Network flows for image processing
Part II: Total Variation and Applications

Jerome Darbon

CNRS / CMLA-ENS Cachan
Mathematics Department UCLA

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Notations

- Discretization

$$\begin{array}{lll} s & \in & S \\ u_s & \in & [0, L-1] \\ s \sim t & \rightarrow & (s, t) \end{array}$$

finite discrete grid / finite number of gray-levels / neighbors $\rightarrow$ cliques (C-connectivity)

- Level sets

$$u^{\lambda} = \{s \in S | \mathbb{1}_{u_s \leq \lambda}\}$$

- We consider level sets as variables

A) Total Variation minimisation with convex fidelity

- Convex problem
- Image restoration
- Reformulation through level sets
- Polynomial algorithm
- Results

TV: Reformulation through level sets

- Total Variation

$$\underbrace{\int_\Omega |\nabla u| = \int_{\mathbb{R}} P(u^\lambda)}_{\text{Co-area formula}} = \sum_{\lambda=0}^{L-2} P(u^\lambda) = \sum_{\lambda=0}^{L-2} \underbrace{\sum_{(s,t)} w_{st}\,|u_s^\lambda - u_t^\lambda|}_{R_{st}(u_s^\lambda, u_t^\lambda)}$$

- Data fidelity

$$D\,(u_s,\ v_s) = \sum_{\lambda=0}^{L-2} \underbrace{(D(\lambda+1, v_s) - D(\lambda, v_s))\,(1-u_s^\lambda)}_{D^\lambda(u_s^\lambda, v_s)} + D(0, v_s)$$

$$212E(u|v) = \sum_{\lambda=0}^{L-2} \underbrace{\left(R_{st}(u_s^\lambda, u_t^\lambda) + D^\lambda(u_s^\lambda, v_s)\right)}_{\text{binary MRF}} + C = \sum_{\lambda=0}^{L-2} E^\lambda(u^\lambda, v)$$

TV: Independent minimization and reconstruction

- Minimize (MAP) independently each binary MRF
  - $E(u|v) \to E(\{u\}^\lambda, v)$
  - Family of minimizers: $\{\hat{u}^\lambda\}_{\lambda=0\dots(L-2)}$
- Reconstruction: $\hat{u}_s = \inf\{\lambda | \mathbb{1}_{\hat{u}_s^\lambda} = 1\}$ provided that

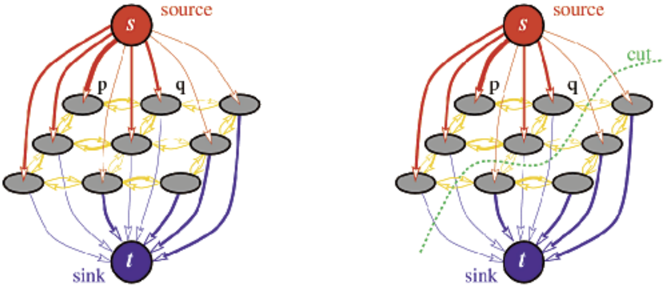
$$u_s^\lambda \leq u_s^\mu \quad \forall \lambda < \mu \quad \forall s \quad (monotony)$$

monotone lemma

If all conditional energies $E(u_s \mid N_s, v_s)$ are convex *functions of grey level $u_s \in]0, L-1[$, for any neighborhood configuration and local observed data, then the monotone property holds.*

$\Rightarrow$ "convex+TV" models satisfies lemma's conditions

- How to minimize a binary Markovian energy ?
- Build a graph such that its maximum flow yields an optimal labelling



- construction of the graph: [*Kolmogorov and Zabih*, PAMI 2004]
- maximum flow algorithm: [*Boykov and Kolmogorov*, PAMI 2004]
- in practice quasi-linear (w.r.t number of pixels)
- in theory $O(n^2\sqrt{m})$ (n=# pixels, m=# arcs)

picture taken from Boykov *et al.* PAMI 2001

TV: Graph construction conditions

- Regularity conditions described in [*Picard et al. Networks 1975*] [*Barahona* J. Physics A 1985] [*Kolmogorov et al.*, PAMI 2004]
- Binary Markovian energy with pairwise interaction

$$E(x_1, \dots, x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i, x_j)$$

- $E^i(x_i)$: always regular
- $E^{i,j}(x_i, x_j)$: regular iff submodular, i.e.

$$E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(1,0) + E^{i,j}(0,1)$$

- TV case: $\sum_{st} w_{st}|u_s - u_t|$

$$0 \leq w_{st} \quad \text{ok}$$

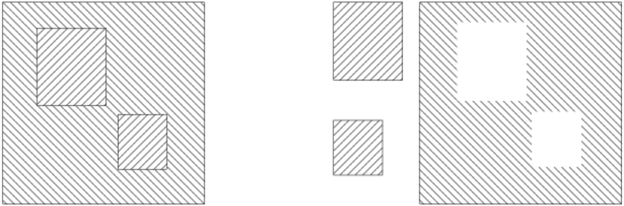
- Decomposition through level sets (recall)

$$E(u|v) = \sum_{\lambda=0}^{L-2} E^{\lambda}(u^{\lambda}|v)$$

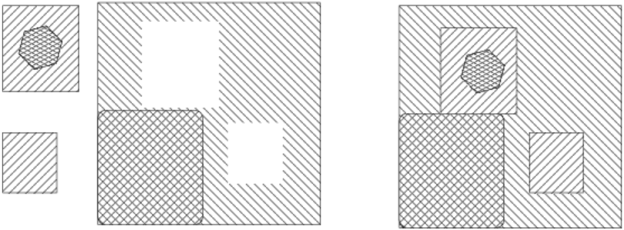
- Direct approach $\Longrightarrow$ $(L-1)$ minimum cost cuts per pixel
- A divide-and-conquer algorithm with dichotomy
  - decompose into independent subproblems
  - solve each subproblem
  - recompose the solution

TV: Minimization algorithm

- **Decomposition**: Solve for a level λ ; connected components



- **Independance**: $w_{st}|u_s^\lambda - u_t^\lambda|$
- **Solving sub-problems and recomposition**



- Thresholding: dichotomy on $[0, L-1] \Longrightarrow \log_2(L)$ maximum flows

TV: Results

TV: Results, additive Gaussian noise

$\mu = 0, \sigma = 12$

restored image ($\beta = 23, 5$)

$\mu = 0, \sigma = 20$

restored image ($\beta = 44, 5$)

TV: Results, time

TV: Results, time

- size (512×512) ; L^2; time in seconds

Image	$\beta = 2$	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$
Lena	2,07	2,24	2,53	3,04	3,40	3,75
Aerial	2.13	2.24	2.45	2.75	3.06	3.28
Barbara	2.07	2.26	2.51	2.87	3.22	3.50

- size (256×256) ; L^2; time in seconds

Image	$\beta = 2$	$\beta = 5$	$\beta = 10$	$\beta = 20$	$\beta = 30$	$\beta = 40$
Lena	0.51	0.54	0.60	0.72	0.80	0.87
Aerial	0.55	0.57	0.61	0.67	0.74	0.78
Barbara	0.53	0.55	0.60	0.69	0.75	0.80
Girl	0.52	0.55	0.64	0.75	0.85	0.92

Experiments performed on a Pentium 4 3 GHz

TV: Partial conclusion

- Exact solution for ”convex+TV” models
- Reformulation through level sets
- Polynomial algorithm. Complexity $\log_2(L) \cdot T(n, (C+2)n)$
- Similar algorithm proposed by
  - Direct approach: Zalesky [*Zalesky* JAM 02]
  - Dichotomy: Chambolle [*Chambolle* EMMCVPR 05]
  - Parametric max-flow: Hochbaum [*Hochbaum* ACM 01]
- More details and generalization to convex priors and ”levelable” priors in
  - J. Darbon and M. Sigelle. *Image Restoration with Discrete Constrained Total Variation*
    Part I: Fast and Exact Optimization
    Part II: Levelable Functions, Convex Priors and Non-Convex Cases
    In Journal of Mathematical Imaging and Vision, December 2006

where $T(n, m)$ is the time required to perform a maximum flow on a graph of n nodes and m edges.

$L^1 + TV$ is a contrast invariant filter

I) Minimum cuts in networks and integer programming

II) Total Variation minimization with convex fidelity

III) $L^1 + TV$ as a constrast invariant filter

- Definition and Theorem
- Experiments

Definitions

- Level sets and threshold decomposition principle
  $L^\lambda(u) = \{x \in \Omega | u(x) \leq \lambda\}$, $U^\lambda(u) = \{x \in \Omega | u(x) > \lambda\}$
- Notation $L^\lambda(u) = u^\lambda$
- Reconstruction from level sets
  $u(x) = \sup\{\lambda | \mathbb{1}_{L^\lambda(u)(x)} = 1\} = \inf\{\mu | \mathbb{1}_{U^\mu(u)(x)} = 1\}$
- Any continuous non-decreasing function g is called a *continuous change of contrast*.
- **Lemma** [Guichard Morel 02 ISMM]:

$$\forall \lambda \, \exists \mu \;\; L_\lambda(g(u)) = L_\mu(u) \;\; .$$

- F is a contrast invariant filter iff

$$g(F(u)) = F(g(u))$$

Links with mathematical morphology

$L^1 + TV$ is invariant with change of contrast

Definition and lemma

• *A continuous and non-decreasing function* $h : \mathbb{R} \mapsto \mathbb{R}$ *, is called a continuous change of contrast.*
• *A filter* $\mathcal{T}$ *is invariant w.r.t. a change of contrast iff it satisfies:*

$$h(\mathcal{T}(u)) = \mathcal{T}(h(u)) \ ,$$

where u is an image and h a change of contrast.

If $\hat{u}$ minimizer for $E^{L^1+TV}(\cdot|v)$ then $h(\hat{u})$ minimizer for $E^{L^1+TV}(\cdot|h(v))$
It is enough to show that for all λ a minimizer for $h(v)^\lambda$ is $h(\hat{u})^\lambda$

$L^1 + TV$: Idea of the proof

- Reformulation of $L^1 + TV$ through level sets
  - Coarea Formula:

$$\int_\Omega |\nabla u| = \int_{\mathbb{R}} \int_\Omega |\nabla \chi_{u^\lambda}|\, d\lambda = \int_{\mathbb{R}} P(u^\lambda) d\lambda$$

  - Fidelity terms:

$$|u(x) - v(x)| = \int_{\mathbb{R}} \left|u^\lambda(x) - v^\lambda(x)\right| d\lambda$$

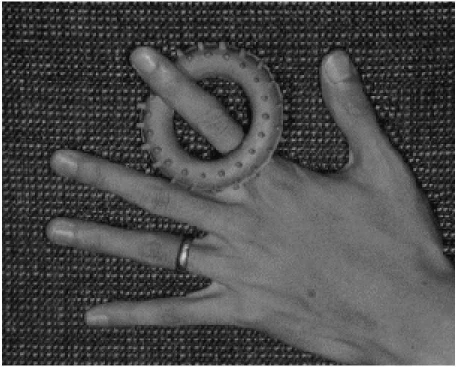
- Finally

$$E_v(u) = \int_{\mathbb{R}} E_v^\lambda(u^\lambda, v^\lambda) d\lambda$$

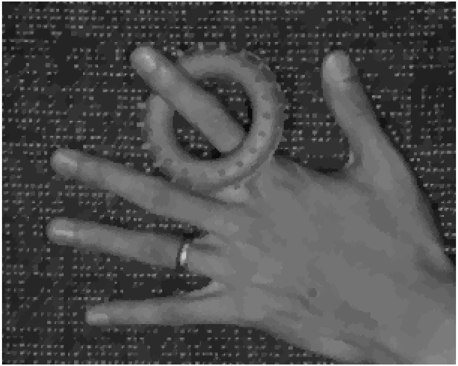
where

$$E_v^\lambda(u^\lambda, v^\lambda) = \int_\Omega \left(\beta\, |\nabla \chi_{u^\lambda}| + \left|u^\lambda(x) - v^\lambda(x)\right| dx\right)$$

$L^1 + TV$: Experiments



original



$\beta = 1,5$

$L^1 + TV$: Results

$\beta = 2,5$

$\beta = 3,0$

$L^1 + TV$: Results

Original image

$L^1 + TV$: Results

[Yin *et al.* 05 CAM report] Result with $\beta = 2$ (left) and difference with the original i

$L^1 + TV$: Partial Conclusion

- Original proof in *J. Darbon. Total Variation Minimization with L^1 Data Fidelity as a Contrast Invariant Filter. 4th International Symposium on Image and Signal Processing and Analysis (ISPA 2005).*
- Proof in the discrete case in *J. Darbon and M. Sigelle. Image Restoration with Discrete Constrained Total Variation: Part I: Fast and Exact Optimization.*

$L^1 + TV$ on the FLST-tree

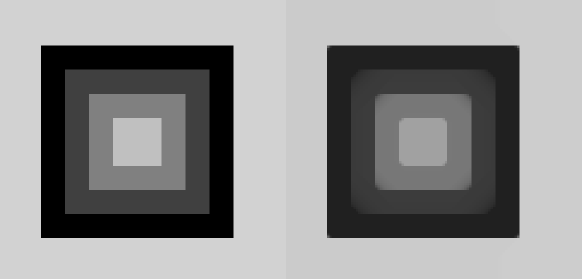
I) Total Variation minimization with convex fidelity

II) Total Variation minimization with convex fidelity

III) $L^1 + TV$ as a constrast invariant filter

VI) $L^1 + TV$ on the FLST-tree

L^2 + TV, contrast and contours



- Loss of contrast $\rightarrow$ use L^1
- How to preserve contours ?

Fast Level Set Transform Tree

- Level Sets
  $L^{\lambda}(u) = \{x \in \Omega | u(x) \leq \lambda\}$, $U^{\lambda}(u) = \{x \in \Omega | u(x) > \lambda\}$
- Inclusion property

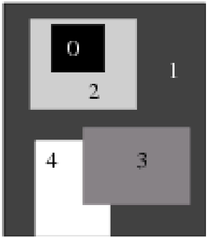
$$U^{\lambda}(u) \subset U^{\mu}(u) \; \forall \lambda \geq \mu$$

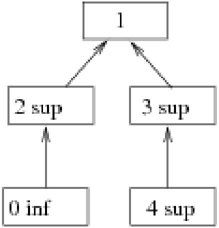
$$L^{\lambda}(u) \subset L^{\mu}(u) \; \forall \lambda \leq \mu$$

- Induce a tree [Salembier *et al.* ITIP 98] :
  - conected components of lower/upper sets
  - $L^{\lambda} \rightarrow$ "Min-Tree" $\rightarrow$ dark objects on light background
  - $U^{\lambda} \rightarrow$ "Max-Tree" $\rightarrow$ light objects on dark background
- "Fast Level Set Transform" (FLST) [Monasse *et al* ITIP 2000]
  - Merge the 2 trees into a single one
  - Need of a criteria : Holes

Fast Level Set Transform Tree

- Shapes = connected components of level sets whose holes have been filed.
- Definition of the tree
  - 1 node = 1 shape
  - Parent = smallest form which contains it
  - children = included forms
- Decomposition of the image into forms S_1 ... S_n





Fast Level Set Transform Tree

- $L^1 + TV$ on the FLST tree
  equivalent to
  $L^1 + TV$ + edge preservation
- Attributes associated to each node
  - gray level u_i
  - area for data fidelity $\to |D_i|$
  - perimeter for TV (co-aire formula) $\to P_i$
- Data fidelity:

$$\sum_{i=1}^{N} |D_i||u_i - v_i|$$

- Total Variation (Dibos *etal* 2000 SIAM NA):

$$\sum_{i=1}^{N-1} P_i|u_i - u_i^p|$$

$L^1 + TV$ on the FLST-tree

- Finally

$$E^{L^1+TV}(u|v) = \sum_{i=1}^{N} |D_i||u_i - v_i| + \beta \sum_{i=1}^{N-1} P_i|u_i - u_i^p|$$

- Sites: nodes of the tree
- Neighborhoods $\rightarrow$ parents et children
- pairwise interactions
- MAP of the Markovian energy$L^1 + TV$
  $\rightarrow L^1 + TV$ algorithm of the first part

Results

Original image

$\beta = 3$

Results

$\beta = 15$ borders of the result ($\beta = 30$)

5 regions

Results

Original image

Pertinent contours ?

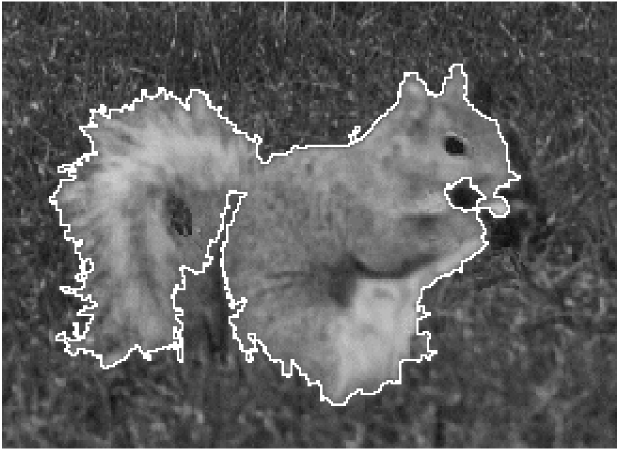
Results

$\beta = 1$

Results

$\beta = 2$

Results



borders of the result ($\beta = 10$) superimposed on the original image

2 regions

Time results

Image	FLST	Minimization
Lena (256x256)	0.18	0.11
Lena (512x512)	1.09	1.04
Woman (522x232)	0.39	0.06
Squirrel (209x288)	0.24	0.19

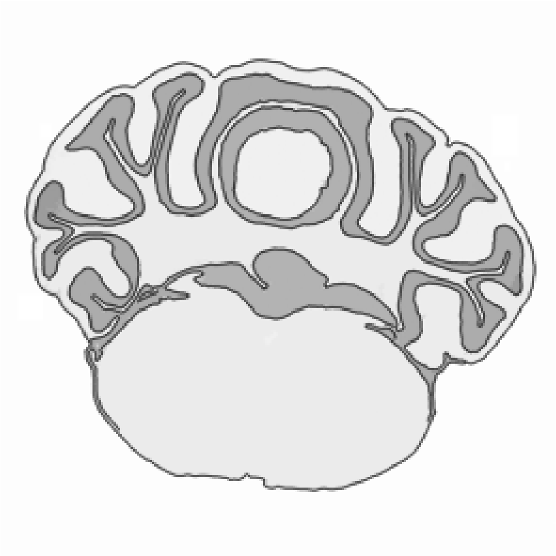
FLST tree computed with the implementation available in Megawave (ENS de Cachan), time in seconds

Results: Biological images

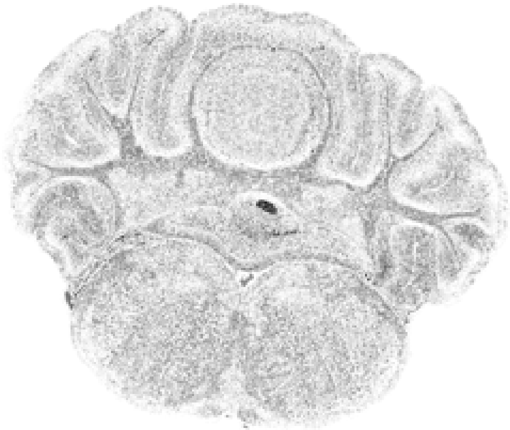
Fast and Accurate Feature Detection and Triangulation Using Total Variation Filtering of Biological Images
Alexandre Cunha, Jérôme Darbon, Tony F. Chan, Arthur Toga
Accepted for oral presentation to ISBI 2007

Results

Results

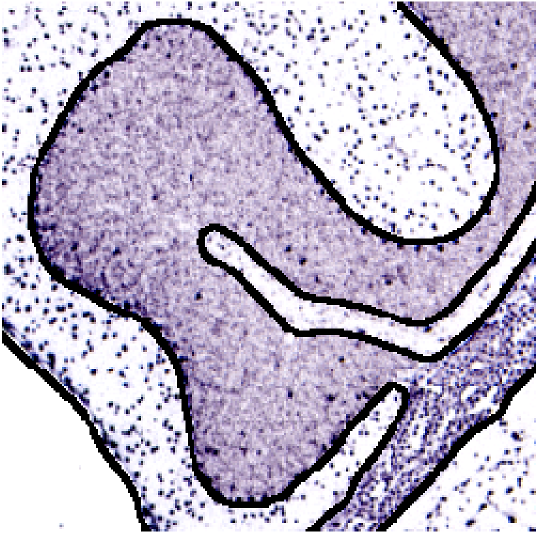


Results

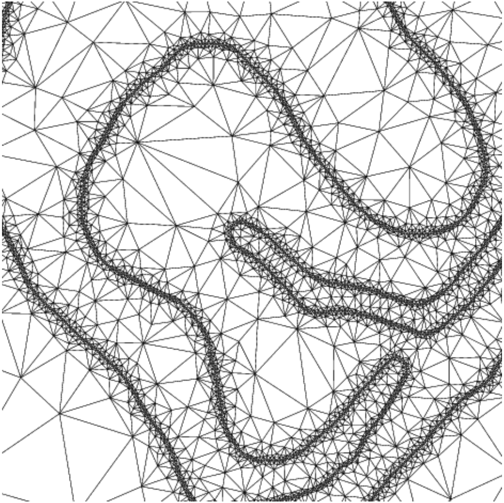


Results

Results



Results



Partial conclusion

- Contrast preservation (since it is morphological)
- Edge preservation
- Fast
- Good simplification for future segmentaion
- Restatement of morphological connected filters as a constrained $L^1 + TV$ minimization

Conclusion

- Exact optimization for
  - Convex + TV (polynomial)
  - Any + Levelable [Darbon Sigelle JMIV 06]
  - Any + Convex [Darbon Sigelle JMIV 06]
- $L^1 + TV$ is invariant with respect to changes of contrast
  - Framework for new connected filters
  - Extension to the vectorial case [Darbon Peyronnet ISVC 05]
- Successfull extension to
  - MRFs with submodular and supermodular priors [Darbon UCLA 07]
  - Exact Optimization of the Chan and Vese Model [Darbon Mirage 07]
  - Reformulation of morphological connected filter as a constrained $L^1 + TV$ minimization problem