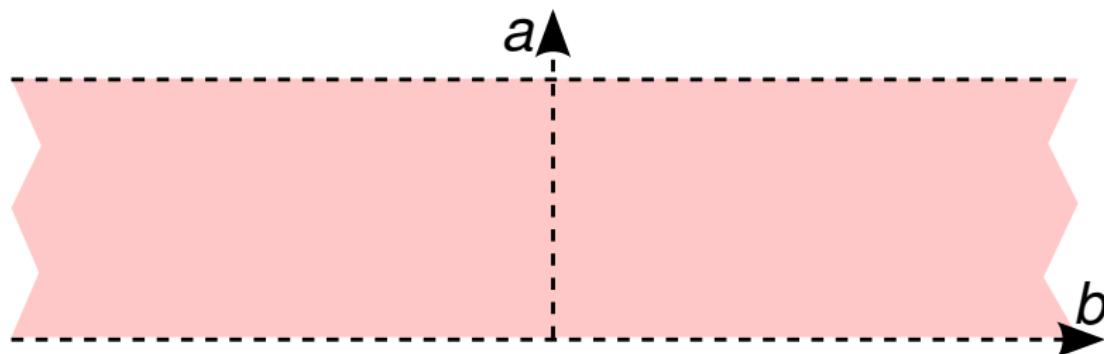
• We propose an algorithm:

Input: An arbitrary error threshold;

Output: Approximation of D_{match} up to the threshold;

The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

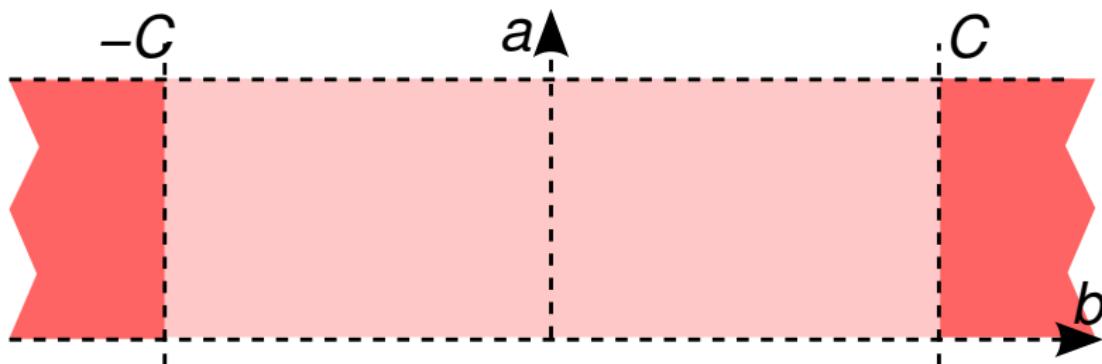


The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

Proposition 1 ($b \geq |C|$)

$$b \leq -C \Rightarrow d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})) \leq d_B(D(\varphi_{(0.5,b)}), D(\psi_{(0.5,b)})) = d_B(D(\varphi_1), D(\psi_1));$$
$$b \geq C \Rightarrow d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})) \leq d_B(D(\varphi_{(0.5,b)}), D(\psi_{(0.5,b)})) = d_B(D(\varphi_2), D(\psi_2)).$$



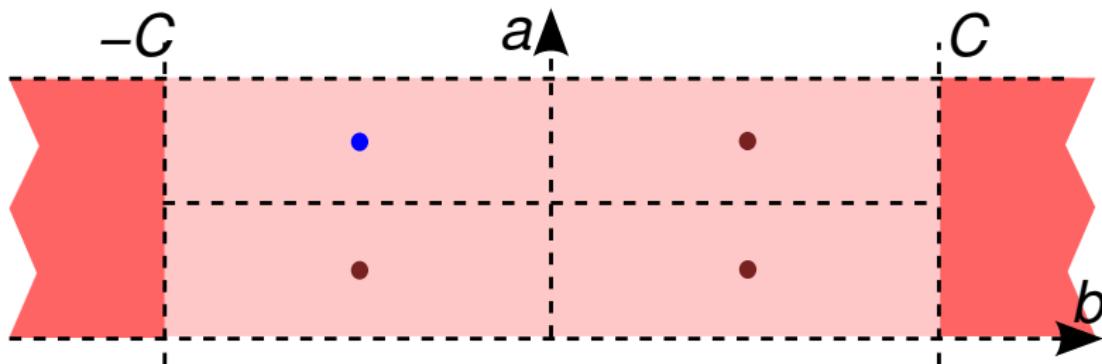
The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

Proposition 2 ($b < |C|$)

If $|a - a'| < \delta$ and $|b - b'| < \delta C$, then

$$|d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})) - d_B(D(\varphi_{(a',b')}), D(\psi_{(a',b')}))| \leq 18\delta C.$$



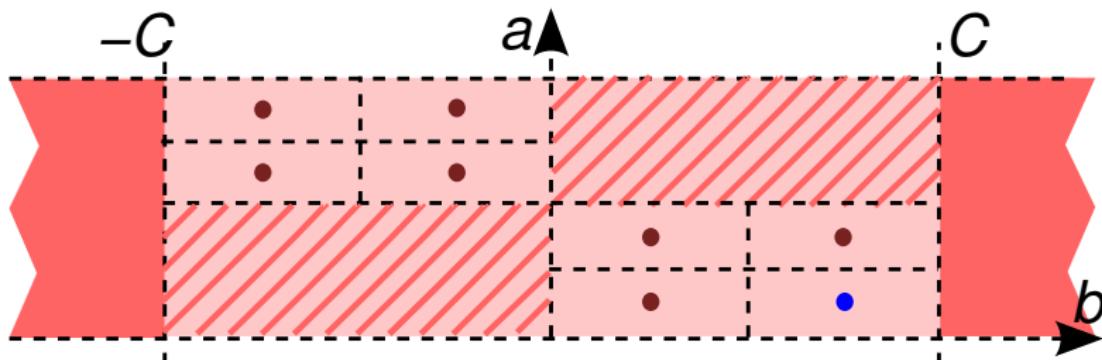
The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

Proposition 2 ($b < |C|$)

If $|a - a'| < \delta$ and $|b - b'| < \delta C$, then

$$|d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})) - d_B(D(\varphi_{(a',b')}), D(\psi_{(a',b')}))| \leq 18\delta C.$$



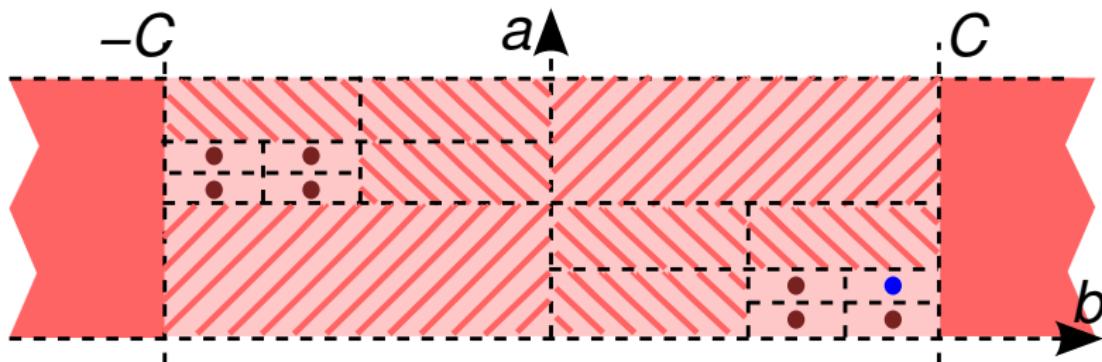
The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

Proposition 2 ($b < |C|$)

If $|a - a'| < \delta$ and $|b - b'| < \delta C$, then

$$|d_B(D(\varphi_{(a,b)}), D(\psi_{(a,b)})) - d_B(D(\varphi_{(a',b')}), D(\psi_{(a',b')}))| \leq 18\delta C.$$



The algorithm, in a nutshell

- $\varphi, \psi : X \rightarrow \mathbb{R}^2$; $C = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$;
- Recall that $\gamma_{(a,b)} \longleftrightarrow (a, b) \in \mathbb{R}^2$, $0 < a < 1$.

Refining procedure

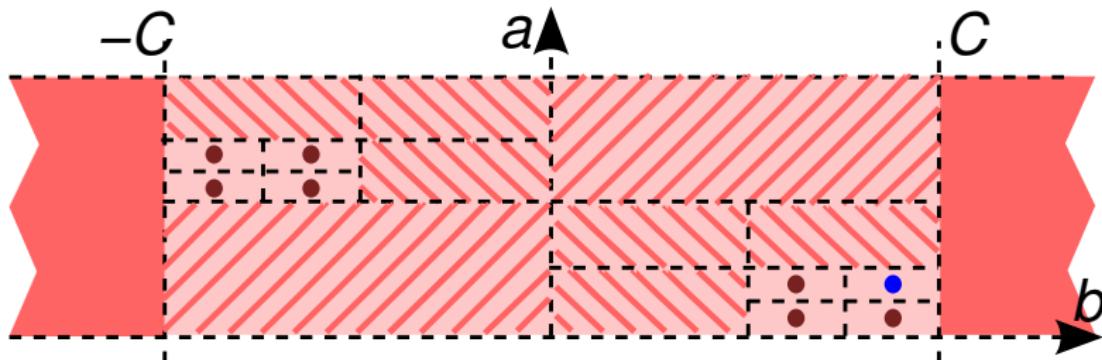
ε input real parameter;

$thresh = 18\delta C$;

while $thresh \geq \varepsilon$ do

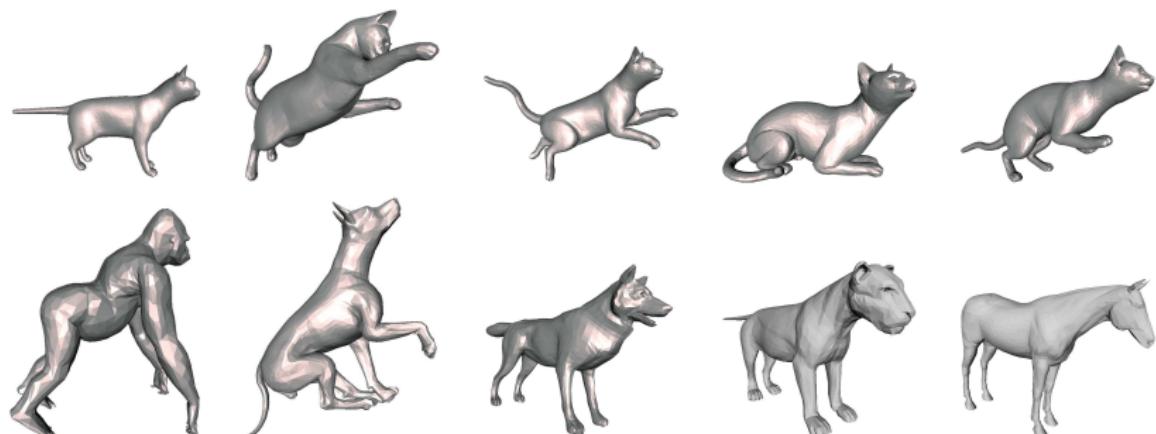
 Refine Subdivision; $\delta \leftarrow \delta/2$; $thresh \leftarrow 18\delta C$;

endwhile



Experimental results: Setting and Databases

- 2-dimensional, 0th rank invariant (size functions²) for shape comparison;
- Databases: triangle meshes from the Non-rigid world Benchmark³.



²Frosini & Landi (1999), Frosini & Mulazzani (1999)...

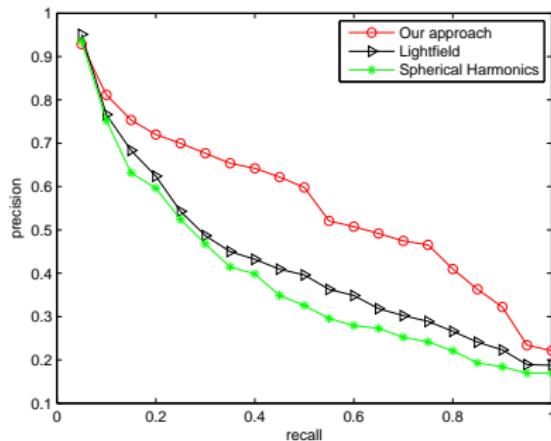
³Non-rigid World, <http://tosca.cs.technion.ac.il/>

Experiment #1: Non-rigid shape similarity

Useful property: Invariance of rank invariant;

Selected measuring function: $\varphi = (\varphi_1, \varphi_2)$ with

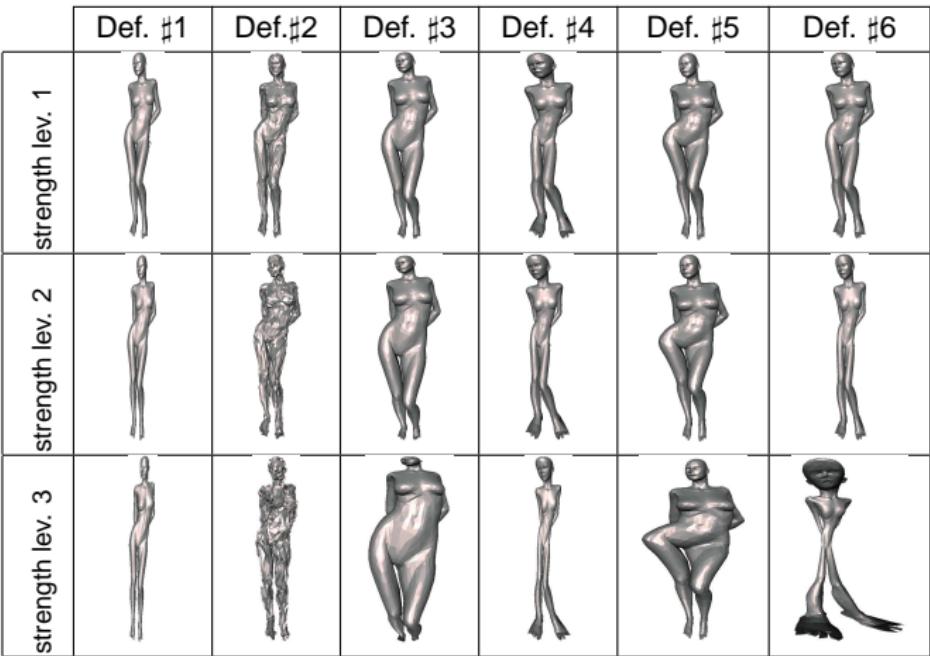
- φ_1 = heat kernel signature ($t = 1000$);
- φ_2 = integral geodesic distance.



Experiment #2: Robustness to non-metric-preserving deformations

Database derived from the **Non-rigid world benchmark**.

Victoria0

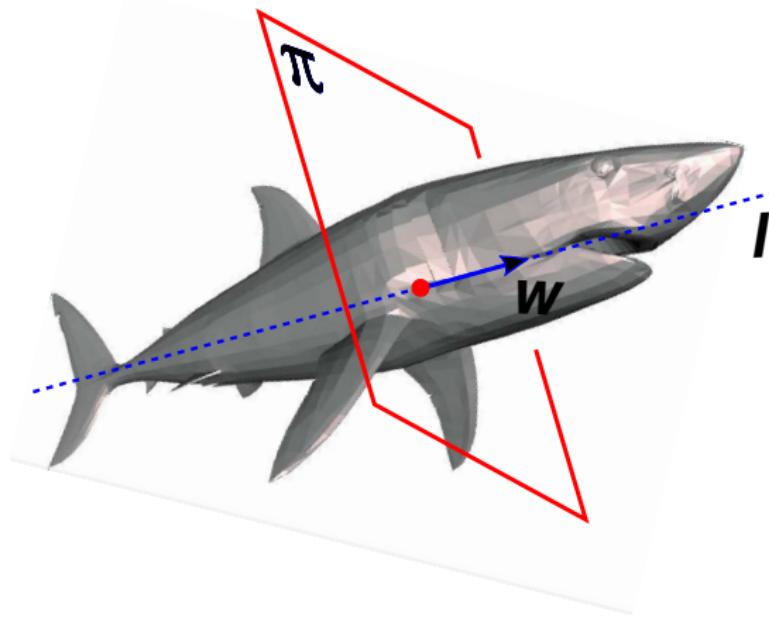


Experiment #2: Robustness to non-metric-preserving deformations

Useful property: Modularity of rank invariant;

Selected measuring function: $\varphi = (\varphi_1, \varphi_2)$ with

- φ_1 = normalized distance from I ;
- φ_2 = normalized distance from π .

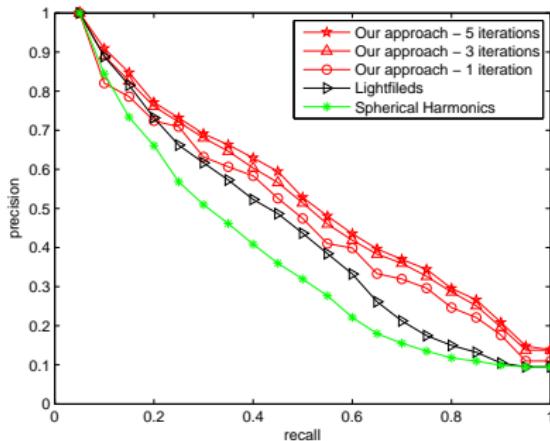


Experiment #2: Robustness to non-metric-preserving deformations

Useful property: Modularity of rank invariant;

Selected measuring function: $\varphi = (\varphi_1, \varphi_2)$ with

- φ_1 = normalized distance from I ;
- φ_2 = normalized distance from π .



1 Our approach to Shape Comparison

2 Persistence

3 Multidimensional persistent Betti numbers (rank invariant)

- Algorithm
- Experimental results

4 Conclusions

- 0th rank invariant and multidimensional matching distance for shape comparison;

Conclusions and future works

- 0th rank invariant and multidimensional matching distance for shape comparison;
- To do: Can we improve the algorithm for the approximation of the multidimensional matching distance?

- 0th rank invariant and multidimensional matching distance for shape comparison;
- To do: Can we improve the algorithm for the approximation of the multidimensional matching distance?
 - Studying the variation of $D(\varphi(a, b))$;

Conclusions and future works

- 0th rank invariant and multidimensional matching distance for shape comparison;
- To do: Can we improve the algorithm for the approximation of the multidimensional matching distance?
 - Studying the variation of $D(\varphi(a, b))$;
 - Lower bounds for d_B :

