Coating Flows on Slowly Rotating Cylinders

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In collaboration with

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Thank you for the invitation!

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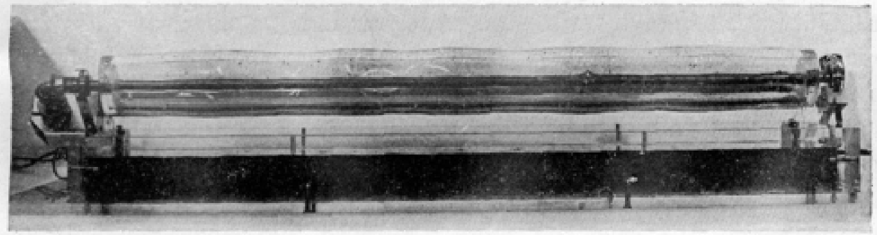
Warning: this talk does not cite all the work that it should cite. Our article is more responsible: "Nonnegative solutions for a long-wave unstable thin film equation with convection" with M. Chugunova and R.M. Taranets, SIAM Journal on Mathematical Analysis 2010.

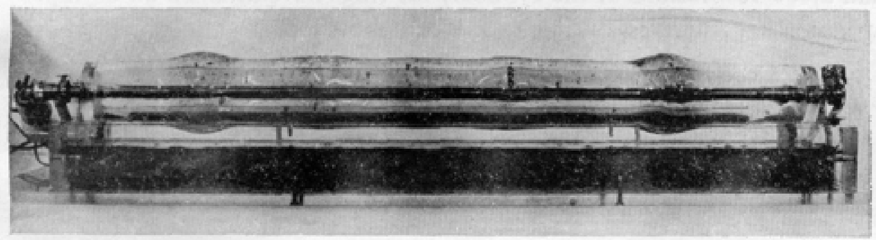
Honey on a knife

"It is a matter of common experience that if a knife is dipped in honey and then held horizontally, the honey will drain off; but that the honey may be retained on the knife by simply rotating it about its length. The question arises: what is the maximum load of honey that can be supported per unit length of knife for a given rotation rate?" — H.K. Moffatt, Journal de Méchanique 16(1977)5:651–673.

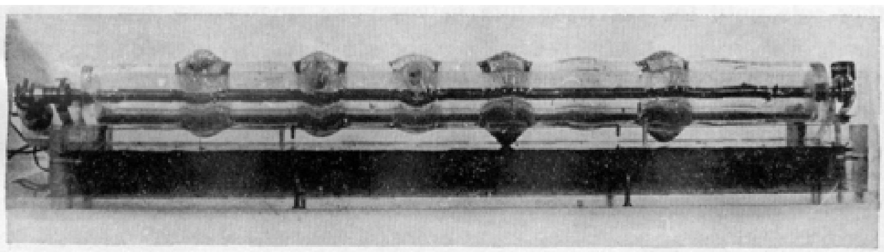
Coating experiments



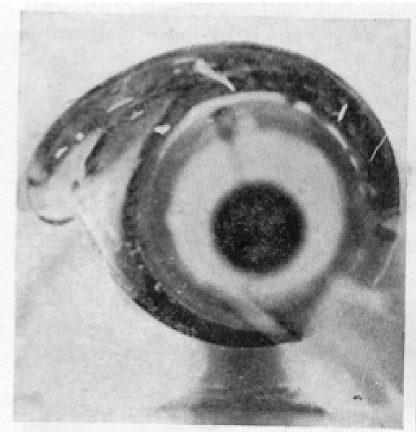
(b) 28.4 rpm



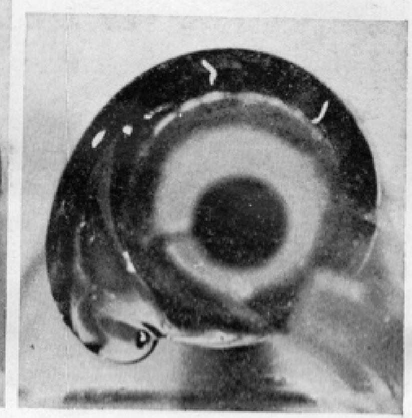
(c) 38.2 rpm



(d) 48.8 rpm



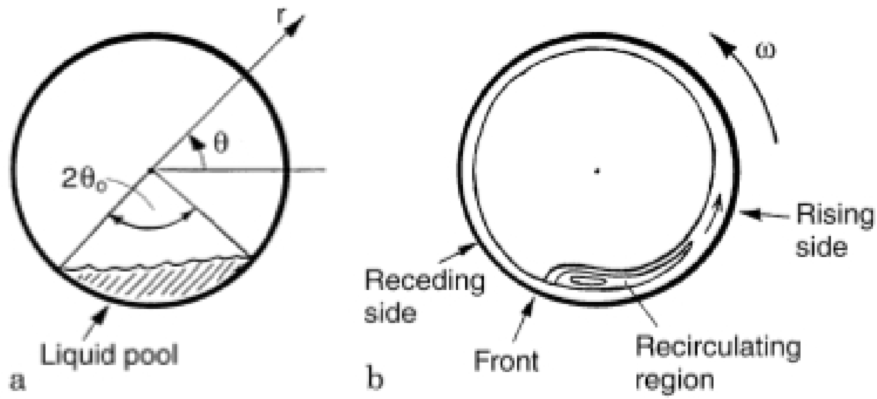
(a)

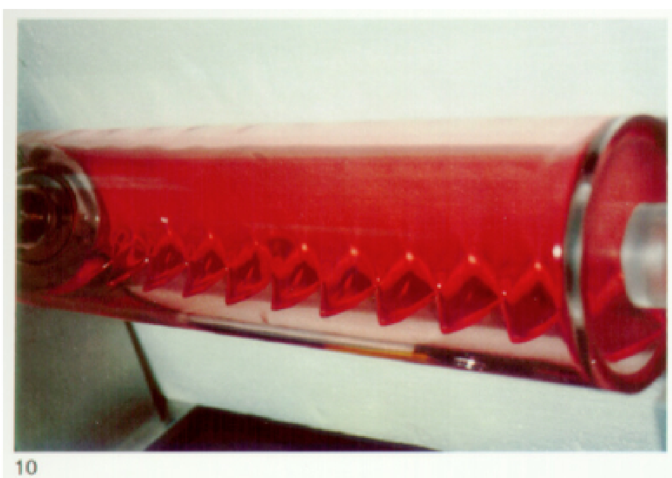


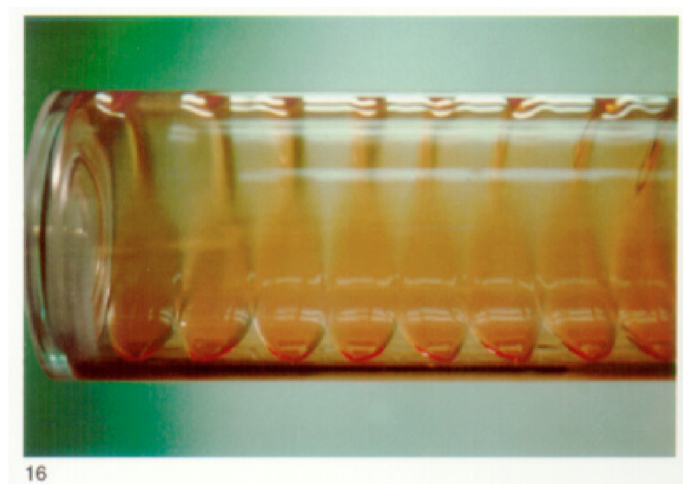
(b)

H.K. Moffatt, Journal de Méchanique 16(1977)5:651–673. Reproduced without author's permission.

Rimming experiments







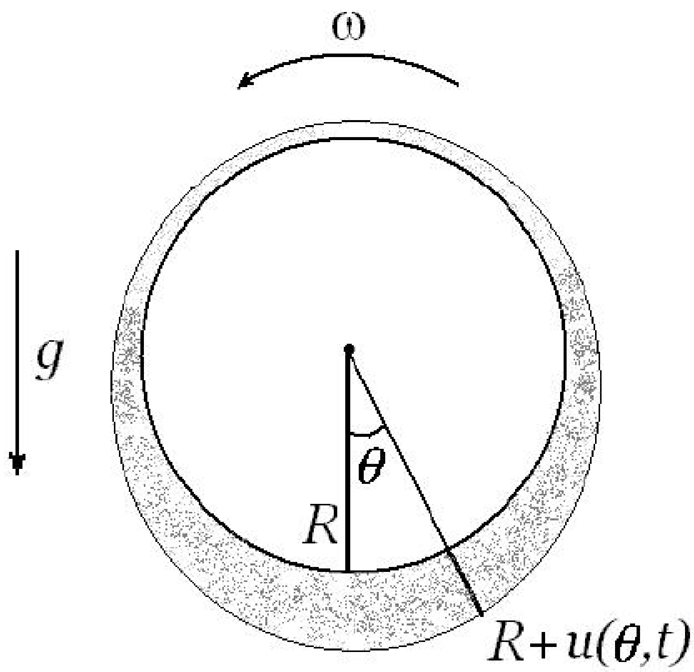
S.T. Thoroddsen and L. Mahadevan Experiments in Fluids 23(1997)1-13. Reproduced without authors' permission.

Movie

Marina and Stephen Morris (UofT Physics) supervised a third-year undergraduate, Gary Yan, who built a demonstration experiment similar to that of Thoroddsen and Mahadevan

Model parameters

Consider a thin liquid film on the outer surface of a cylinder:



R **is the radius of the cylinder.** ω **is the rate of rotation.** g **is the acceleration due to gravity.** ν **is the kinematic viscosity.** ρ **is the density.** σ **is the surface tension.**

Lubrication approximation model

Three dimensionless quantities: the Reynolds number $\mathrm{Re} = \frac{R^2\omega}{\nu}$, $\gamma = \frac{g}{R\omega^2}$, **and the Weber number** $\mathrm{We} = \frac{\rho R^3 \omega^2}{\sigma}$.

Modelling assumptions:

- **The fluid flow is modelled by the Navier Stokes equations**
- **There is no slip at the liquid/solid interface**
- **There is surface tension at the liquid/air interface**
- **If** $\bar{u}$ **is the average thickness of the fluid then** $\varepsilon = \bar{u}/R$ **is small**
- $\chi = \frac{\mathrm{Re}}{\mathrm{We}}\varepsilon^3$ **and** $\mu = \gamma\,\mathrm{Re}\,\varepsilon^2$ **have finite, nonzero limits as** $\varepsilon \longrightarrow 0$.

Lubrication approximation model

Assume the flow is constant along the length of the cylinder.

Pukhnachov, Journal of Applied Mechanics and Technical Physics 18(1977)3:344–351:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) + \frac{\chi}{3}\frac{\partial}{\partial \theta}\left(u^3\left[\frac{\partial u}{\partial \theta} + \frac{\partial^3 u}{\partial \theta^3}\right]\right) = 0$$

$$\theta \in [-\pi, \pi], \quad \frac{\partial^i u}{\partial \theta^i}(-\pi, t) = \frac{\partial^i u}{\partial \theta^i}(\pi, t) \text{ for } t > 0, \ \ i = \overline{0,3}$$

where $\mu = \gamma \text{ Re } \varepsilon^2 = \frac{gR}{\omega\nu}\varepsilon^2$ **and** $\chi = \frac{\text{Re}}{\text{We}}\varepsilon^3 = \frac{\sigma}{\nu\rho R\omega}\varepsilon^3$ **.**

Moffatt (1973, 1977) found the same evolution equation for the zero surface tension ($\chi = \sigma = 0$) case:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) = 0.$$

Steady States: zero surface tension

$$\frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) = 0 \quad \Longrightarrow \quad u - \frac{\mu}{3}u^3 \sin(\theta) = q$$

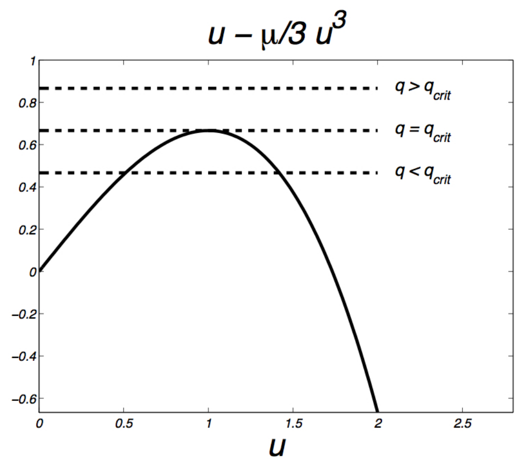
for some fixed q.

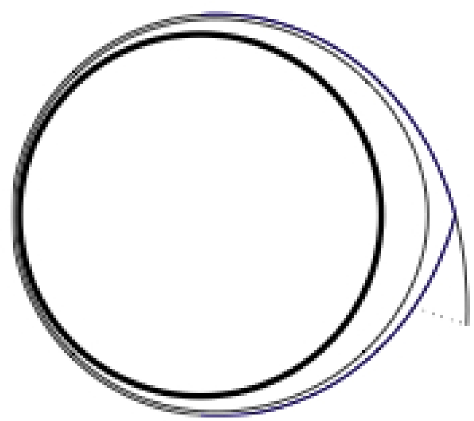
$$\theta = 0, \theta = \pi \quad \Longrightarrow \quad u(0) = u(\pi) = q$$

$$\theta = \pi/2 \quad \Longrightarrow \quad u(\pi/2) \textbf{ is a root of } u - \frac{\mu}{3}u^3 = q$$

$$\theta = 3\pi/2 \quad \Longrightarrow \quad u(3\pi/2) \textbf{ is a root of } u + \frac{\mu}{3}u^3 = q$$

At $\theta = \pi/2$ there might be no positive root if q is too big.





If $q < q_{crit}$ there is a smooth steady state, at $q = q_{crit}$ there's a corner, if $q > q_{crit}$ the steady state is discontinuous/shocks.

Steady states: nonzero surface tension

If there's no surface tension then $q_{crit} = \frac{2}{3\sqrt{\mu}} = \frac{2}{3\varepsilon}\sqrt{\frac{\omega\nu}{gR}}$ **The total amount of "honey" in the steady state is closely related to the value of** q **and so we see that the larger** ω **is, the more "honey" you can hold on your knife.**

If there is surface tension, Pukhnachov[1] **proved that** $q_{crit} \leq 2\sqrt{3/\mu} \approx 3.464/\sqrt{\mu}$**. We improve on this:**

Theorem(ChugPughTara 2010) For nonzero surface tension, there is no strictly positive 2π **periodic steady state with flux** $q > \frac{2}{3}\sqrt{\frac{2}{\mu}} \approx 0.943/\sqrt{\mu}$**. Hence** $q_{crit} \leq \frac{2}{3}\sqrt{\frac{2}{\mu}}$**.**

NB: our upper bound on q_{crit} **doesn't depend on surface tension, but** q_{crit} **will.**

[1] V.V. Pukhnachov, Mathematics and Continuum Mechanics (2004)191-199.

Steady states: nonzero surface tension, with rotation

The steady state satisfies

$$u - \frac{\mu}{3} u^3 \sin(\theta) + \frac{\chi}{3} u^3 \left(u_{\theta\theta\theta} + u_\theta \right) = q.$$

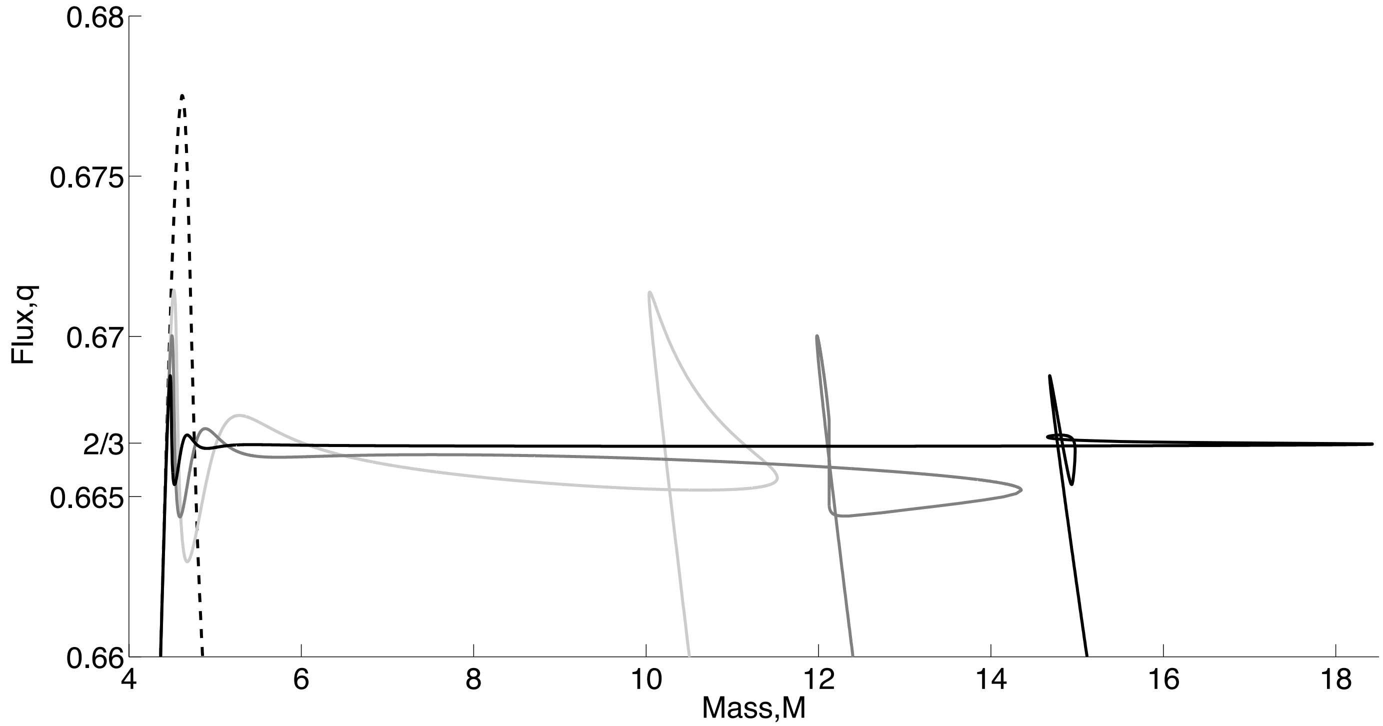
For a given flux, is there a smooth periodic solution? If yes, is it unique? For a given Mass, is there a solution? If yes, is it unique? How does surface tension affect things?

For zero surface tension, given a Mass, if there's a solution then it's unique. Benilov et al.[2] did extensive numerics & asymptotics and found that for some surface tension values, there are certain Masses which yield two solutions and others that yield three solutions. Marina Chugunova & Dmitry Pelinovsky[3], working with two undergraduates (Daniel Badali & Steven Pollack) found even smaller surface tension values for which a given Mass can yield up to five steady states.

[2] E.S. Benilov, M.S. Benilov, and N. Kopteva, J Fluid Mechanics 597(2008)91-118

[3] D. Badali, M. Chugunova, D.E. Pelinovsky, and S. Pollack, Physics of Fluids, 23(2011)

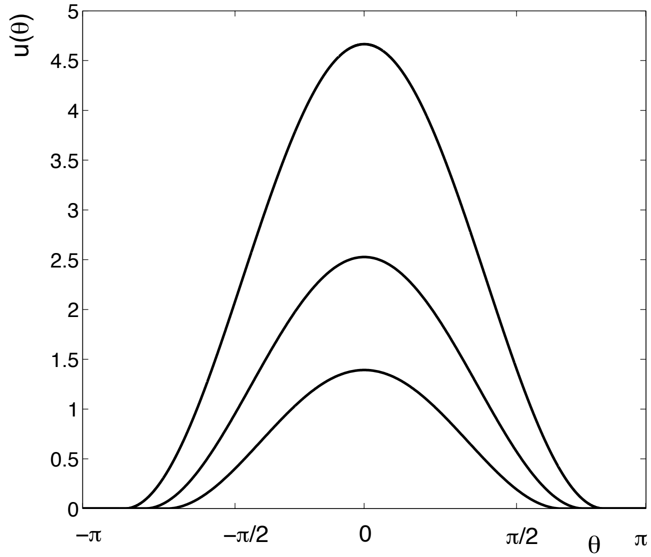
Steady states: nonzero surface tension, with rotation



$$u - \frac{1}{3}u^3 \sin(\theta) + \frac{\chi}{3}u^3 \left(u_{\theta\theta\theta} + u_\theta\right) = q.$$

Chugunova et al. $\chi = 0.005$ **(dashed),** 0.001 **(light gray),** 0.0005 **(dark gray),** 0.0001 **(black). Curves were generated using a custom-written turning-point algorithm and implemented in MATLAB.**

Steady states: nonzero surface tension, no rotation



Almut Burchard, Marina Chugunova, Ben Stephens (2010)

Analytical results for the nonzero surface tension, zero rotation case. 1) Given a mass, there is a unique nonnegative function of minimal energy. The minimizer is symmetric decreasing about $\theta = 0$, compactly supported, and meets the dry region at zero contact angle. The larger the mass, the smaller the dry region. 2) The minimizer is globally attractive for all strong solutions, however the H^1 distance from a strong solution to the minimizer can not decay faster than a power law $\sim t^{-2/3}$.

Steady states: nonzero surface tension, with rotation

Daniel Ginsberg & Gideon Simpson (2011) **Analytical results for the nonzero surface tension, nonzero rotation case. If there's rotation then the only steady states are strictly positive. The entire cylinder surface is coated.**

Global existence, rimming flow

Theorem(ChugPughTara 2010) Consider nonnegative initial data $u_0 \in H^1$ which has finite entropy. Then given a time $T < \infty$ there is a nonnegative strong generalized solution $u \in L^2(0,T; H^2_{per}(\Omega))$ for:

$$u_t + \Big(|u|^3(a_0\, u_{\theta\theta\theta} + a_1 u_\theta + a_2 w'(\theta))\Big)_\theta + a_3 u_\theta = 0$$

where a_1, a_2, a_3 are arbitrary constants, constant $a_0 > 0$, and $w(\theta)$ is periodic.

Pukhnachov's model

$$u_t + \Big[|u|^3\,(u_{\theta\theta\theta} + \alpha^2\, u_\theta - \sin\theta) + \omega u\Big]_\theta = 0,\ \ \theta \in \Omega = (-\pi, \pi)$$

is a special case of the equation above.

Sketch of proof

1. **Find dissipated quantities.**
2. **Prove short–time existence of solution.**
3. **Prove that short–time solution can be continued in time.**

Finding Dissipated Quantities **Unlike for** $u_t = -(|u|^n u_{xxx})_x$ **one cannot show that the energy and entropy are dissipated independent of one another. We show that a linear combination of the Dirichlet energy and the entropy is dissipated, and find a uniform upper bound that holds for short times.**

Energy and Entropy

Energy for approximate solutions:

$$\mathcal{E}_\epsilon(T) = \frac{1}{2}\int a_0 h_{\epsilon x}^2(x,T) - a_1 h_\epsilon^2(x,T) - 2a_2 w(x) h(x,T)\, dx$$

α-Entropy for approximate solutions:

$$\int G_\epsilon^{(\alpha)}(h_\epsilon(x,T))\, dx \quad \textbf{where} \quad G_\epsilon^{(\alpha)\prime\prime}(y) = \frac{y^\alpha}{f_\epsilon(y)}.$$

Given nonnegative initial data that has finite entropy there is a time T_{loc} such that for $T \in [0, T_{loc}]$

$$\mathcal{E}_\epsilon(T) \le C + KT$$

and

$$\int h_{\epsilon,x}^2(x,T) + \tilde{C}\, G_\epsilon^{(\alpha)}(h_\epsilon(x,T))\, dx + \iint_{Q_T} \beta h_\epsilon^\alpha h_{\epsilon,xx}^2 + \gamma h_\epsilon^{\alpha-2} h_{\epsilon,x}^4\, dxdt \le C$$

where $\alpha \in (-1/2, 0)$ and $\tilde{C}$, C, and K are finite.

Continuation argument

Going from local to global in time. **Having take $\epsilon \to 0$, we prove the strong, nonnegative solutions satisfy the bound**

$$\mathcal{E}_0(T_{loc}) \leq C + K\,T_{loc}$$

at the "final" time T_{loc}. This can then be used to prove a bound for the H^1 norm of the solution at time T_{loc}:

$$\frac{a_0}{4}\|h(\cdot, T_{loc})\|_{H^1}^2 \leq \mathcal{E}_0(T_{loc}) + K\,T_{loc} + K_2.$$

This is then used in a proof by contradiction: assume the solution cannot be extended globally in time. Then either the H^1 norm or the α-Entropy diverges. The H^1 norm cannot diverge by the above. Since the α-Entropy is controlled by the H^1 norm, it cannot diverge either. Hence the solution can be continued in time.

Thank you !

THANK YOU FOR YOUR INTEREST