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Workshop on Computational Topology

Persistence based signatures for compact metric spaces

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Joint work with Vin de Silva and Steve Oudot

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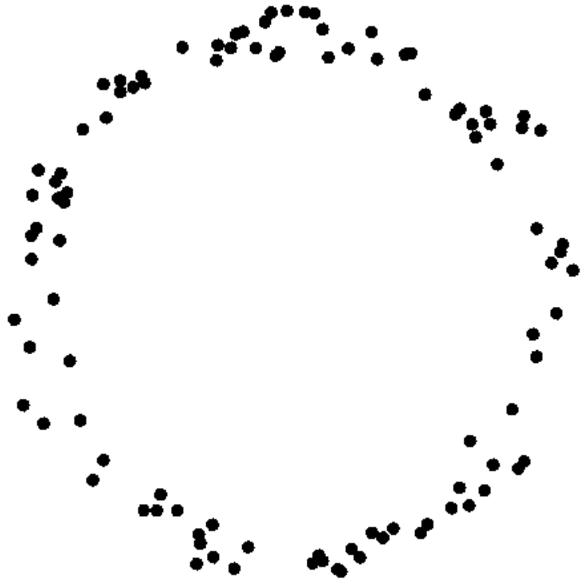
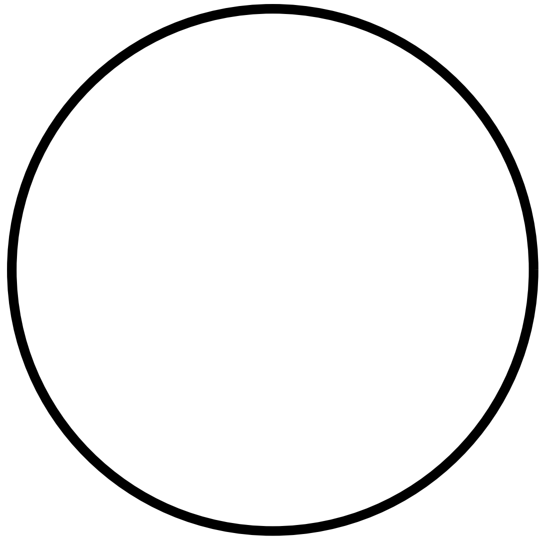
GEOMETRICA

Introduction

Problem: how to compare topological properties of close metric spaces (X, d_X) and (Y, d_Y)?





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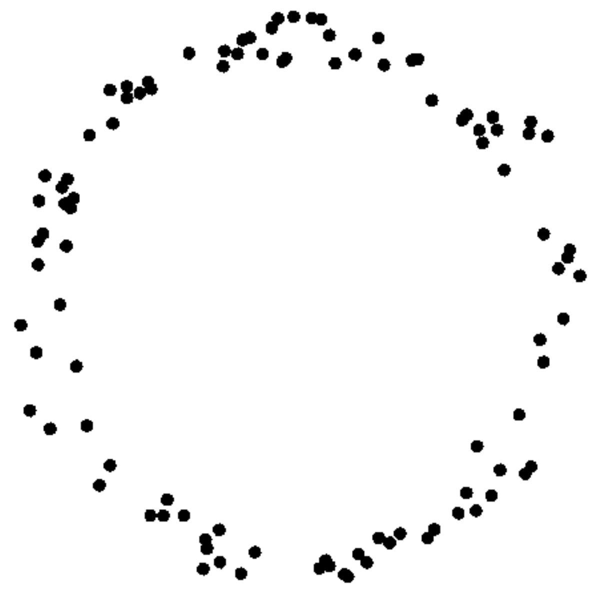

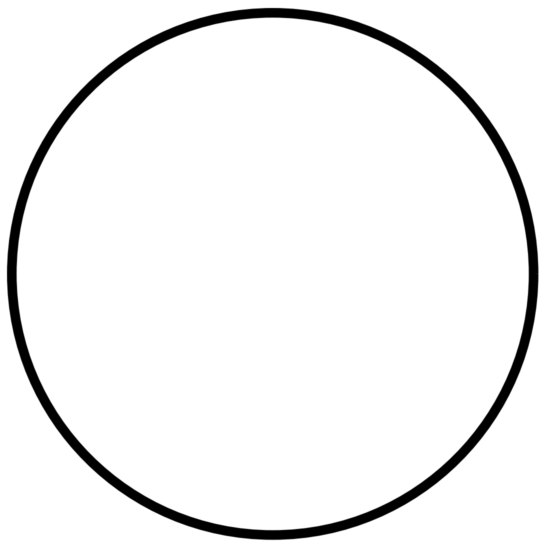

- Gromov-Hausdorff distance between (X, d_X) and (Y, d_Y)

For $\varepsilon \geq 0$, $C \subset X \times Y$ is an ε-correspondence if
i) $\forall x \in X$, $\exists y \in Y$ s. t. $(x, y) \in C$;
ii) $\forall y \in Y$, $\exists x \in X$ s. t. $(x, y) \in C$;
iii) $\forall (x, y), (x', y') \in C$, $|d_X(x, x') - d_Y(y, y')| \leq \varepsilon$.

$$d_{GH}(X, Y) = \frac{1}{2} \inf\{\varepsilon \geq 0 : \text{there exists an } \varepsilon\text{-correspondence between } X \text{and } Y\}$$

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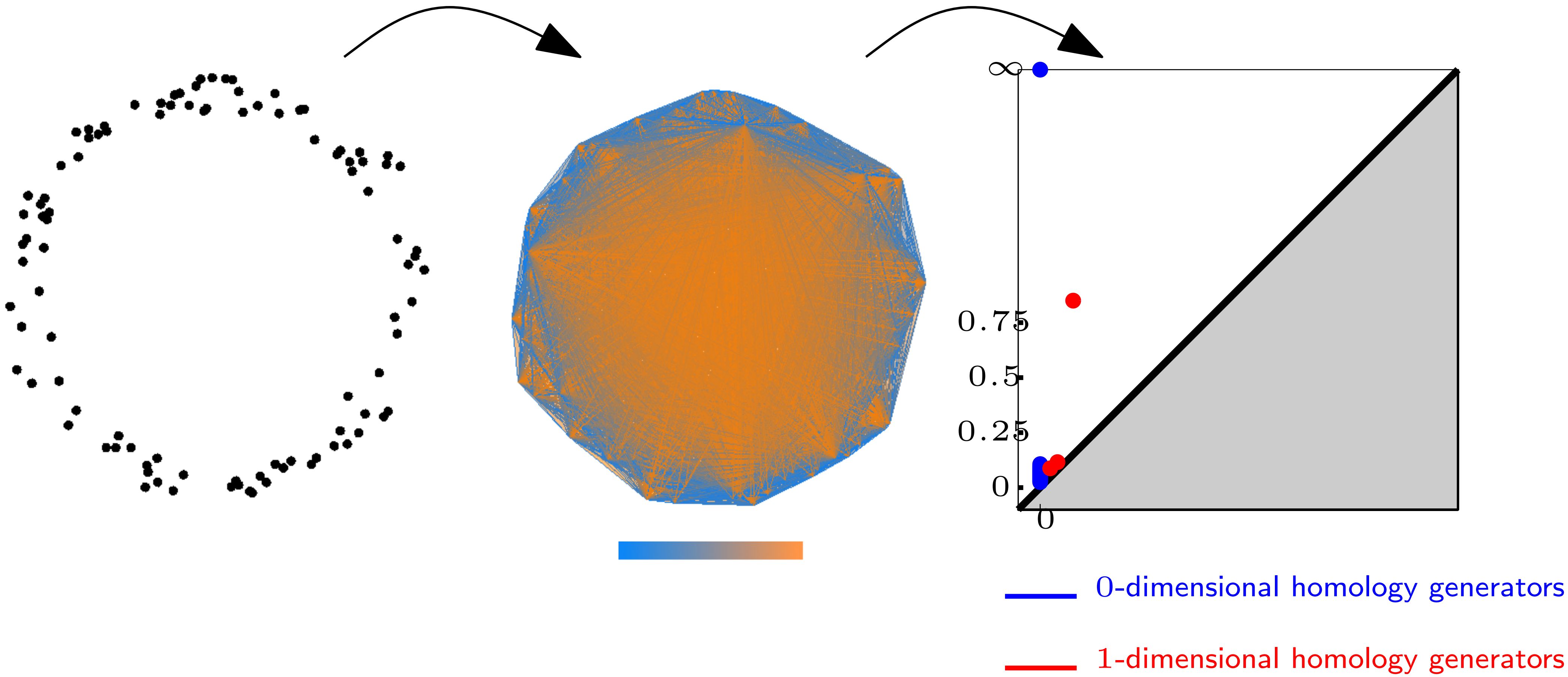

- Scale dependent approach considering the Vietoris-Rips complexes built on X and Y.

  For $a \in \mathbb{R}$, $\mathrm{Rips}(X, a)$ is the simplicial complex with vertex set X defined by

$$[x_0, x_1, \cdots, x_k] \in \mathrm{Rips}(X, a) \Leftrightarrow d_X(x_i, x_j) \leq a, \quad \text{for all } i, j$$

- Compare the persistence of the filtrations $\mathbf{H}\mathrm{Rips}(X)$ and $\mathbf{H}\mathrm{Rips}(Y)$ where $\mathbf{H} = \mathbf{H}_p(-; \mathbb{F})$ is the simplicial homology functor in dimension p over a field $\mathbb{F}$.

Previous works

- J.-C. Hausmann 1995
- J. Latschev 2001
- D. Attali, A. Lieutier 2011

When (X, d_X) is a riemannian manifold (or a sufficiently regular subset of $\mathbb{R}^d$), Rips(X, a) is homotopy equivalent to X for small enough $a > 0$.

- F. C., D. Cohen-Steiner, L. Guibas, F. Mémoli, S. Oudot 2009

If X and Y are **finite** metric spaces, the bottleneck distance between the persistence diagrams of $\mathbf{H}\mathrm{Rips}(X)$ and $\mathbf{H}\mathrm{Rips}(Y)$ is a lower bound of $2d_{GH}(X, Y)$.

persistence diagrams used as shape signatures for classification

Tame persistent modules

Definition: A persistent module is a family $\mathcal{M} = (M^a, a \in \mathbb{R})$ of vector spaces together with linear maps $\xi_a^b : M^a \to M^b$ for all $a \leq b$ where $\xi_a^a = id_{M^a}$ and $\xi_a^c = \xi_b^c \circ \xi_a^b$ for all $a \leq b \leq c$.

Example: Given a metric space (X, d_X), the Vietoris-Rips filtration $\mathbf{H}\mathrm{Rips}(X)$ is a persistence module with linear maps the morphisms ξ_a^b induced at the homology level by the inclusion maps $\mathrm{Rips}(X, a) \hookrightarrow \mathrm{Rips}(X, b)$.

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In the "classical" setting (finite dimentional spaces), the definition of persistence only involves the rank of the maps ξ_a^b, $a < b$.

Definition: A persistence module $\mathcal{M}$ is tame if for any $a < b$, ξ_a^b has a finite rank.

Theorem [CCGGO'09]: tame persistence modules have well-defined persistence diagrams.

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A map $\Phi : \mathcal{M} \to \mathcal{N}$ of degree t is a family of morphisms $\phi_a : M^a \to N^{a+t}$ for all a s.t. $\eta_{a+t}^{b+t} \circ \phi_a = \phi_b \circ \xi_a^b$.

$$\begin{tikzcd}
M^a \arrow[rr] \arrow[dr] & & M^b \arrow[dr] & \\
& N^{a+t} \arrow[rr] \arrow[ur, phantom, "\circlearrowright"] & & N^{b+t}
\end{tikzcd}$$

An ε-interleaving between $\mathcal{M}$ and $\mathcal{N}$ is specified by two maps $\Phi^\varepsilon : \mathcal{M} \to \mathcal{N}$ and $\Psi^\varepsilon : \mathcal{N} \to \mathcal{M}$ of degree ε s.t. $\Phi^\varepsilon \circ \Psi^\varepsilon$ and $\Psi^\varepsilon \circ \Phi^\varepsilon$ are the shifts of $\mathcal{N}$ and $\mathcal{M}$.

$$\begin{tikzcd}
\cdots \arrow[rr] & & M^a \arrow[rr, "\xi_a^{a+2\varepsilon}"] \arrow[dr, "\phi_a"'] & & M^{a+2\varepsilon} \arrow[rr] \arrow[dr] & & \cdots & \\
& \cdots \arrow[rr] \arrow[ur] & & N^{a+\varepsilon} \arrow[rr, "\eta_{a+\varepsilon}^{a+3\varepsilon}"'] \arrow[ur, "\psi_{a+\varepsilon}"'] & & N^{a+3\varepsilon} \arrow[rr] & & \cdots
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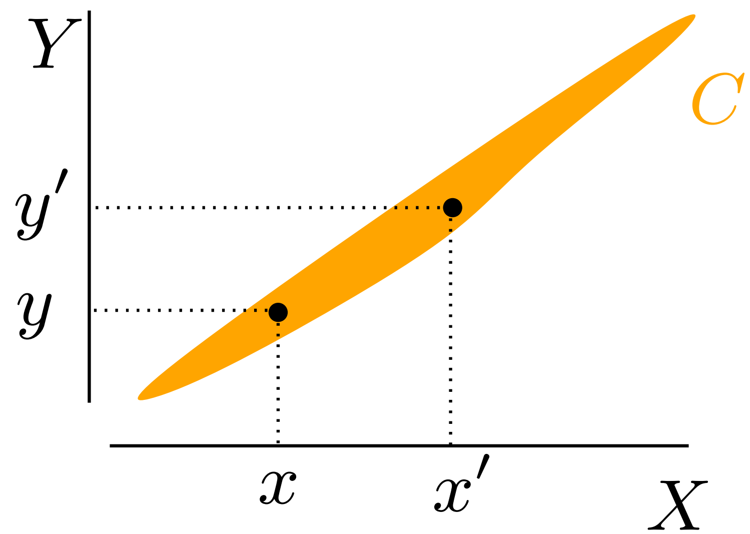
A persistence module $\mathcal{M}$ is tame if for any $a < b$, ξ_a^b has a finite rank.

Theorem [CCGGO'09]: If $\mathcal{M}$ and $\mathcal{N}$ are tame and ε-interleaved for some $\varepsilon \geq 0$ then

$$d_B(\mathrm{Dgm}(\mathcal{M}), \mathrm{Dgm}(\mathcal{N})) \leq \varepsilon$$

Correspondences and interleaving

Proposition: Let X, Y be metric spaces and let $C \subset X \times Y$ be an ε-correspondence between X and Y. Then C induces a canonical ε-interleaving between $\mathbf{H}\mathrm{Rips}(X)$ and $\mathbf{H}\mathrm{Rips}(Y)$.



$$|d_X(x,x') - d_Y(y,y')| < \varepsilon$$

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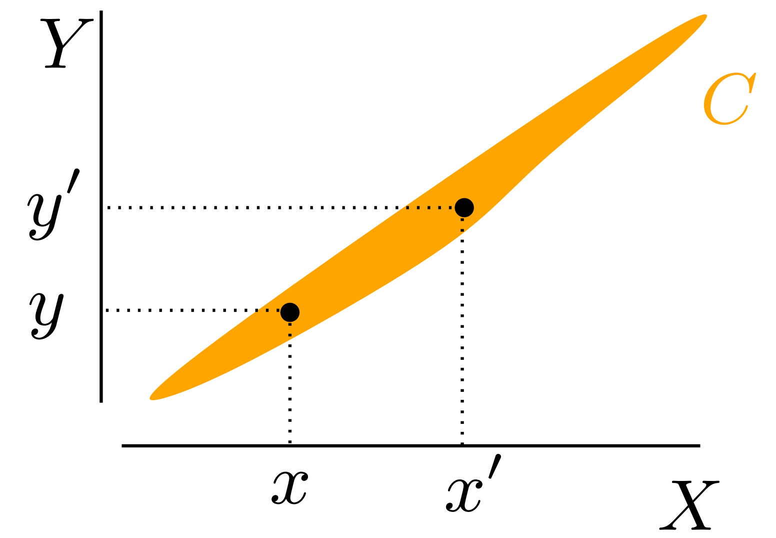
Lemma: Let $f : X \to Y$ be compatible with C in the sense that $(x, f(x)) \in C$ for all $x \in X$. Then f induces a simplicial map $\overline{f} : \mathrm{Rips}(X, a) \to \mathrm{Rips}(Y, b)$ whenever $b \geq a + \varepsilon$. Moreover, if $g : X \to Y$ is compatible with C, $\overline{f}, \overline{g}$ are contiguous maps.



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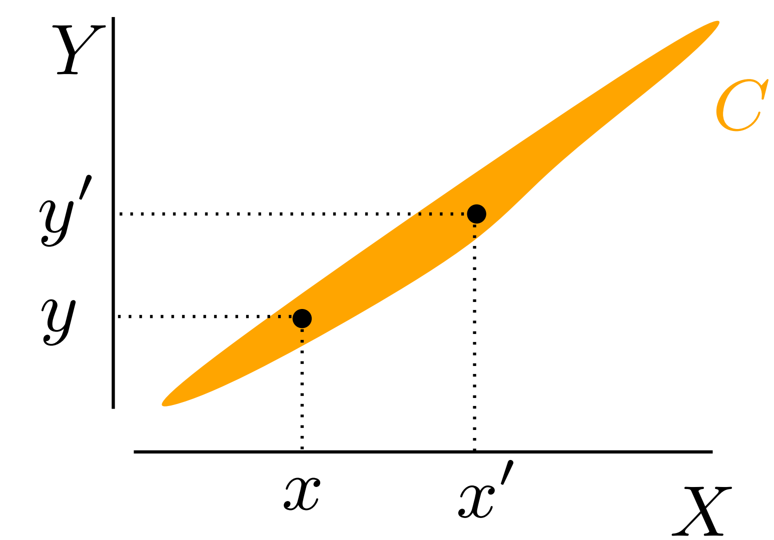
Lemma: Let $f : X \to Y$ be compatible with C in the sense that $(x, f(x)) \in C$ for all $x \in X$. Then f induces a simplicial map $\overline{f} : \mathrm{Rips}(X, a) \to \mathrm{Rips}(Y, b)$ whenever $b \geq a + \varepsilon$. Moreover, if $g : X \to Y$ is compatible with C, $\overline{f}, \overline{g}$ are contiguous maps.

Proof of lemma:

1. if $\sigma = [x_0, \cdots x_k] \in \mathrm{Rips}(X, a)$ then $\overline{f}(\sigma) = [f(x_0), \cdots f(x_k)] \in \mathrm{Rips}(Y, b)$

2. $\overline{f}(\sigma)$ and $\overline{g}(\sigma)$ are contained in a simplex of $\mathrm{Rips}(Y, b)$:

$$d_Y(f(x_i), f(x_j)) \leq d_X(x_i, x_j) + \varepsilon \leq a + \varepsilon \leq b$$
$$d_Y(g(x_i), g(x_j)) \leq d_X(x_i, x_j) + \varepsilon \leq a + \varepsilon \leq b$$
$$d_Y(f(x_i), g(x_j)) \leq d_X(x_i, x_j) + \varepsilon \leq a + \varepsilon \leq b$$



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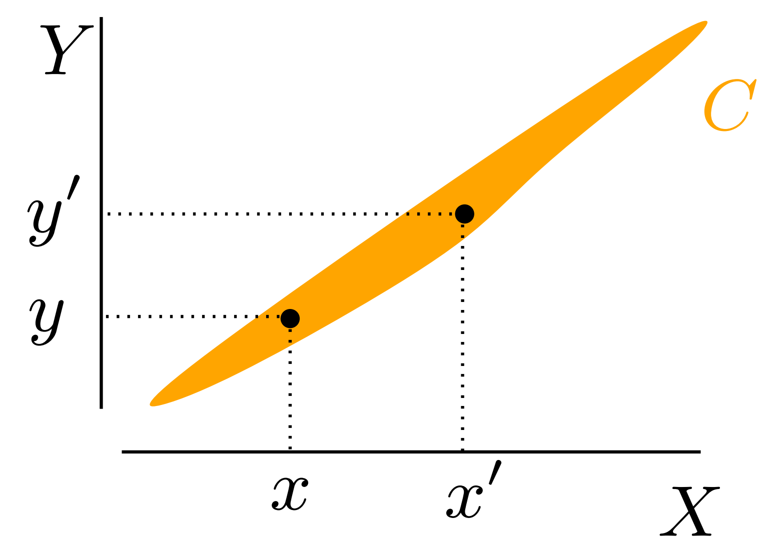
Proof of the proposition:

- Let $f : X \to Y$ and $g : Y \to X$ be compatible.
- $C^T C = \{(x, x') \in X \times X | \exists y \in Y \text{ s.t. } (x, y) \in C, (x', y) \in C\}$ is a 2ε-correspondence and $h = g \circ f$ and id_X are compatible functions $X \to X$.
- It follows from the lemma that h and id_X are contiguous, so they induce the same maps at homology level.



$|d_X(x, x') - d_Y(y, y')| < \varepsilon$

Tameness of the Rips filtration

A metric space (X, d_X) is totally bounded (or precompact) if for any $\varepsilon > 0$ there exists a finite subset $F_\varepsilon \subset X$ such that $d_H(X, F_\varepsilon) < \varepsilon$ (i.e. $\forall x \in X, \exists p \in F_\varepsilon$ s.t. $d_X(x, p) < \varepsilon$).

Theorem: Let X be a totally bounded metric space. Then $\mathbf{H}\mathrm{Rips}(X)$ is tame.
$\Rightarrow \mathrm{Dgm}(\mathbf{H}\mathrm{Rips}(X))$ is well-defined!

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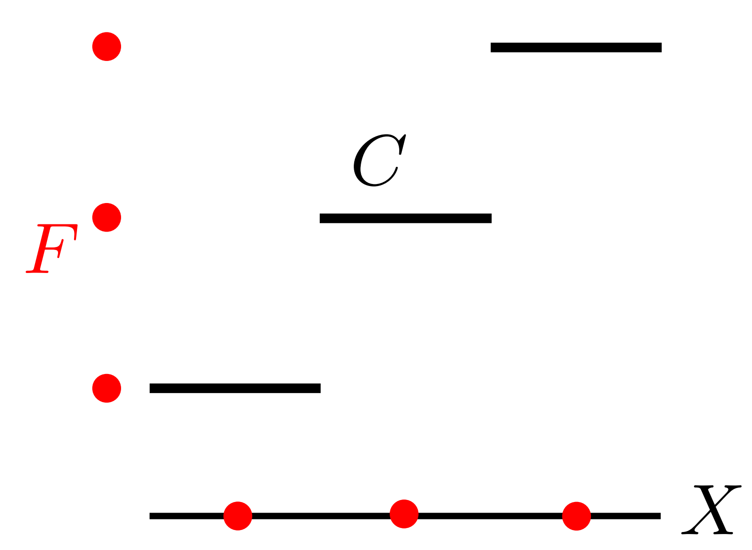
Proof: show that $I_a^b : \mathbf{H}\mathrm{Rips}(X,a) \to \mathbf{H}\mathrm{Rips}(X,b)$ has finite rank whenever $a < b$.

Let $\varepsilon = (b-a)/2$ and let $F \subset X$ be finite s. t. $d_H(X,F) \leq \varepsilon/2$.

Then $C = \{(x,f) \in X \times F | d(x,f) \leq \varepsilon\}$ is an ε-correspondence.



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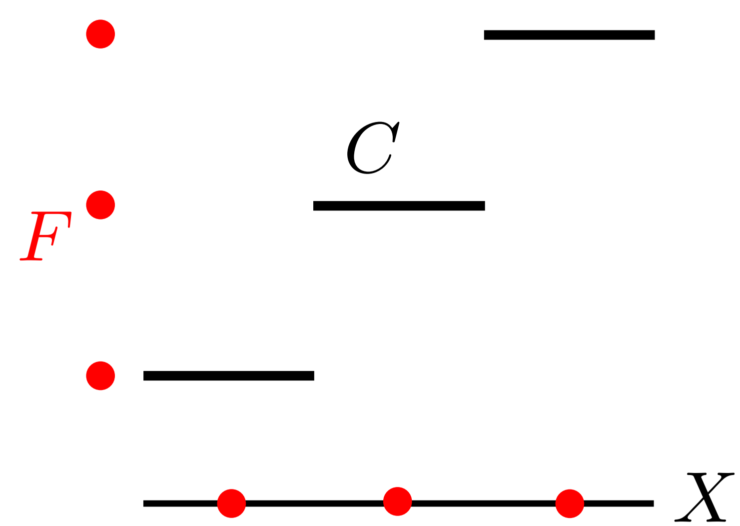

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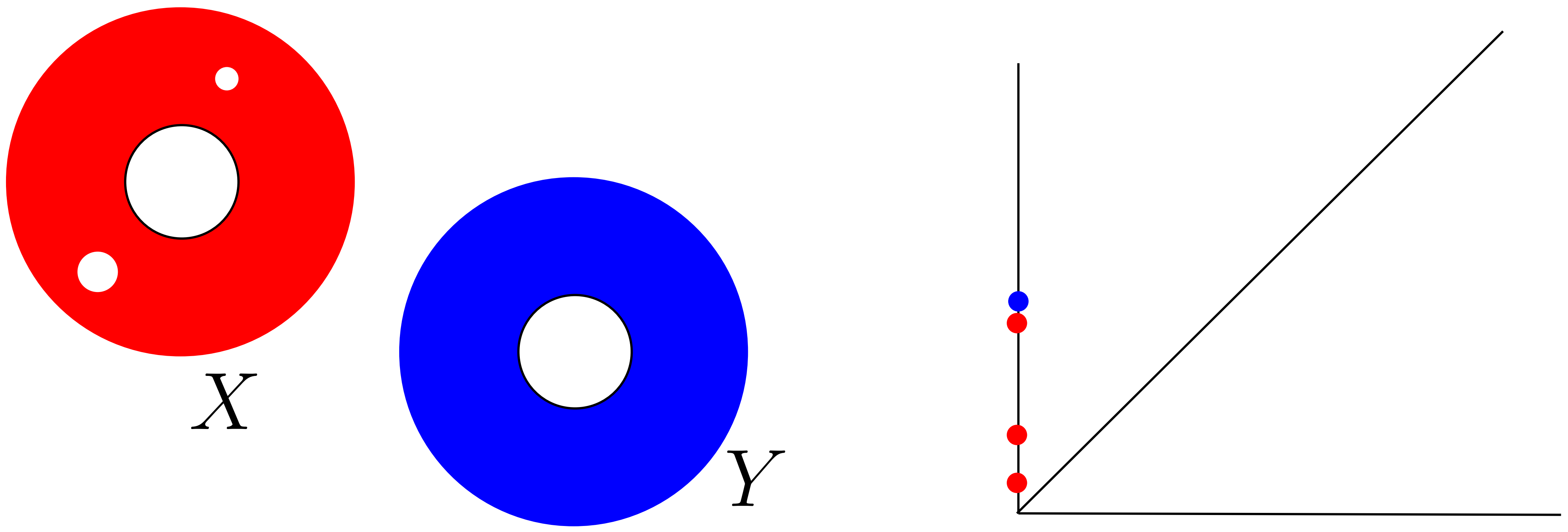
Then $C = \{(x,f) \in X \times F | d(x,f) \leq \varepsilon\}$ is an ε-correspondence.

Using the interleaving map, I_a^b factorizes as

$$\mathbf{H}\mathrm{Rips}(X,a) \to \mathbf{H}\mathrm{Rips}(F, a+\varepsilon) \to \mathbf{H}\mathrm{Rips}(X, a+2\varepsilon) = \mathbf{H}\mathrm{Rips}(X,b)$$

finite dimensional

Stability of persistence diagrams



Theorem: Let X, Y be totally bounded metric spaces. Then

$$\mathbf{d}_b(\mathrm{Dgm}(\mathbf{H}\mathrm{Rips}(X)), \mathrm{Dgm}(\mathbf{H}\mathrm{Rips}(Y))) \leq 2\mathbf{d}_{GH}(X, Y)$$

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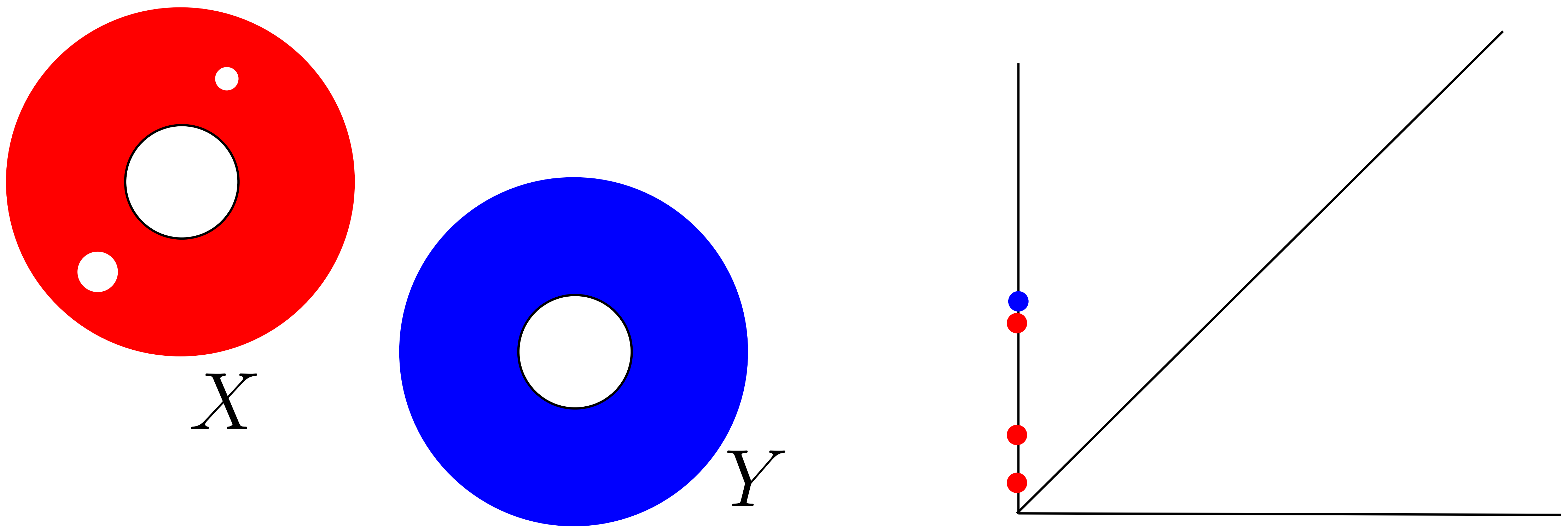

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Proof:

For any $\varepsilon > 2\mathbf{d}_{GH}(X, Y)$ there exists an ε-correspondence C between X and Y.

C induces a canonical ε-interleaving between $\mathbf{H}\mathrm{Rips}(X)$ and $\mathbf{H}\mathrm{Rips}(Y)$ that are tame modules.

It follows that

$$\mathbf{d}_b(\mathrm{Dgm}(\mathbf{H}\mathrm{Rips}(X)), \mathrm{Dgm}(\mathbf{H}\mathrm{Rips}(Y))) \leq \varepsilon$$

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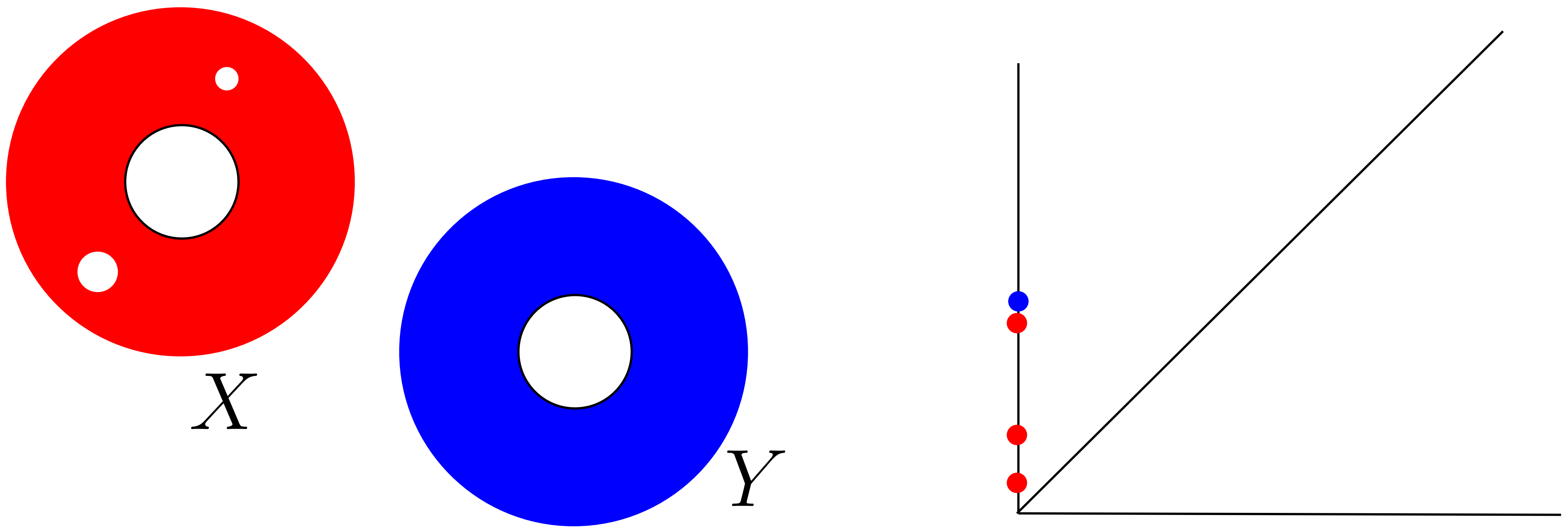

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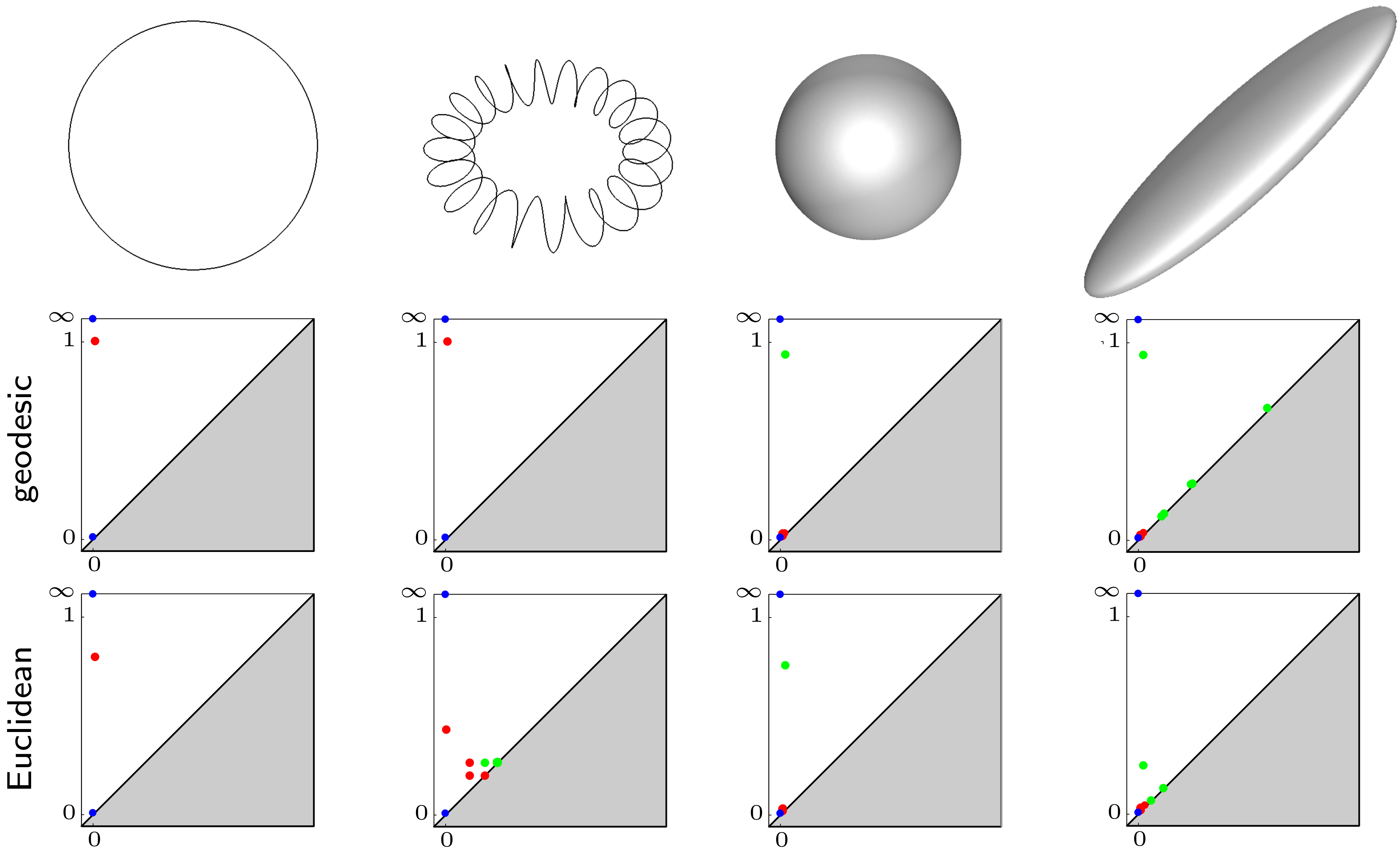
- Robust multiscale signature for totally bounded spaces.
- A (topological) lower bound of the Gromov-Hausdorff distance.
- Same result with Cech filtrations.
- Similar results for pairs (X, f) where $f : X \to \mathbb{R}$ is a Lipschitz function.
- No need of the triangle inequality! $\to$ Also works for "dissimilarity measures"

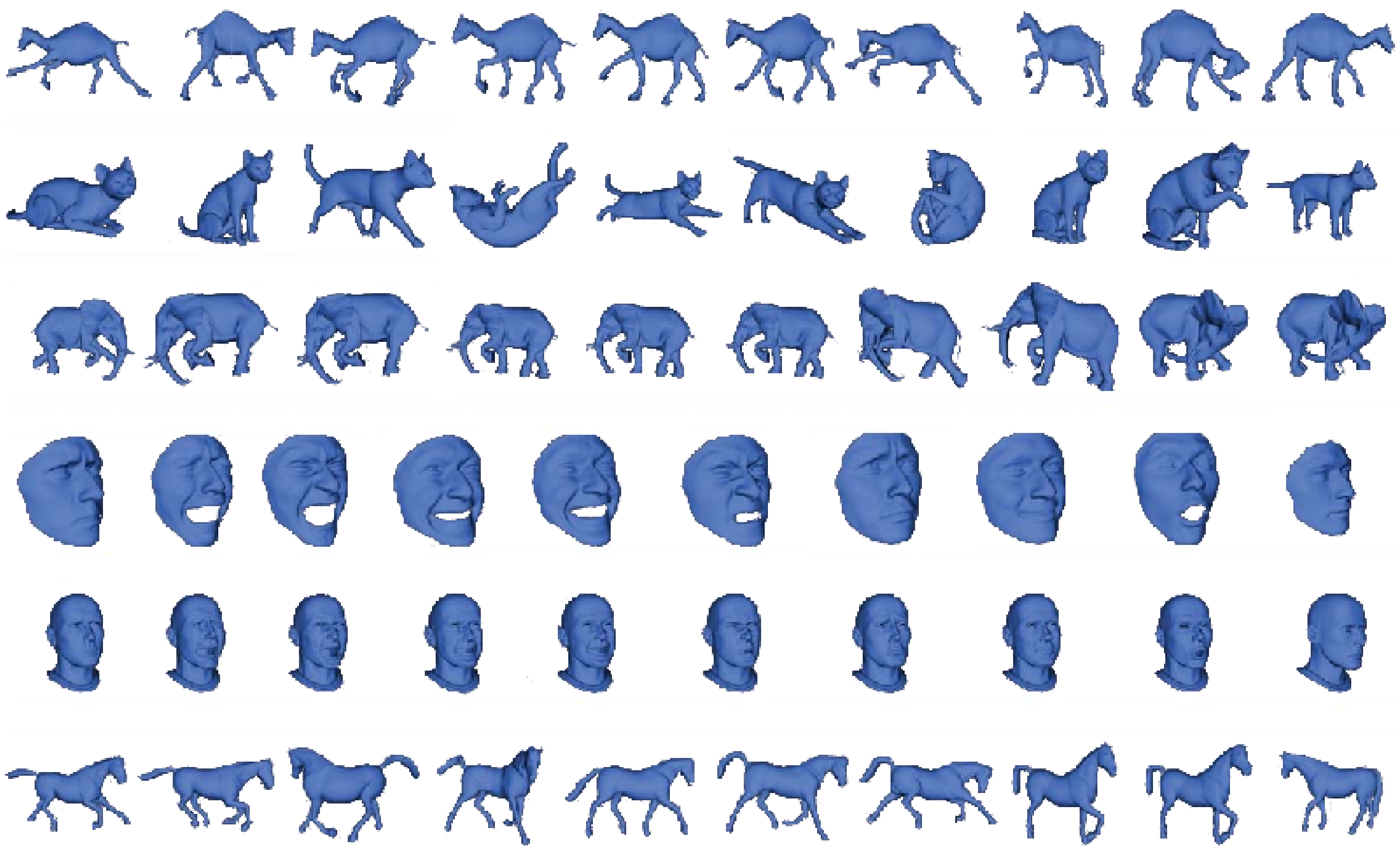
Persistence-based signatures

Signatures of some elementary shapes (approximated from finite samples):



Persistence-based signatures

Experimental results:



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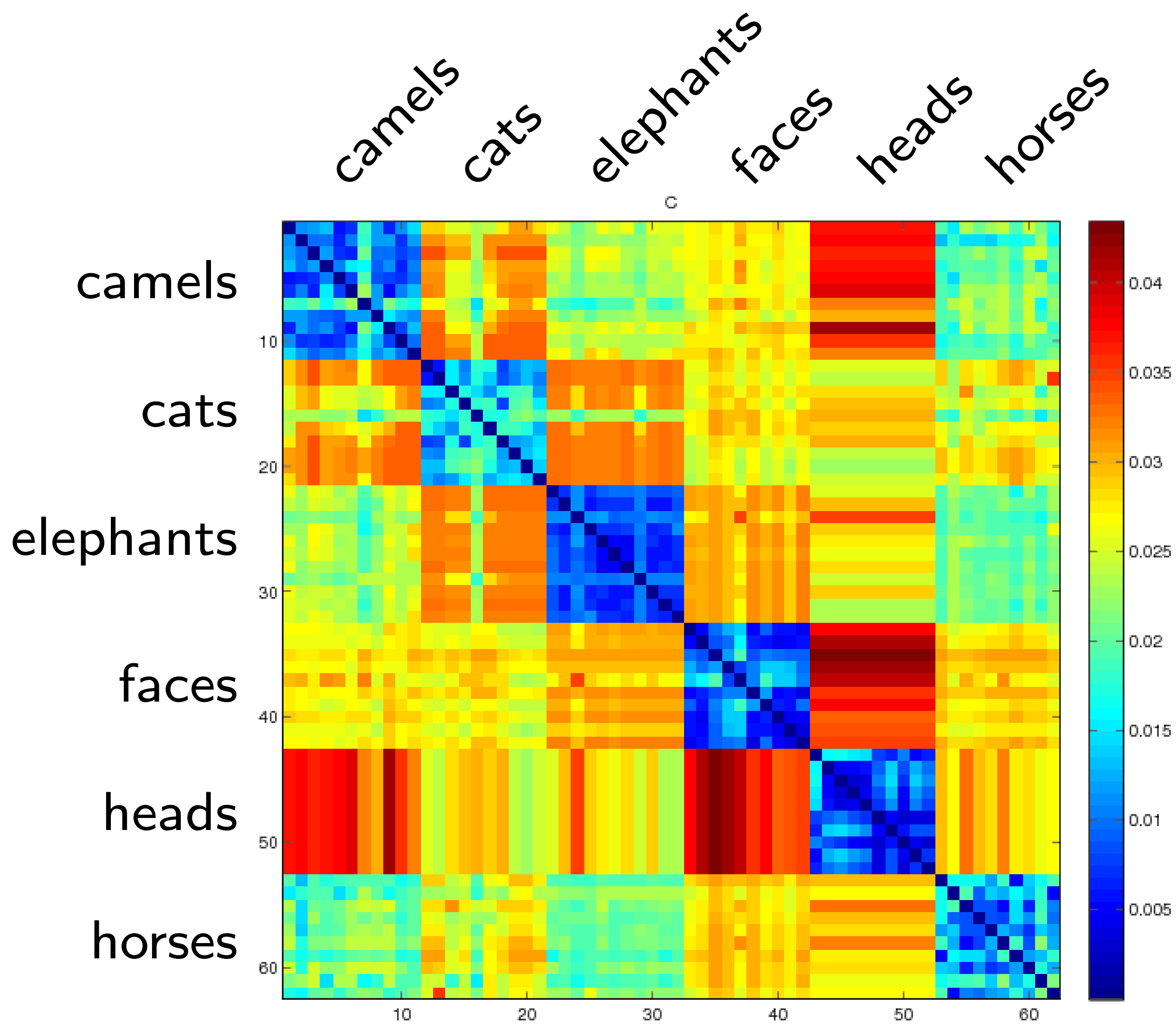

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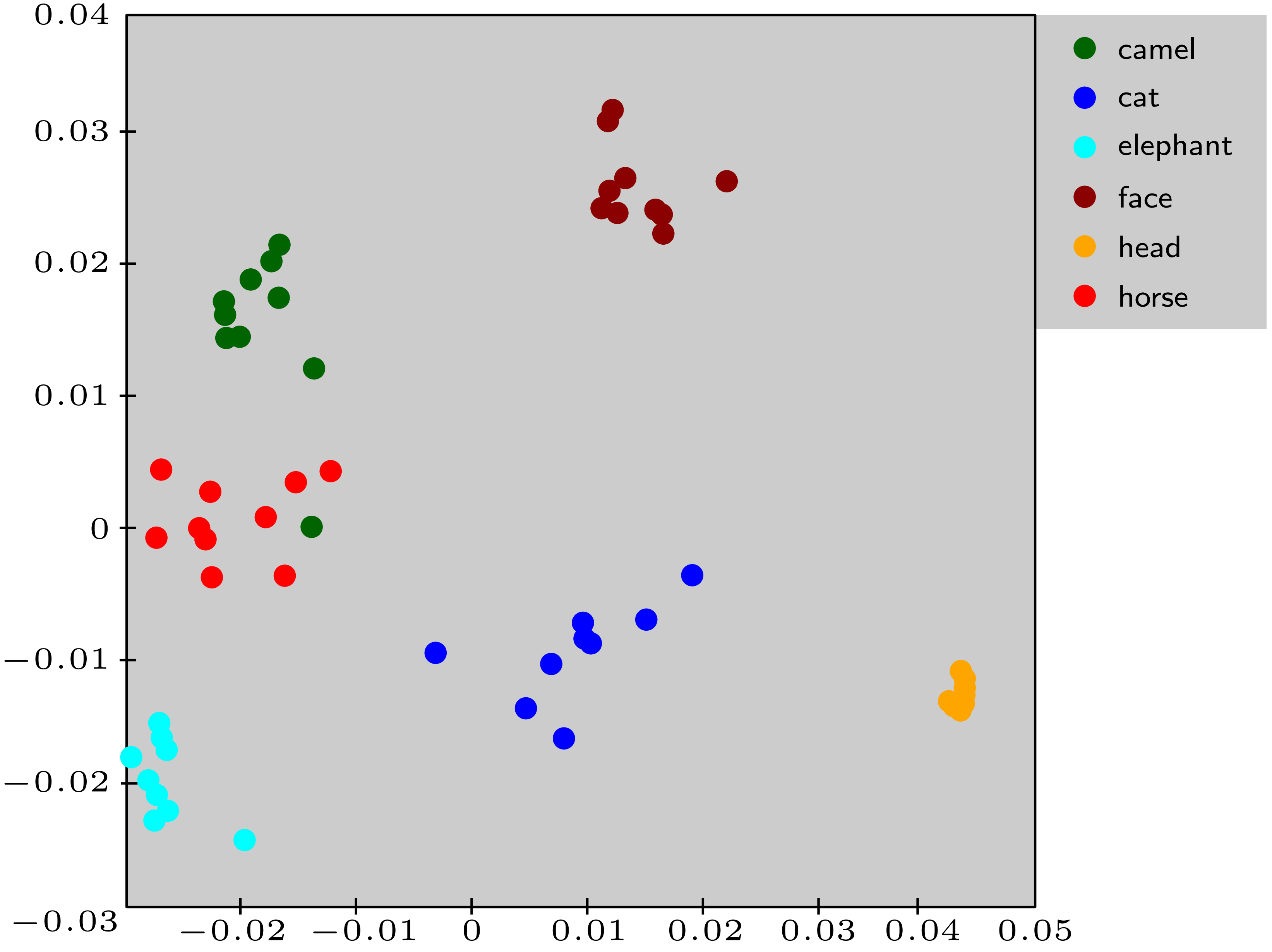
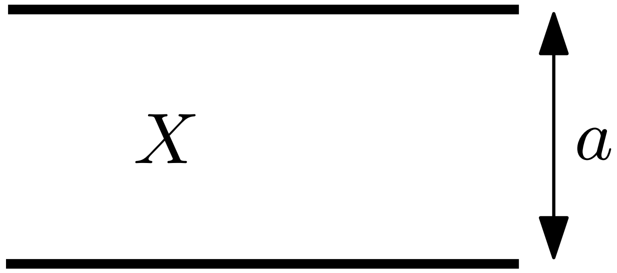


Homework (remarks and open questions)

- Even when X is compact, $\mathbf{H}_p(\mathrm{Rips}(X,a))$, $p \geq 1$, might be infinite dimensional for some value of a:



It is also possible to build such an example with the "open" Rips complex:

$$[x_0, x_1, \cdots, x_k] \in \mathrm{Rips}(X,a) \Leftrightarrow d_X(x_i, x_j) < a, \quad \text{for all } i,j$$

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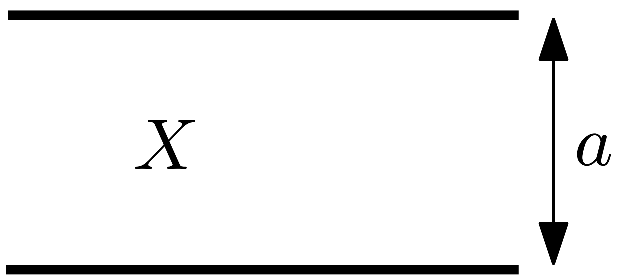

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- If X is compact, then $\dim \mathbf{H}_1(\mathrm{Cech}(X,a)) < +\infty$ for all a ([Smale-Smale, C.-de Silva]).

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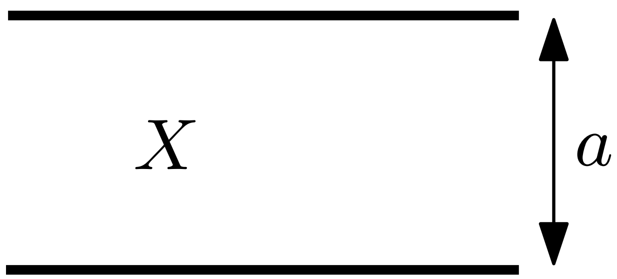

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- If X is compact, then $\dim \mathbf{H}_1(\mathrm{Cech}(X, a)) < +\infty$ for all a ([Smale-Smale, C.-de Silva]).

- If X is geodesic, then $\dim \mathbf{H}_1(\mathrm{Rips}(X, a)) < +\infty$ for all $a > 0$ and $\mathrm{Dgm}(\mathbf{H}_1 \mathrm{Rips}(X))$ is contained in the vertical line $x = 0$.

- Open problem (probably hard): for a given compact set X, how "big" can the set S of values a such that $\dim \mathbf{H}_p(\mathrm{Rips}(X, a)) = +\infty$ be?

Thank You