Two Phase Flow in Porous Media: Stability of Fronts

Michael Shearer
Kim Spayd, Zhengzheng Hu

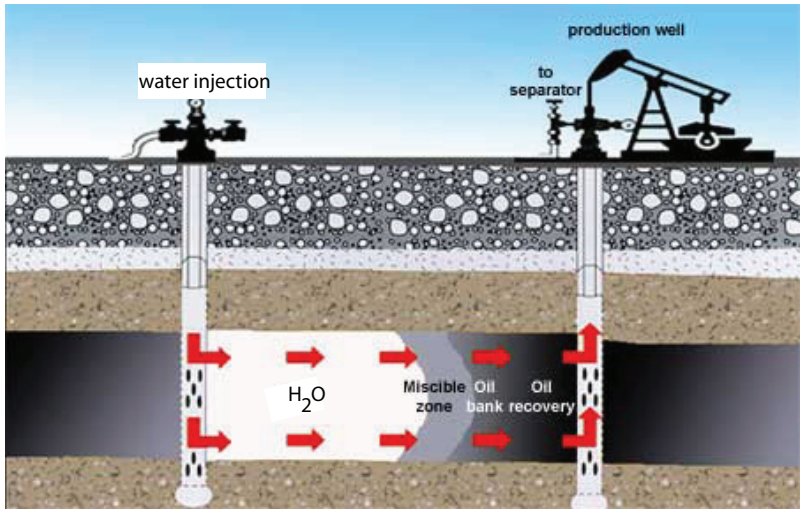
North Carolina State University
Department of Mathematics

Fields Institute 2012

Supported by NSF grants DMS 0604047, DMS 0968258, DMS 0636590 RTG

Introduction

Secondary oil recovery:
water flood



Buckley and Leverett, 1942
scalar 1st order PDE in 1-d

Gray and Hassanizadeh, 1990's
dynamic capillary pressure: rate dependence

Plane waves: undercompressive shocks; Pop, Cuesta, Peletier et al 2006-2011

Fingering instability: Saffman, Taylor, 1958
New insights; Yortsos and Hickernell, 1989

2-D Model w: water; o: oil; T: total.

$u(x, y, t)$: **saturation** (vol. fraction) of water, $(1 - u)$: oil saturation
$p(x, y, t)$: **pressure** in water

Conservation of mass with Darcy's law: velocity $\mathbf{v} = -\lambda^w(u)\nabla p$:

$$\boxed{\phi \frac{\partial u}{\partial t} - \nabla \cdot (\lambda^w(u)\nabla p) = 0} \qquad \phi = \text{porosity}, \ \lambda(u) = \frac{Kk(u)}{\mu}$$

K = absolute permeability, $k(u)$ = relative permeability, μ = viscosity

Incompressibility: $\nabla \cdot \mathbf{v}^{\mathbf{T}} = 0$

$$\boxed{\nabla \cdot \Big(\lambda^T(u)\nabla p + \lambda^o(u)\nabla p_c(u)\Big) = 0}$$

$\lambda^T = \lambda^w + \lambda^o, \quad p^c(u) = p^o - p^w$: capillary pressure;

For simplicity, neglect gravity

Plane waves

$p_e = p_e(u)$: equilibrium capillary pressure; decreasing function

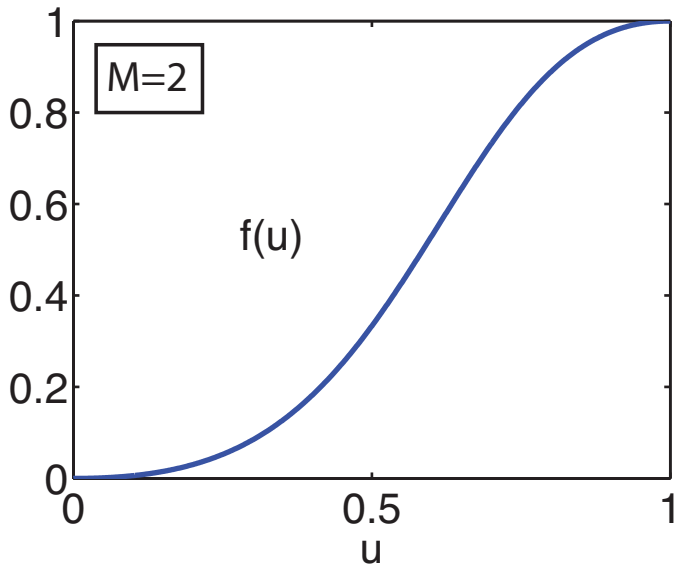
One-dimensional equation: $\partial_x \mathbf{v}^{\mathbf{T}} = 0$: $\mathbf{v}^{\mathbf{T}} = (V, 0)$ constant
eliminate pressure gradient $\partial_x p$

Relative permeability functions: $k^w(u) = \kappa^w u^2$; $k^o(u) = \kappa^o (1-u)^2$

$$\boxed{u_t + f(u)_x = -\left(p_e'(u) H(u) u_x\right)_x}$$

$$f(u) = V \frac{u^2}{u^2 + M(1-u)^2}$$

$M = m^o / m^w$ mobility ratio; $m^j = \kappa^j / \mu^j$

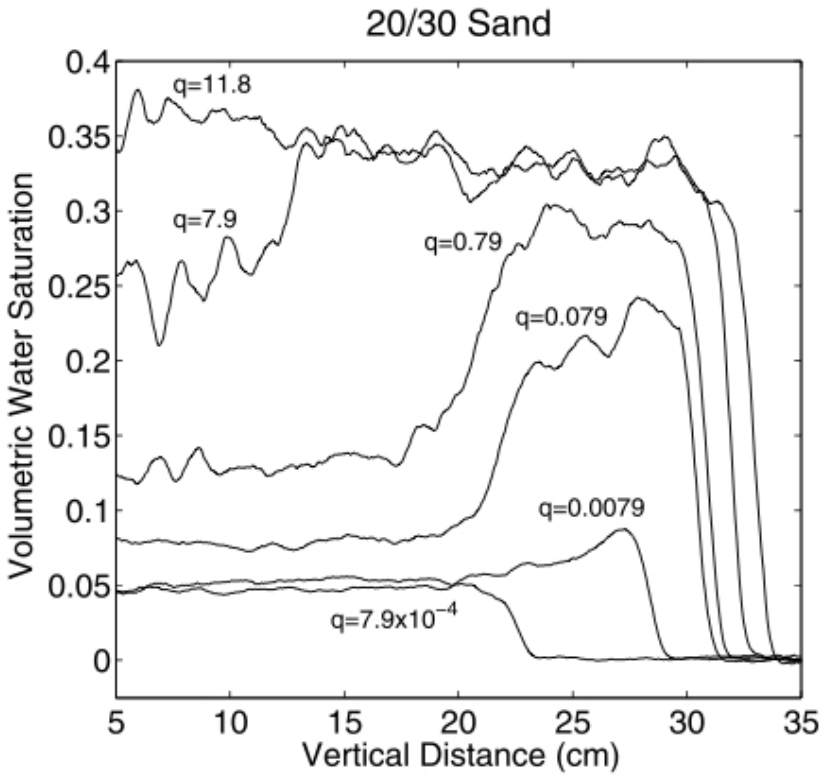


Dynamic Capillary Pressure

Gray and Hassanizadeh (1990, 1993) propose that capillary pressure should be rate dependent:

$$\boxed{p_c(u, u_t) = p_e(u) - \tau u_t}$$

p_e : equilibrium capillary pressure; $p_e(u) = -u$ for simplicity



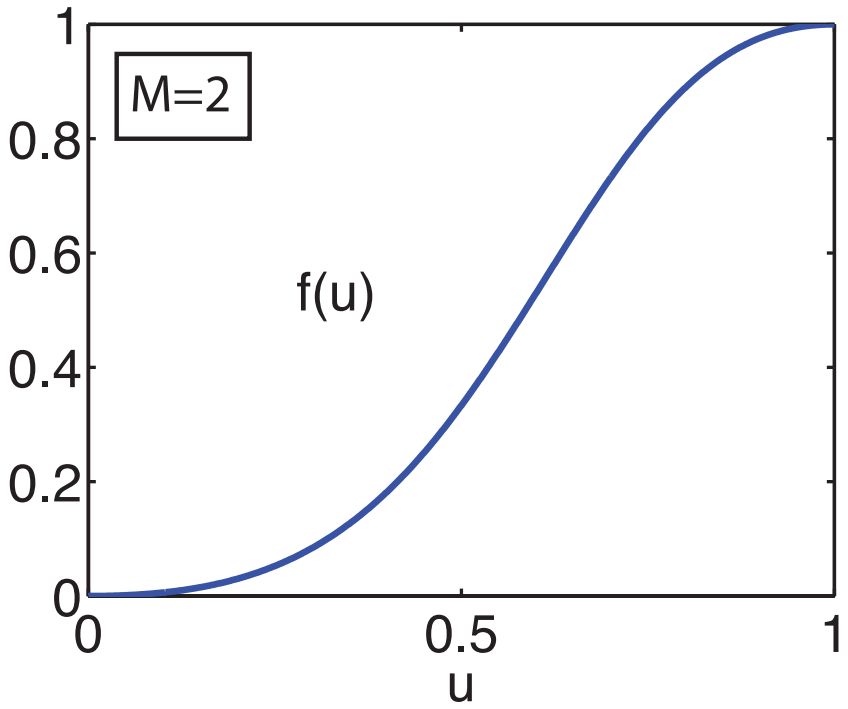
DiCarlo: Water Resources Research, 2004: Experiments (with gravity) show nonmonotonic saturation profiles

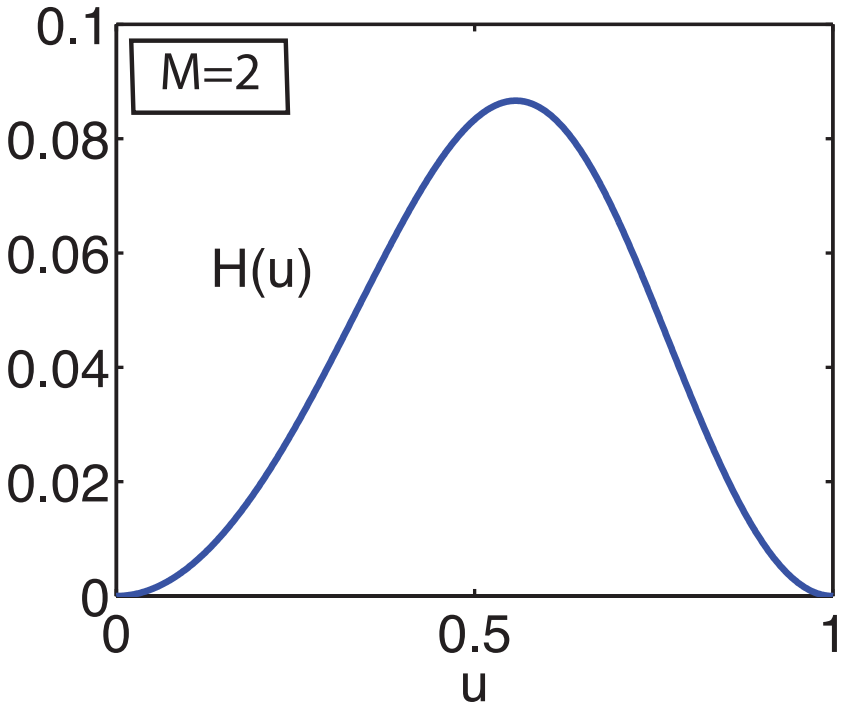
Modified Buckley-Leverett Equation (1-D)

$$\boxed{u_t + f(u)_x = (H(u)u_x)_x + \tau\,(H(u)u_{tx})_x}$$

$$f(u) = V\frac{u^2}{u^2 + M(1-u)^2} \qquad H(u) = \frac{u^2(1-u)^2}{u^2 + M(1-u)^2}$$

$M = m^o/m^w$ mobility ratio; $m^j = \kappa^j/\mu^j$





Scalar conservation law: $u_t + f(u)_x = 0$

Idealization: no capillary pressure; characteristic speed $f'(u)$

Scale invariant solutions: building blocks for solving initial value problem

Rarefactions

$$u(x,t) = \begin{cases} u_- & \text{if } x < f'(u_-)t \\ r(\frac{x}{t}) & \text{if } f'(u_-)t \le x \le f'(u_+)t \\ u_+ & \text{if } x > f'(u_+)t \end{cases}$$

Shocks

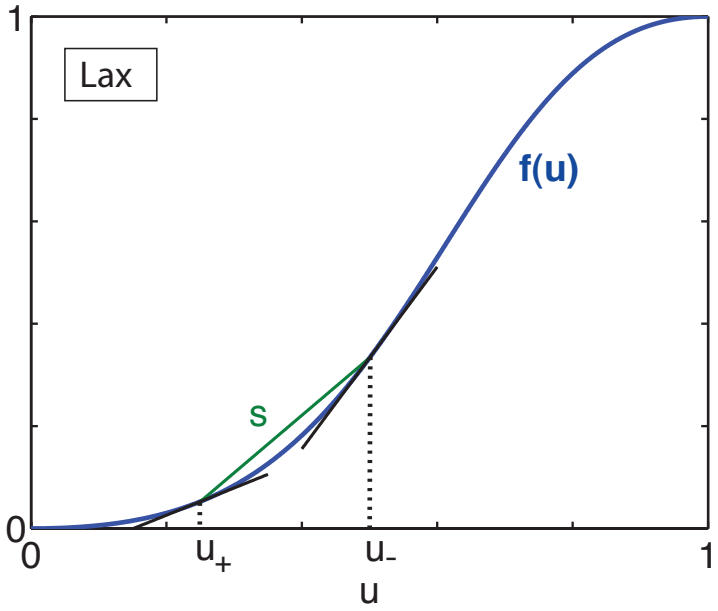
$$u(x,t) = \begin{cases} u_- & \text{if } x < st \\ u_+ & \text{if } x > st \end{cases}$$

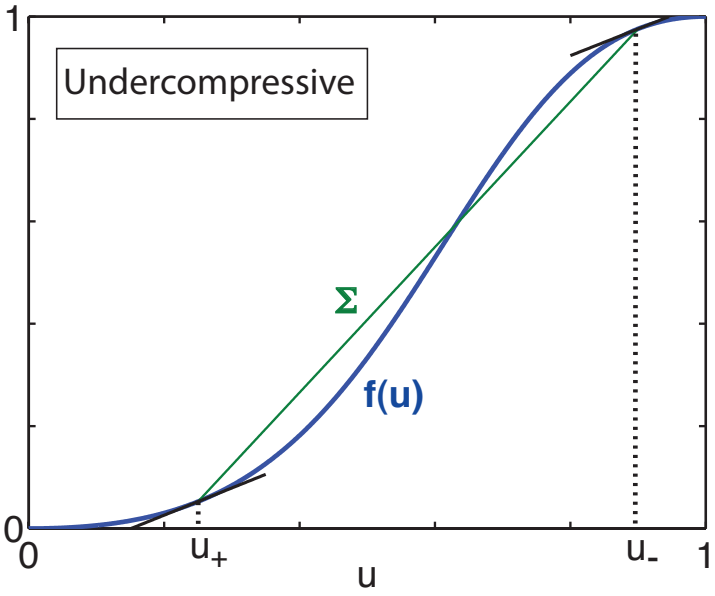
Rankine-Hugoniot condition: shock speed $\quad s = \dfrac{f(u_+) - f(u_-)}{u_+ - u_-}$

Admissible Shocks

$f'(u_+) < s < f'(u_-)$ $\boxed{Lax}$ shock is admissible if there is a traveling wave from u_- (unstable node) to u_+ (saddle point)

Dynamic capillary pressure admits $\boxed{undercompressive}$ shocks Σ : $s > f'(u_\pm)$ **PLUS** corresponding traveling wave (saddle-to-saddle)
Jacobs, McKinney, Shearer (1995), LeFloch Book (2002)

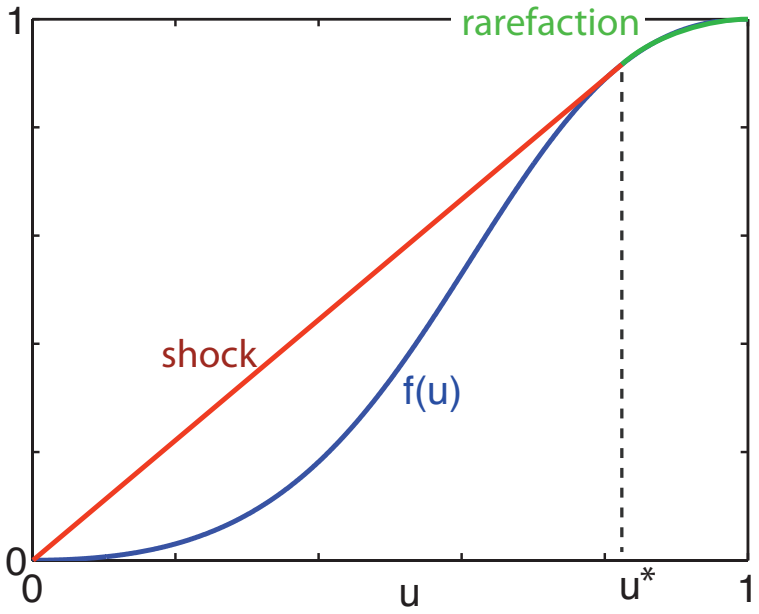




Buckley-Leverett solution 1942

Solve conservation law with initial jump from all water $u_- = 1$ to all oil $u_+ = 0$: water flooding:

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

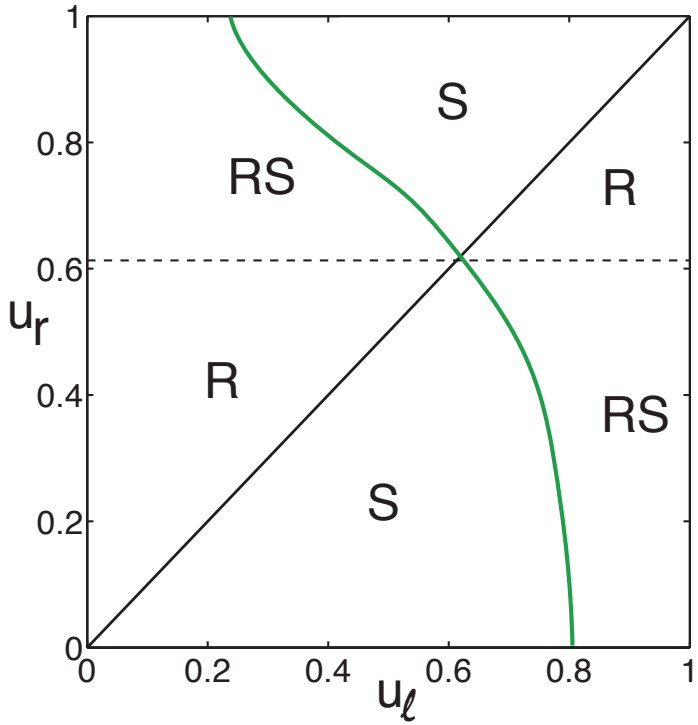


Solution: rarefaction from $u_- = 1$ to u^*; Lax shock from u^* to $u_+ = 0$: *rarefaction-shock.*

The Riemann Problem: Classical Solution

Solve conservation law with initial jump from u_ℓ to u_r

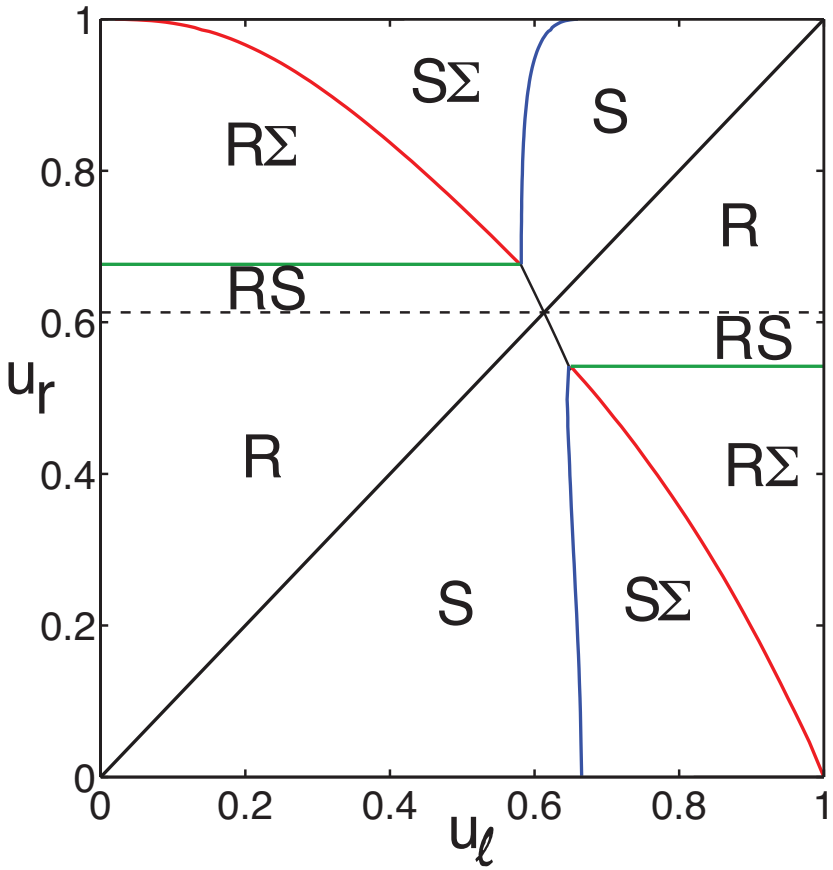
$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_\ell & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$



R: Rarefaction Wave
S: Lax Shock
RS : Rarefaction - Shock

The Riemann Problem; dynamic capillary pressure

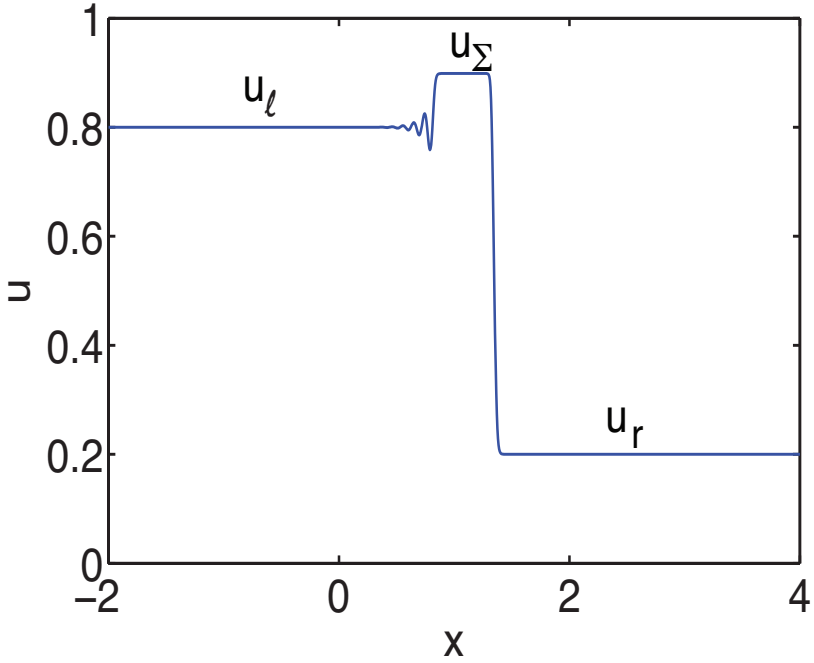
$$(RP): \quad u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_\ell & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$



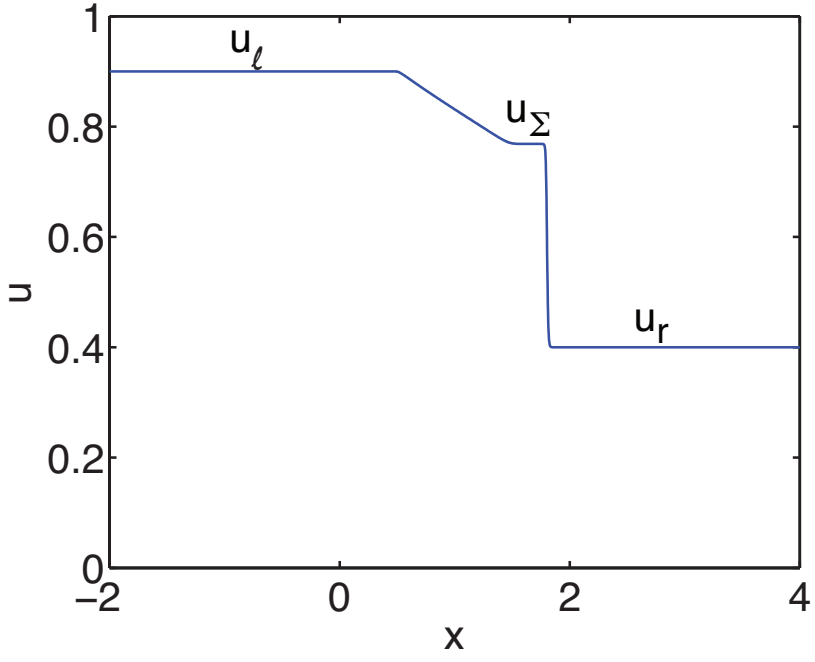
R: Rarefaction Wave
S: Admissible Lax Shock
Σ : Undercompressive Shock

PDE Simulations - Nonclassical Solutions

$u_\ell \in (u_{mid}, u_\Sigma)$:
Lax shock u_ℓ to u_Σ
undercompressive shock
u_Σ to u_r



$u_\ell \in (u_\Sigma, 1)$:
rarefaction wave u_ℓ to u_Σ
undercompressive shock
u_Σ to u_r

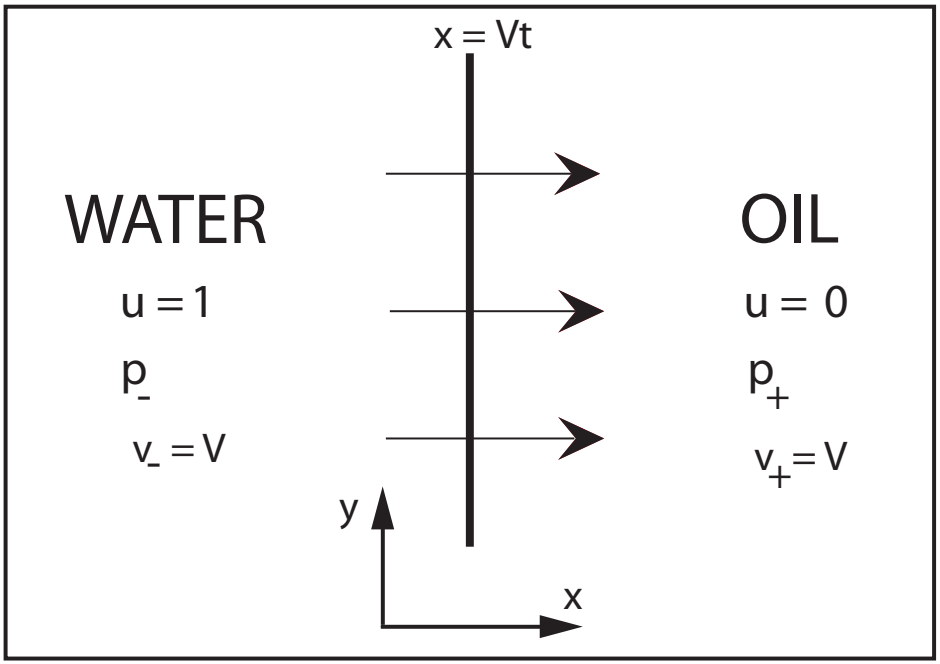


Classical Result: Saffman-Taylor Instability (1958)

Pure fluids: all water ($u = 1$) displacing all oil ($u = 0$)

No capillary pressure ($p_c \equiv 0$) $\Rightarrow$ sharp interface

Does not capture rarefaction-shock solution of Buckley-Leverett



Perturb pressure and interface,
$x = Vt + ae^{i\alpha y + \sigma t}$
wave number α, growth rate $\sigma(\alpha)$

Saffman-Taylor result:

$$\sigma = \sigma_1 \alpha + h.o.t., \; \sigma_1 = V \frac{1 - M}{1 + M}$$

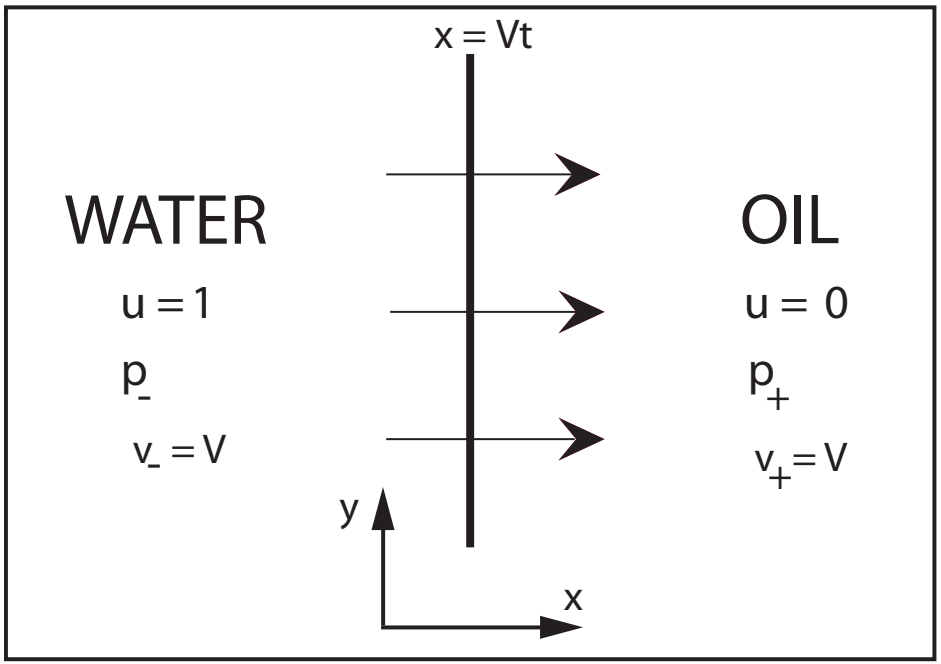
Fingering instability:
Mobility ratio $M = \frac{\mu^w}{k^w} / \frac{\mu^o}{k^o} < 1$

Saffman-Taylor Analysis: 1-dimensional base state

Interface $\bar{x} = Vt$; velocity $V = -m_\pm \partial_x \bar{p}_\pm$; mobility $m_\pm = k_\pm/\mu_\pm$

Continuous pressure $\bar{p}_\pm = -\frac{V}{m_\pm}(x - Vt) = -\frac{V}{m_\pm} z, \quad z = x - Vt$;

shock location: $\bar{z} = \bar{x} - Vt = 0$



Saffman-Taylor perturbation analysis

2-d equations: $v_\pm = -m_\pm \nabla p_\pm; \quad \nabla \cdot v_\pm = 0$

Thus, the pressure is harmonic: $\Delta p_\pm = \partial_z^2 p_\pm + \partial_y^2 p_\pm = 0 \qquad (1)$

$p_\pm(z, y, t) = -\frac{V}{m_\pm} z + q_\pm(z) e^{i\alpha y + \sigma t}$ interface: $z = \hat{z}(y, t) = a e^{i\alpha y + \sigma t}$

From (1): $\quad q_\pm'' - \alpha^2 q_\pm = 0, \quad q_\pm(\pm\infty) = 0$ (resp.)

Hence, $\quad q_\pm(z) = b_\pm e^{\mp \alpha z}$

Next: continuity of velocity and pressure at interface $z = \hat{z}$

Saffman-Taylor dispersion relation $\sigma(\alpha)$

$p_\pm(z,y,t) = -\frac{V}{m_\pm}z + b_\pm e^{\mp\alpha z}e^{i\alpha y+\sigma t}$ interface: $z = \hat{z}(y,t) = ae^{i\alpha y+\sigma t}$

Continuity of velocity and pressure at interface $z = \hat{z}$, retaining linear terms in coefficients $a, b_\pm$

Horizontal velocity:
$\frac{\partial x}{\partial t} = a\sigma e^{i\alpha y+\sigma t} + V = -m_\pm \frac{\partial p_\pm}{\partial z}|_{\hat{z}} = V \pm m_\pm b_\pm \alpha e^{i\alpha y+\sigma t}$

Thus, $b_\pm = \pm\frac{\sigma}{\alpha}a/m_\pm$

Similarly, $p_+ = p_-$ at $z = \hat{z}$: $-\frac{V}{m_+}a + b_+ = -\frac{V}{m_-}a + b_-$

3 linear equations for $a, b_\pm$, parameter $\frac{\sigma}{\alpha}$

Nonzero solution: $\boxed{\frac{\sigma}{\alpha} = V\frac{1-M}{1+M}}$ $M = \frac{m_+}{m_-} = \frac{\mu_-}{k_-}/\frac{\mu_+}{k_+} < 1$

Stability of Lax shocks

Variable saturation $u = u(x, y, t)$, pressure $p(x, y, t)$ $\quad p_c \equiv 0$, linearized equations

$$\sigma = \sigma_1 \alpha + \dots \quad \sigma_1 = V \frac{\lambda^T(u_-) - \lambda^T(u_+)}{\lambda^T(u_-) + \lambda^T(u_+)}$$

$\lambda^T =$ total mobility; shock $u = u_\pm$; $V =$shock speed

Yortsos and Hickernell, 1989; stability of smooth traveling wave matched asymptotics (with $p_c(u)$)

Conclusion: Long-wave stability $\iff \lambda^T(u_-) < \lambda^T(u_+)$

2-dimensional stability

2-d equations with $p_c \equiv 0$ variables u, p saturation, pressure:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (\lambda^w(u) \nabla p) &= 0 \\ \nabla \cdot \left(\lambda^T(u) \nabla p\right) &= 0 \end{aligned} \tag{1}$$

Interface $x = \hat{x}(y, t)$, normal in t, x, y : $(-\hat{x}_t, 1, -\hat{x}_y)$
Jump condition at shock: $([q] = q_+ - q_-)$

$$\begin{aligned} -\hat{x}_t[u] - [\lambda^w(u) p_x] + \hat{x}_y[\lambda^w(u) p_y] &= 0 \\ -[\lambda^T(u) p_x] + \hat{x}_y[\lambda^T(u) p_y] &= 0 \end{aligned} \tag{2}$$

Base shock: $u = \bar{u}_\pm, p = \bar{q}_\pm(x - Vt), \hat{x} = Vt$, constants $\bar{u}_\pm, \bar{q}_\pm, V$

$$V = \frac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-}, \quad f(u) = v^T \frac{\lambda^w(u)}{\lambda^T(u)} \quad \bar{q}_\pm = -\frac{v^T}{\lambda^T(\bar{u}_\pm)}$$

2-d stability: perturb variables and linearize equations

$$u = \bar{u}_\pm + u_\pm(z)e^{i\alpha y+\sigma t}, \quad p = \bar{q}_\pm z + q_\pm(z)e^{i\alpha y+\sigma t}$$
$$\hat{z} = \hat{x} - Vt = ae^{i\alpha y+\sigma t}, \quad z = x - Vt$$

Linearized equations: $(' = \frac{d}{dz})$

$$\begin{aligned} \sigma u - Vu' - \lambda^w(\bar{u}_\pm)\left(q'' - \alpha^2 q\right) + \frac{d\lambda^w}{du}(\bar{u}_\pm)\bar{q}_\pm u' &= 0 \\ \lambda^T(\bar{u}_\pm)\left(q'' - \alpha^2 q\right) + \frac{d\lambda^T}{du}(\bar{u}_\pm)\bar{q}_\pm u' &= 0 \end{aligned} \tag{3}$$

Relevant solutions for small α :

$u = 0, \quad q_\pm = b_\pm e^{\mp\alpha z}, \quad \pm(z - \hat{z}) > 0,$ – as for Saffman-Taylor!

Now linearize the jump conditions and find solvability condition for $b_\pm, a$

2-dimensional stability continued

$$\sigma a[\bar{u}] + [\lambda^w(\bar{u})q'] = 0 \tag{4}$$

$$[\lambda^T(\bar{u})q'] = 0 \tag{5}$$

Equation (5):

$$\boxed{\lambda^T(\bar{u}_+)b_+ = -\lambda^T(\bar{u}_-)b_-} \tag{6}$$

Then (4) implies

$$b_-\lambda^T(\bar{u}_-)(f(\bar{u}_+) - f(\bar{u}_-))v^T = -\frac{\sigma}{\alpha}a(\bar{u}_+ - \bar{u}_-)$$

But $(f(\bar{u}_+) - f(\bar{u}_-))/(\bar{u}_+ - \bar{u}_-) = V$, the shock speed, so

$$\boxed{b_-\lambda^T(\bar{u}_-)V = -a\frac{\sigma}{\alpha}v^T} \tag{7}$$

2-dimensional stability continued

Third equation comes from continuity of pressure

$$p = \bar{q}_\pm(x - Vt) + q_\pm(x - Vt)e^{i\alpha y + \sigma t}, q_\pm = b_\pm e^{\mp z} \quad (\pm(z - \hat{z}) > 0)$$

at $z = x - Vt = \hat{z}(y, t)$. Consequently,

$$\boxed{\bar{q}_+ a + b_+ = \bar{q}_- a + b_-} \tag{8}$$

Thus, $\quad (\bar{q}_+ - \bar{q}_-)a = -\frac{\sigma}{\alpha}\frac{v^T}{V}a\left(\frac{1}{\lambda^T(\bar{u}_+)} + \frac{1}{\lambda^T(\bar{u}_-)}\right), \quad$ (from (6,7))

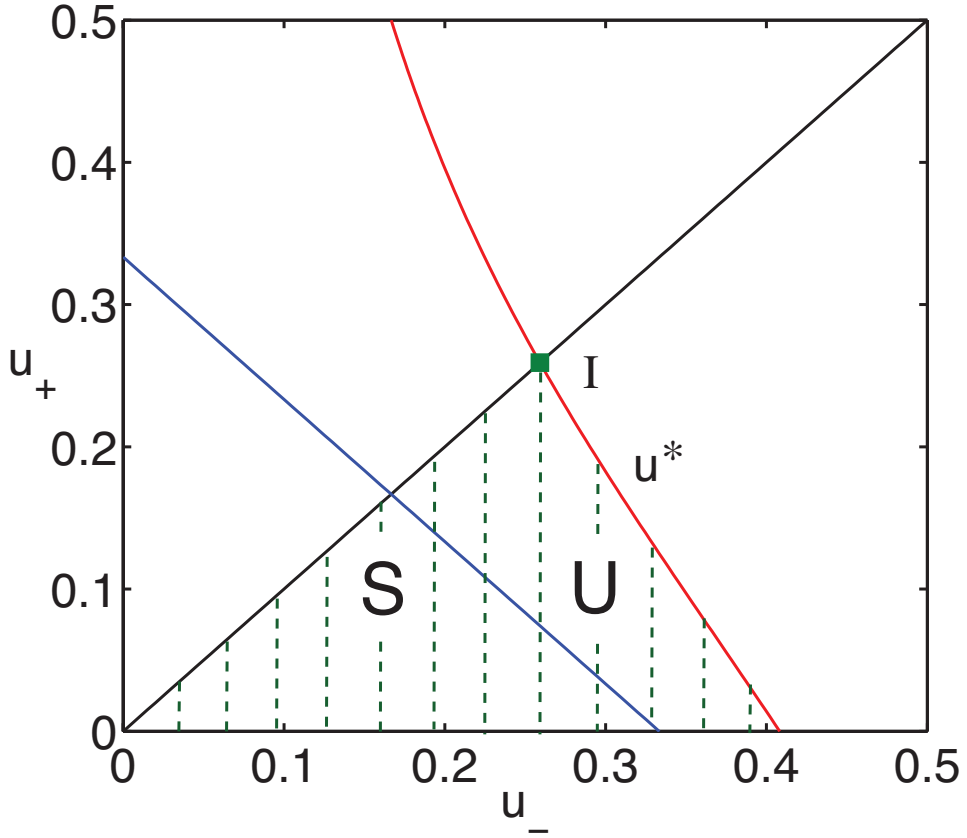
Since $\bar{q}_\pm = -\frac{v^T}{\lambda^T(\bar{u}_\pm)}$, we obtain

$$\boxed{\frac{\sigma}{\alpha} = V\frac{\lambda^T(\bar{u}_-) - \lambda^T(\bar{u}_+)}{\lambda^T(\bar{u}_-) + \lambda^T(\bar{u}_+)}}$$

Interpretation of stability condition: quadratic relative permeabilities: $k(u) = \kappa u^2$

Lax shocks for $u_+ < u_- \leq u_\alpha^*$

Stability boundary: $u_+ = -u_- + \frac{2M}{M+1}$



$M = 0.2 \quad \dfrac{2M}{M+1} = \dfrac{1}{3}$

Inflection point **I** of $f(u)$ at $u_I = 0.2591$

S: Stable Lax shocks

U: Unstable Lax shocks

Undercompressive shocks are all unstable

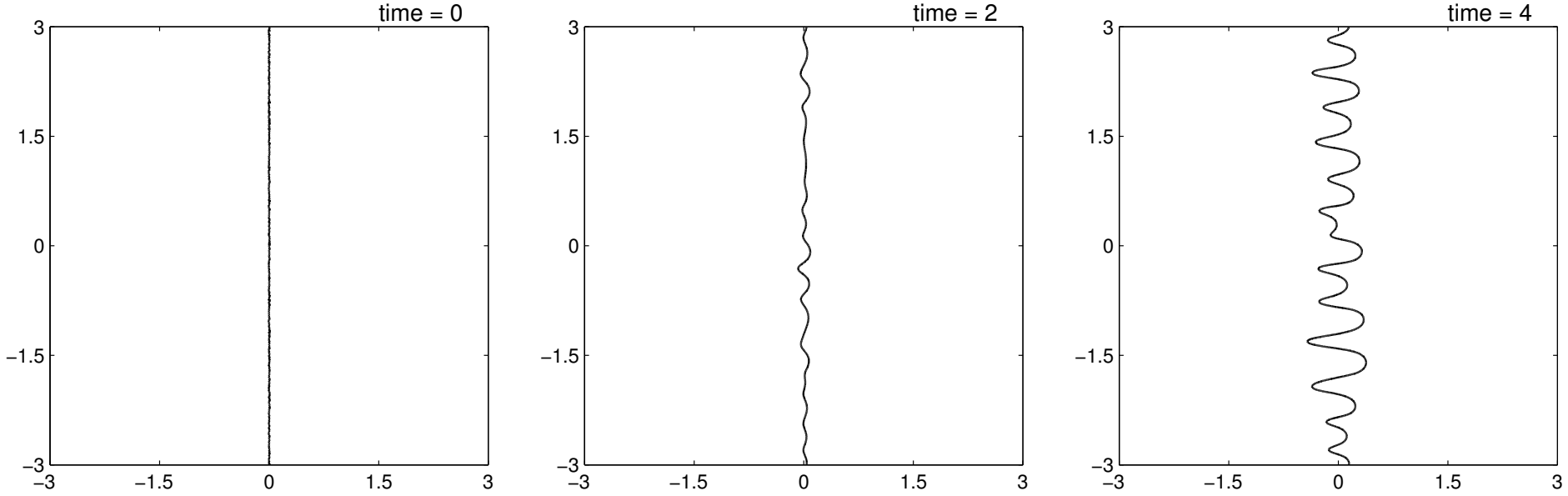
Fingering Instability - Zhengzheng's Simulations

Crank-Nicolson time step, centered difference spatial discretization, first-order upwind scheme for advection term with periodic side boundary conditions, moving frame

$\Delta t = O(10^{-3})$, $\Delta x = \Delta y = O(10^{-2})$

Initial condition: randomly perturbed hyperbolic tangent

$u_- = 0.2, u_+ = 0, M = 0.05$



Numerical Simulations - Stable case

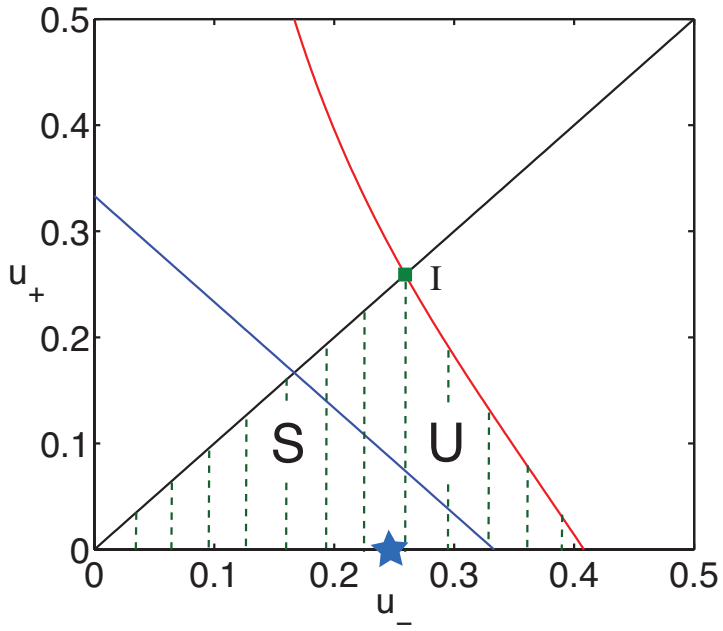
$M = 0.2$

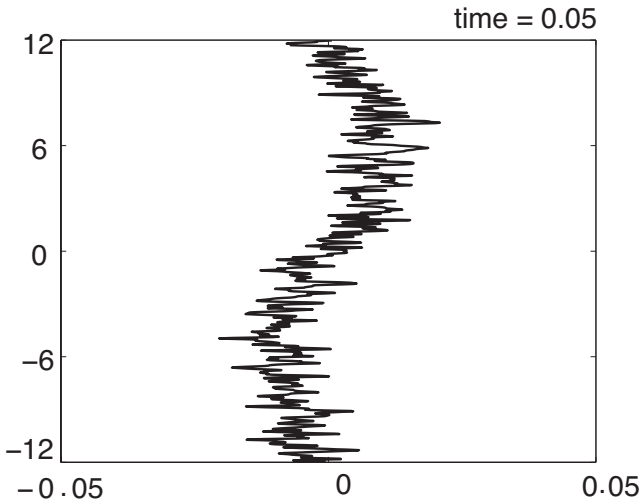
$u_- = 0.25, u_+ = 0$

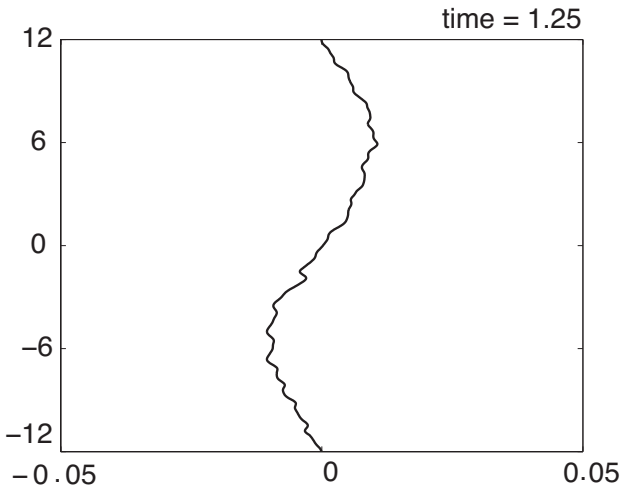
Oil-water mixture displaces oil

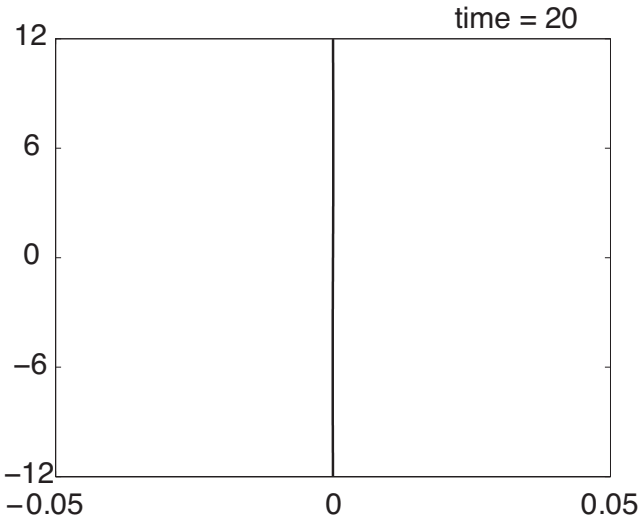
Lax shock

Initial perturbation decays



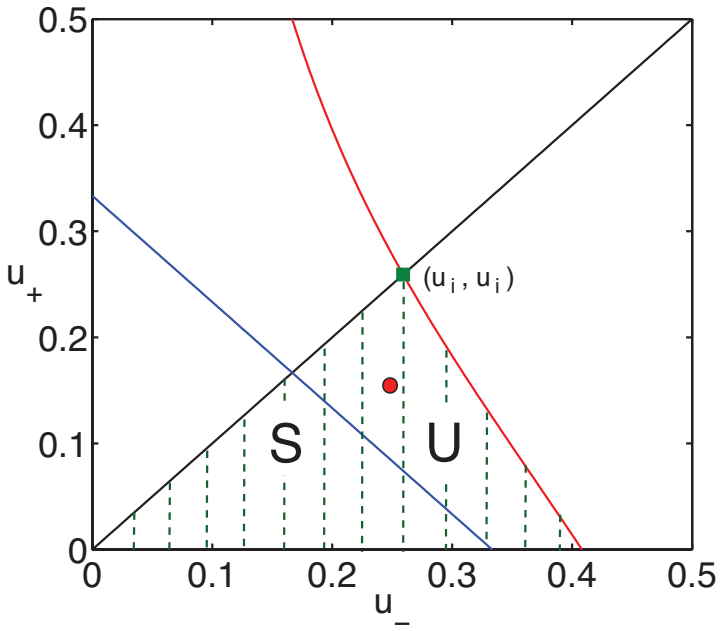


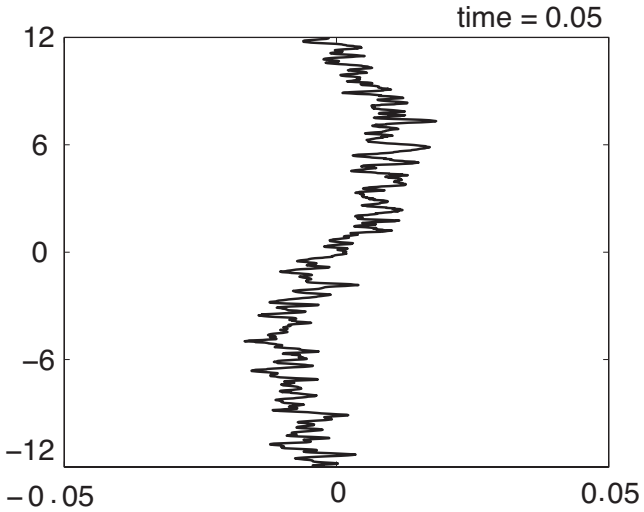


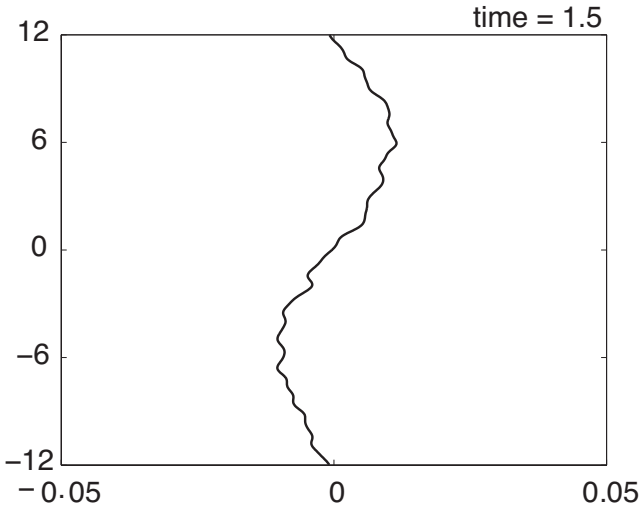


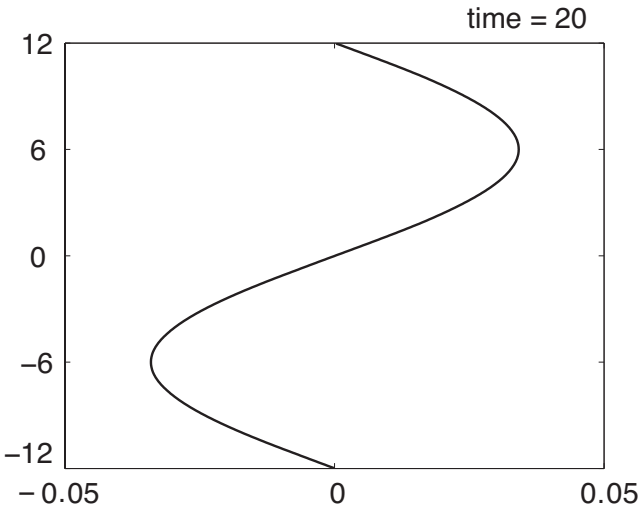
Numerical Simulations: Unstable case: Fingering Instability

- $M = 0.2$
- $u_- = 0.25, u_+ = 0.15$
- Lax shock
- Initial perturbation grows $\Rightarrow$ fingering instability









Conclusions

Undercompressive 1-d shocks with dynamic capillary pressure: non-monotone solutions

Analysis of stability/fingering instability in 2-d, connection to Saffman-Taylor instability

Surprising linear dependence of growth rate on wave number for long waves: distinguishes stable waves from unstable

Numerical simulations of full parabolic/elliptic system;
Riaz and Tchelepi (2006) also conducted numerical experiments

Oil/water mixture displacing oil can be stable.

K. Spayd and M. Shearer, SIAM J. Appl. Math. (2011)

K. Spayd, M. Shearer and Z. Hu, Applicable Analysis (2012).