Network flows for image processing
Part II-Bis: Total Variation and Applications

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- Marc Sigelle (Telecom Paris)
- Antonin Chambolle (Ecole Polytechnique)

Separable TV minization

I) Total Variation Minimization and parametric max-flow

- Both continuous and discrete point of view
- Separable cases: Rudin-Osher-Fatemi, curve evolution

Generalized perimeters and Coarea-Formula

- Idea: $\mathcal{J}$ is a "generalized perimeter"
  Let $\Omega \subset \mathbb{R}^N$. Let $\mathcal{J}$ be nonnegative, defined on measurable subsets of Ω be such that
  - $\mathcal{J}$ is l.s.c. with respect to the L^1 convergence
  - $\mathcal{J}(A \cup B) + \mathcal{J}(A \cap B) \;\leq\; \mathcal{J}(A) + \mathcal{J}(B)$ [*Submodular* Lovasz 82]
  - $\mathcal{J}(\emptyset) = \mathcal{J}(\Omega) = 0$
- Idea: J is a "Total Variation"
  Extension to any $u \in L^1(\Omega) \quad \to J(u)$
  by the Generalized coarea formula [Choquet,Lovasz,Visintin]:

$$J(u) \;=\; \int_{-\infty}^{\infty} \mathcal{J}(\{u > \lambda\})\, d\lambda$$

  where $\{u > \lambda\} = \{x \in \Omega \,|\, u(x) > \lambda\}$ (level sets)

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Links

- Proposition (Bouchitté)
  J is 1-homogeneous, convex, l.s.c., $\forall c\, J(\cdot + c) = J(\cdot)$
- Conversely:
  If J is a convex, l.s.c., nonnegative functional wich satisfies the generalized coarea formula
  then $A \mapsto J(\chi_A)$ is a “generalized perimeter” in our sense.

A matter of wording

- Same concepts but different words:
  - Analysis / Level sets / Coarea formula
  - Combinatorics / unary representation / Lovasz extension
  - Game Theory/Fuzzy Logic / $\alpha-$cut / Choquet Integral
- Similarity with Mathematical Morphology [Guichard Morel], ”stack filters”
  - $\mathcal{J} \equiv$ monotone filter on sets
  - $J \equiv$ monotone filter on functions
  - Example: $J \equiv \| \cdot - v\|_1 + TV(\cdot)$ is morphological [D. 05]

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Examples

- Total variation/Perimeter: $P(A,\Omega) = \int_\Omega |D\chi_A|$ with

$$\int_\Omega |Du| \;=\; \sup\left\{ \int_\Omega u \mathrm{div}\psi \;:\; \psi \in C_c^\infty(\Omega;\mathbb{R}^N)\,, |\psi(x)| \leq 1 \;\forall x \right\}$$

- Nonlocal functionals (k nonnegative) [Buades et al. 05]:

$$J(u) \;=\; \int_{\Omega\times\Omega} k(x,y)|u(x) - u(y)|\, dxdy$$

- Discrete energies of the form (computational point of view)

$$J(u) = \sum_{i,j} w_{ij}|u_i - u_j|$$

- Note that $\sqrt{u_x^2 + u_y^2}$ is NOT submodular

- We minimize the following energy wrt. A (with $v \in L^1(\Omega)$)

$$\mathcal{E}(A|v) \;=\; \mathcal{J}(A) - \int_A v\,dx$$

Lemma $v' < v \Rightarrow A' \subseteq A$

- Several approaches for the proof
  - Stochastically-based [D. Sigelle 04]
  - Variational approach [Chambolle 05] $\leftarrow$ the simplest
  - Algorithmically-based [Gallo-Grigoriadis-Tarjan 89]

ROF Comparison Principle

- Let u minimize Rudin-Osher-Fatemi (ROF)

$$E(u|v) \;=\; J(u) \;+\; \frac{1}{2}\int_\Omega (u-v)^2\,dx\,.$$

  Then: for any s, $\{u > \lambda\}$ and $\{u \geq \lambda\}$ are the minimal and maximal solution of

$$\min_A \mathcal{E}(A, v-\lambda)$$

- If u, u' respectively minimize $E(u|v)$ and $\mathcal{E}(u'|v')$ then $v \leq v' \Rightarrow u \leq u'$.
- Optimization point of view:
  - To minimize $\mathcal{E}(\cdot|v) \rightarrow$ Minimize the associated ROF pb and threshold
  - To minimize ROF $E(\cdot|v) \rightarrow$ Solve a series a $\mathcal{E}(\cdot|\lambda - v)$ and reconstruct u from its level sets

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Minimizing ROF via maximum-flow (1/4)

- Reduction of ROF to series a ”generalized perimeter” functionals
- Pairwise interactions perimeters $\mathcal{J}(A) = \sum_{i,j} w_{ij}|\theta_i - \theta_j|$
  where $\theta = \mathbb{1}_A$, i.e., $\theta_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$
- Each binary problem to solve has the form

$$\min_{\theta} \sum_{i,j} w_{ij}|\theta_i - \theta_j| + \sum_i (\lambda - v_i)\theta_i$$

$\rightarrow$ Perimeter + data fidelity

$\Rightarrow$ Globally solvable using maximum-flow/minimum-cut

Minimizing ROF via maximum-flow (2/4)

Strategies of reconstruction, i.e., order for the series of $\{\lambda_l\}$

- Dyadic (dichotomy/bitonic) search:
  $\rightarrow \|\hat{u} - u^*\|_\infty \leq \epsilon$ in $O(\log_2 \epsilon)$ [D. Sigelle 06]
- A much better idea: Re-use the maximum-flow result
  - Inclusion property says:
    - the set a nodes connected to the source is increasing (as $\lambda \nearrow$)
    - equivalent to capacity arcs "Source $\rightarrow$ nodes" $\nearrow$ (parametric max-flow)
    - equivalent to convexity of data fidelity of TV problems
  - [Gallo-Grigoriadis-Tarjan 89] showed
    - the comparison principle
    - time complexity of a parametric max-flow = time of a one max-flow
  - Algo Parametric: lowest to greastest
  - Algo Parametric: dyadic search [Chambolle D. 06/07] [Hochbaum 01]
    $\rightarrow$ Solution $\epsilon = 0$ in *strongly* polynomial time ($\log \frac{1}{\epsilon}$ becomes $\log n$)

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Minimizing ROF via maximum-flow Time results (3/4)

$\frac{1}{2\lambda}\|u - v\|^2_{l^2}$, 8-neighbors, 256 gray-levels
original, $\lambda = 10$, $\lambda = 20$, $\lambda = 60$




Minimizing ROF via maximum-flow Time results (4/4)

$\frac{1}{2\lambda}\|u-v\|_{l^2}^2$, 8-neighbors, 256 gray-levels, Intel Core 2 Duo 2.4 Gz

PR: Push/relabel FIFO max-flow algorithm

BK: [Boykov Kolmogorov 04] max-flow algorithm (efficient for sparse graph)

Images (size)	Approach	$\lambda = 10$	$\lambda = 20$	$\lambda = 60$
Girl (256^2)	Parametric PR	5.32	9.24	12.67
	Parametric BK	2.97	3.25	4.04
	Dyadic Parametric PR	1.49	2.05	3.42
	Chambolle/Darbon-Sigelle	0.55	0.72	1.08
	Dyadic Parametric BK	0.42	0.55	0.81
Girl (512^2)	Parametric PR	41.14	71.39	140.79
	Parametric BK	11.90	13.19	17.08
	Dyadic Parametric PR	10.15	12.92	22.17
	Chambolle/Darbon-Sigelle	2.34	3.05	5.01
	Dyadic Parametric BK	1.86	2.54	4.19

Geometric evolutions (1/2)

- Goal: Computing anisotropic mean curvature flows
- Almgren-Taylor-Wang's implicit approach [93]:
  - Given a time-step $h > 0$
  - A current set $A^{n-1} \approx A((n-1)h)$, the curve is ∂A^{n-1}
  - Find the new curve evolved by mean curvature by solving

$$\min_A \mathcal{J}(A) \;+\; \frac{1}{h}\int_A d_{A^{n-1}}(x)\,dx$$

where $\mathcal{J}$ is an isotropic or anisotropic perimeter and d_A the signed distance to ∂A.

Geometric evolutions (2/2)

- Parametric-based approach
  a) minimize an approximation of min $E(u, d_{A^{n-1}})$ (around the level 0)
  c) find the new set $A^n = \{u \geq 0\}$.
- DEMO

Solving Stefan's equation system using [Almgren 93]
($\rightarrow$solve one step evolution + update temperature)