

Geodesic Minimal Paths

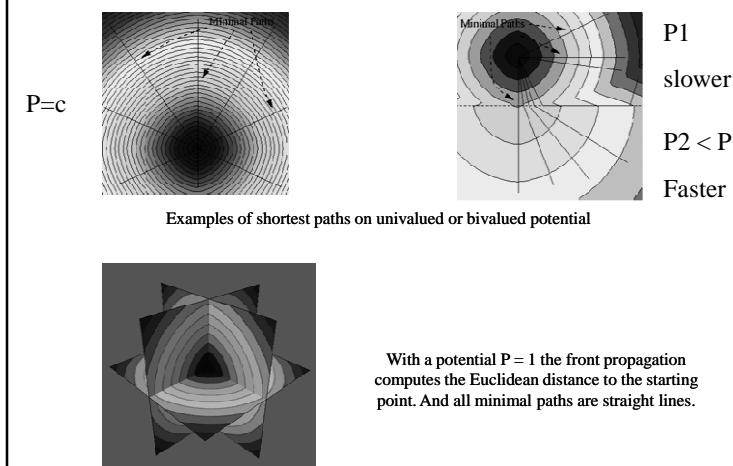
- Minimal paths, Eikonal Equation , Fast Marching and Front Propagation
- 3D Fast Marching, some examples
- Fast Marching on a surface and adaptive Remeshing
- Anisotropic Fast Marching
- Closed Contour as a set of minimal paths. Perceptual Grouping. Key points method
- Geodesic Voting and tree structure segmentation
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Minimal path - 2D and 3D synthetic examples



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3D FAST MARCHING:

Extension to 3D by the same numerical resolution of Eikonal Equation

$U_{i,j,k}$ solution of the discrete problem

$$\|\nabla U\| = \tilde{P} \quad (\max\{u - U_{i-1,j,k}, u - U_{i+1,j,k}, 0\})^2 + (\max\{u - U_{i,j-1,k}, u - U_{i,j+1,k}, 0\})^2 + (\max\{u - U_{i,j,k-1}, u - U_{i,j,k+1}, 0\})^2 = \tilde{P}_{i,j,k}^2$$

2nd degree Equation,
action U at $\{i,j,k\}$ depends only on smaller action neighbors

Fast marching : order in the selection of points to solve the local scheme.

Starting point p_0 with $U = 0$.

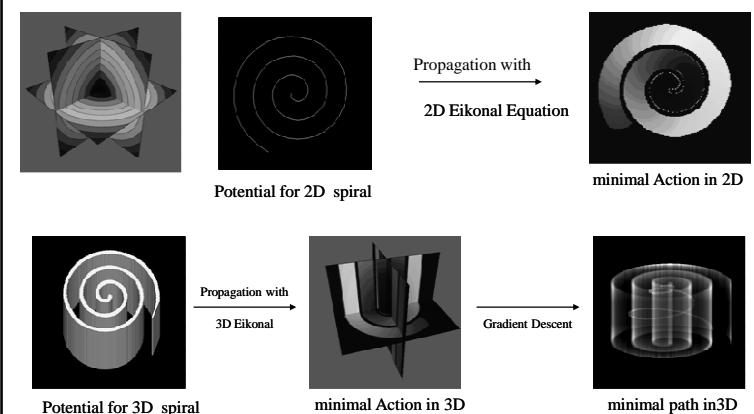
Level sets of U can be seen as a front propagation outwards.

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Extension to 3D – synthetic example

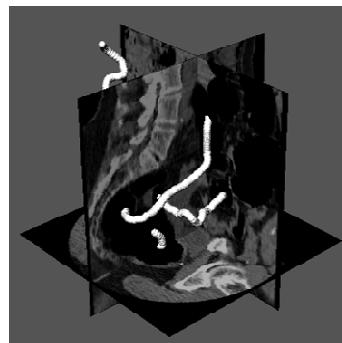


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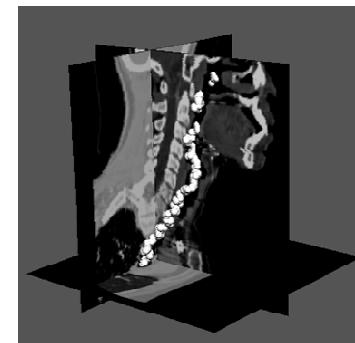
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Exemples de Chemins Minimaux 3D



Colon 3D CT



Trachée 3D CT

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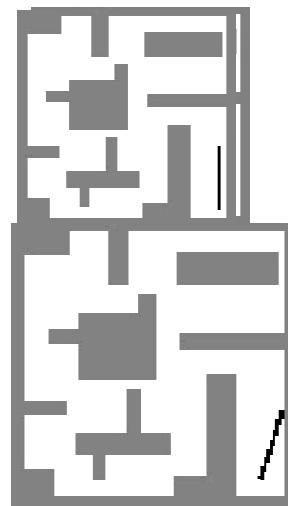
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Navigation and Robotics:

Finding a minimal path in
larger dimension space:

(location, size, orientation)



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3D Minimal Path for tubular shapes in 2D

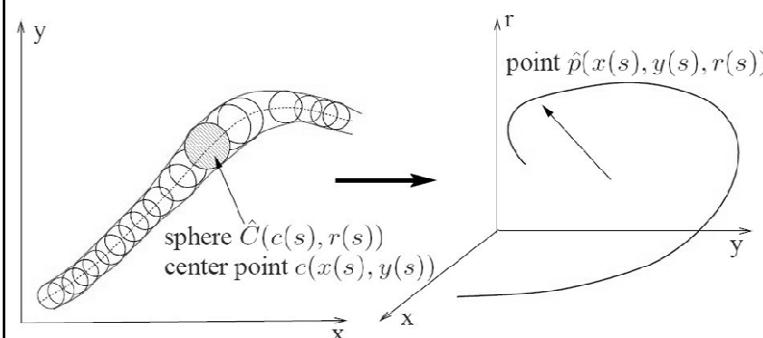


Figure 1. A tubular surface is presented as the envelope of a family of spheres with continuously changing center points and radii.

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3D Minimal Path for tubular shapes in 2D

2D in space , 1D for radius of vessel

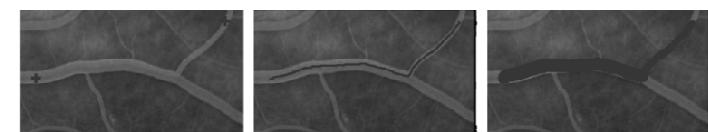


Fig. 2. Vessel segmentation for an angiogram 2D projection image based on the proposed method

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Fast Marching sur une surface et lignes géodésiques



■ [DEMO](#)

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Fast Marching sur une surface



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Fast Marching sur une surface

- [maillage.ppt](#)
- [presfr](#)

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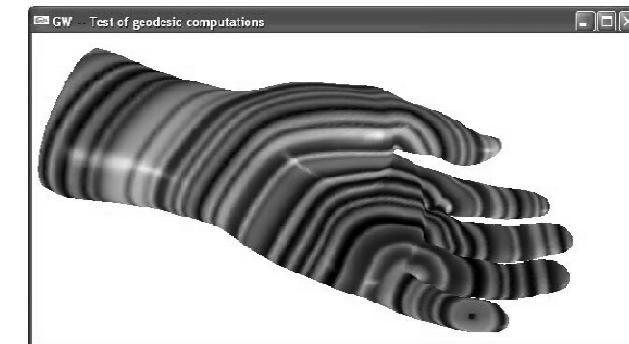
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Fast Marching on a surface and Remeshing

Front Propagation on a surface from one point.



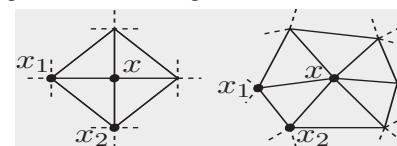
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Fast Marching on a surface and Remeshing

From orthogonal grid to triangulation



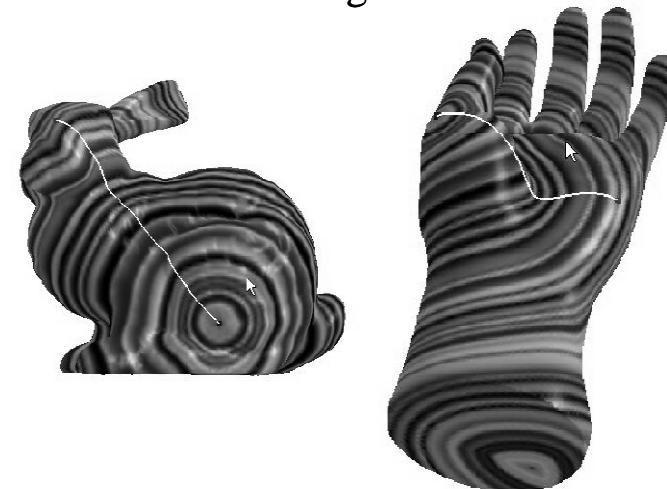
- Consider each triangle in the 1-ring.
- Choose for $U(x)$ the minimum *valid* solution.
- Stability issues for obtuse angles:
 - Subdivision of these angles.
 - Local support via unfolding.

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Fast Marching on a surface



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Geodesic lines on a surface



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Uniform and Adaptive Remeshing

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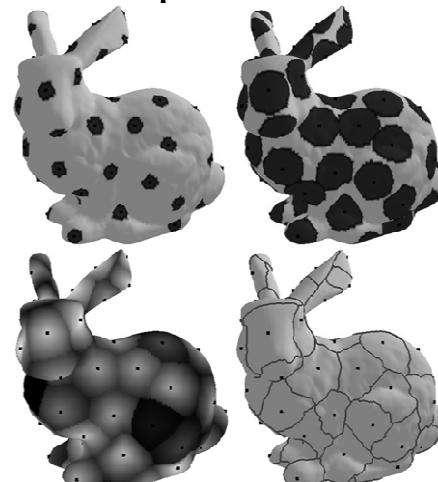
[SKIP SLIDES for VIDEO](#)

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Example of Voronoi



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Sampling with uniform distribution

Choose first point anywhere

choose the furthest point

update the geodesic distance

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The two new furthest points

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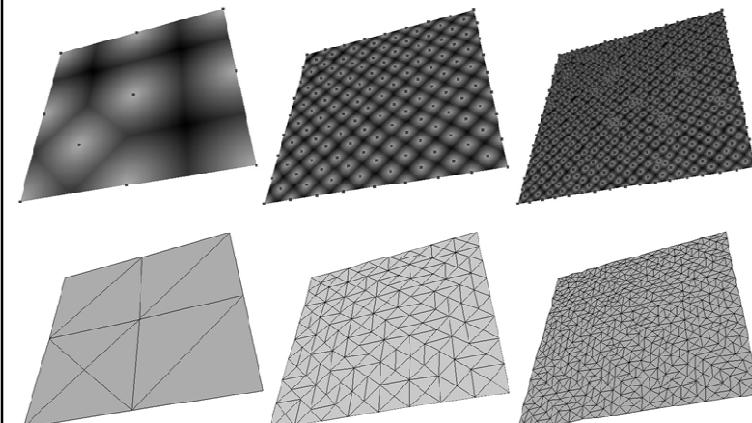
Sampling with uniform distribution



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Sampling on a plane



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Point Sampling Algorithm

- *Initialization:* choose $S \leftarrow \{x_1\}$
compute U_1 distance to x_1 .
- *Loop on n:* choose $x_{n+1} = \arg \max_x (U_n(x))$
Update $S \leftarrow S \cup \{x_{n+1}\}$ and $U_{n+1} = \min(U_n, U_{x_{n+1}})$
- Stop: if $U_n(x_{n+1}) \leq \delta$

Propagation for update is limited to

$$\{x / U_n(x) \geq U_{x_{n+1}}(x)\}$$

Fast algorithm, order $O(N \log(N)^2)$ operations.

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How to Triangulate ?

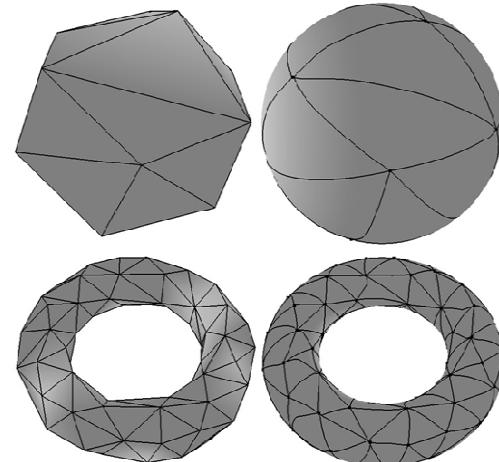
- Keep track of Voronoi neighborhood information.
- Construction geodesic Delaunay triangulation.
- We can draw the corresponding geodesic triangles on the surface.

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Geodesic Delaunay Triangulation

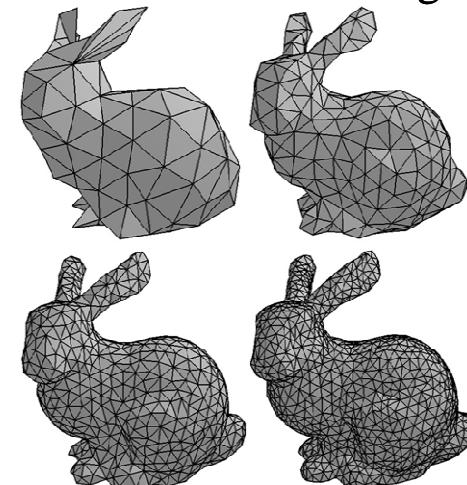


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Uniform Remeshing

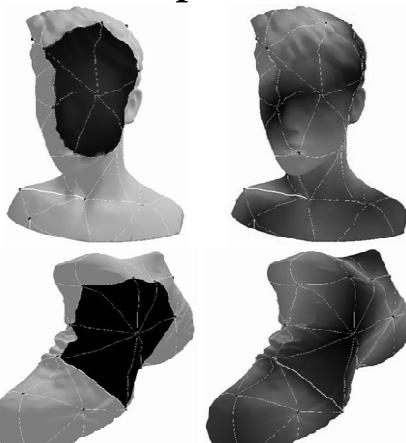


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Examples of Parameterization



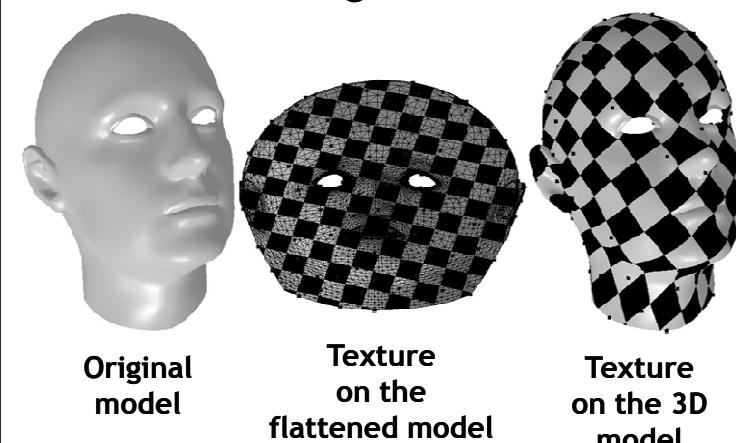
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- Perform local propagations.
- Compute 3 distances for each vertex.
- Use Heron formula.
- Ex: interpolate color along coordinates.

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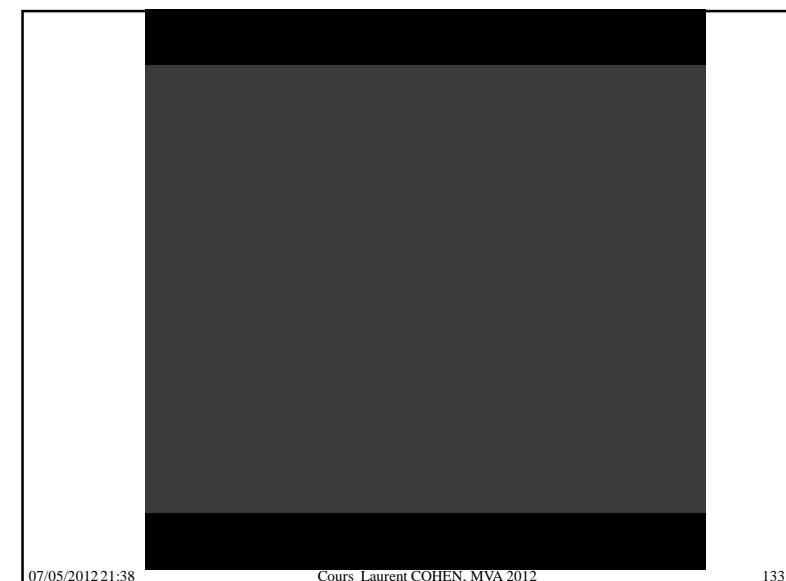
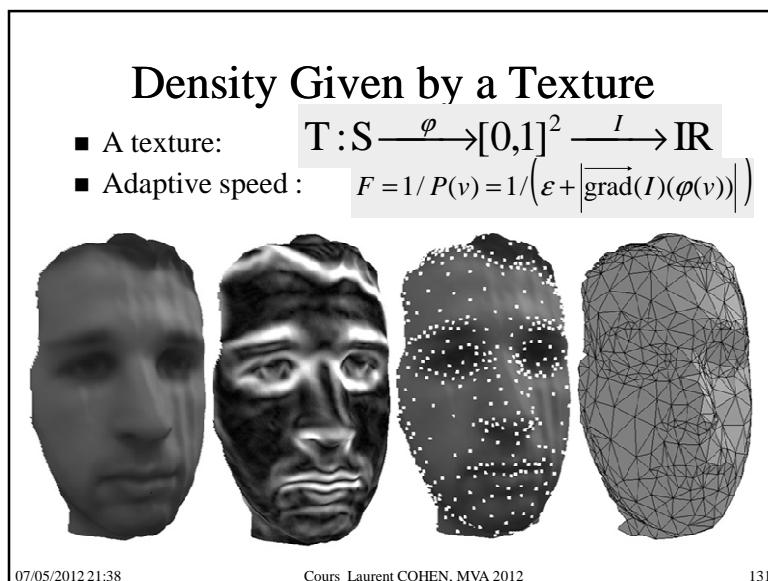
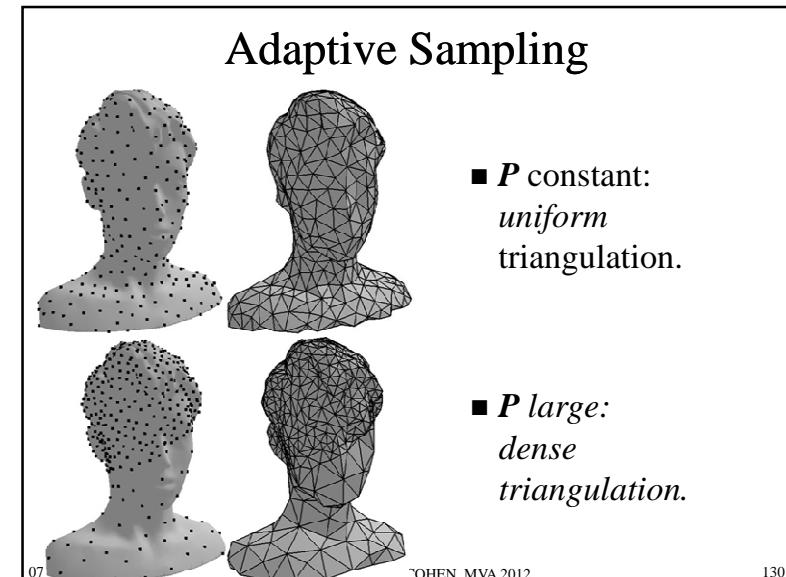
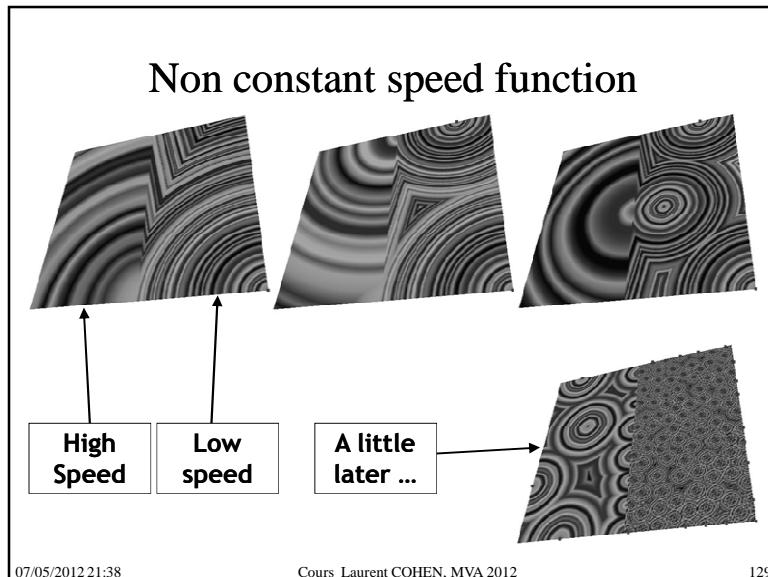
Flattening and texture

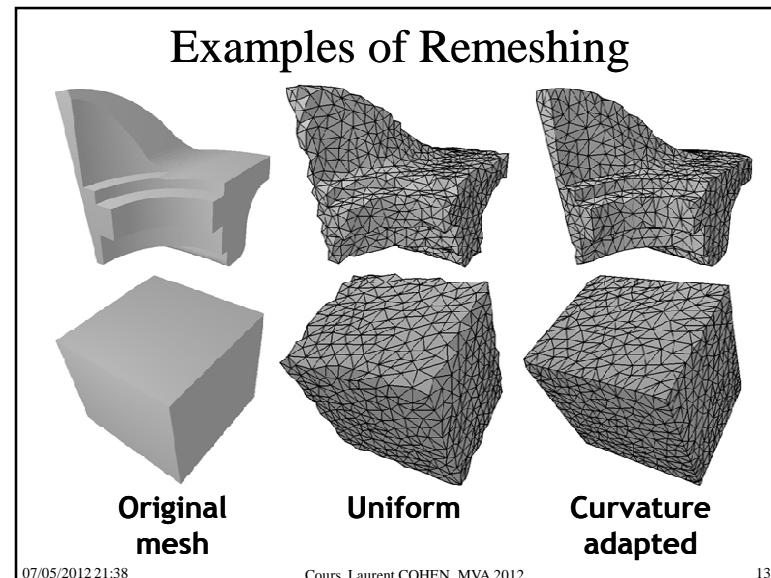


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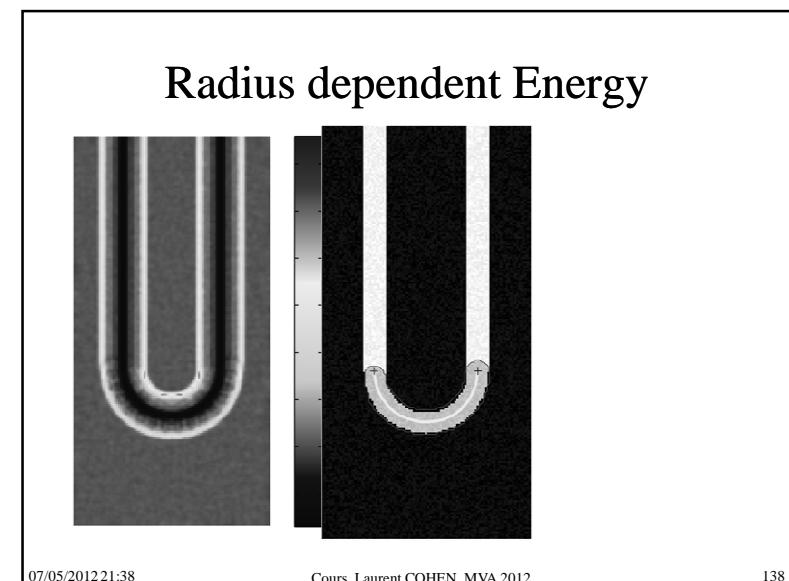
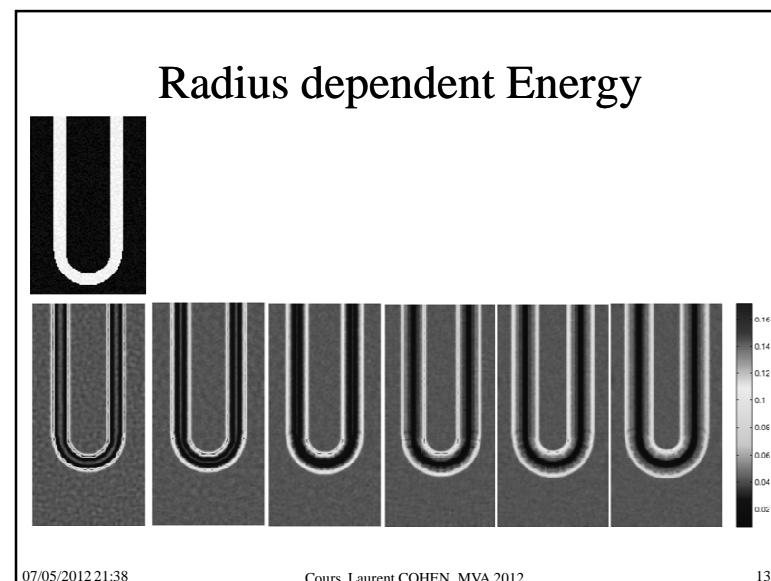
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Riemannian Manifolds, Anisotropy and Geodesic Distances

- 2D Riemannian manifolds defined over a compact planar domain $\Omega \subset \mathbb{R}^2$
- Length of a curve $[0,1] \rightarrow \Omega$

$$L(\gamma) \stackrel{\text{def.}}{=} \int_0^1 \sqrt{\gamma'(t)^T H(\gamma(t)) \gamma'(t)} dt.$$

with $H: \Omega \rightarrow \mathbb{R}^{2 \times 2}$ a metric tensor field of anisotropy $\alpha: \Omega \rightarrow [0,1]$

- Geodesic distance

$$d(x, y) = \min_{\gamma \in \mathcal{P}(x, y)} L(\gamma), \quad \forall (x, y) \in \mathbb{R}^2$$

- Distance map $U_S: \Omega \rightarrow \mathbb{R}$ of a point set $S = \{x_k\}_k$

$$U_S(x) = \min_{x_k \in S} d(x, x_k), \quad \forall x \in \Omega$$

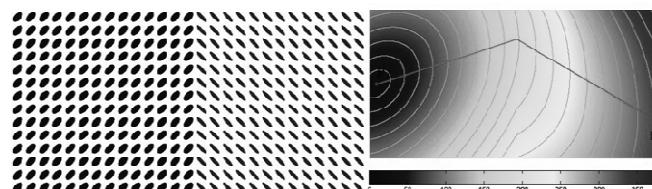
Geodesic Methods for Shape and Surface Processing, Gabriel Peyre and Laurent D. Cohen in Advances in Computational Vision and Medical Image Processing: Methods and Applications, Springer, 2009.

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Anisotropy and Geodesics



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Anisotropy and Eikonal Equation

Theorem: U_{x_0} is the unique viscosity solution of the Hamilton-Jacobi equation

$$\|\nabla U_{x_0}\|_{H(x)^{-1}} = 1 \quad \text{with} \quad U_{x_0}(x_0) = 0,$$

where $\|v\|_A = \sqrt{v^T A v}$.

Geodesic curve γ between x_1 and x_0 solves

$$\gamma'(t) = -\frac{H(\gamma(t))^{-1} \nabla U_{x_0}}{\|H(\gamma(t))^{-1} \nabla U_{x_0}\|} \quad \text{with} \quad \gamma(0) = x_1.$$

Example: isotropic metric $H(x) = W(x) \text{Id}_x$,

$$\|\nabla U_{x_0}\| = W(x) \quad \text{and} \quad \gamma'(t) = -\frac{\nabla U_{x_0}}{\|\nabla U_{x_0}\|}.$$

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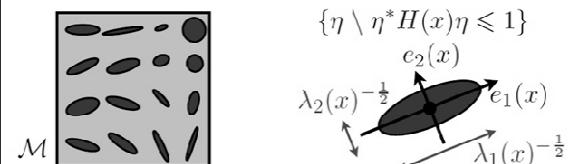
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Anisotropy and Geodesics

Tensor eigen-decomposition:

$$H(x) = \lambda_1(x)e_1(x)e_1(x)^T + \lambda_2(x)e_2(x)e_2(x)^T \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2.$$



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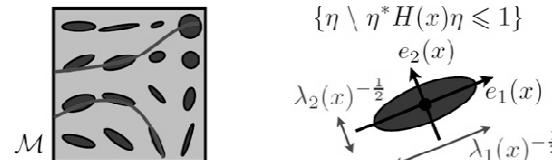
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Anisotropy and Geodesics

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Geodesics tend to follow $e_1(x)$.

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Anisotropy and Geodesics

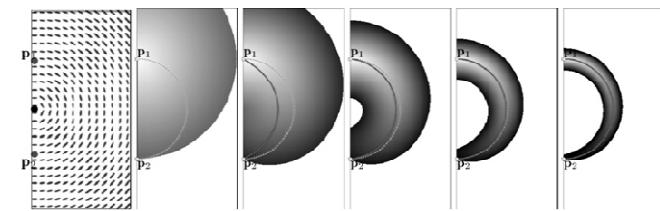


FIG. 2.14: Given an elliptic metric $M = w_1^2 e_r e_r^T + w_2^2 e_\theta e_\theta^T$ with standard polar notations, influence of anisotropy ratio $\frac{w_2}{w_1}$ is shown.

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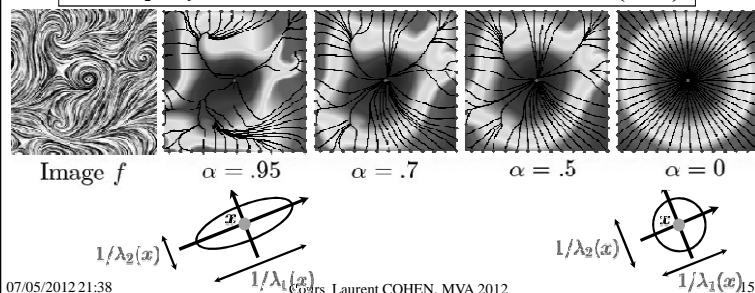
Anisotropy and Geodesics

Tensor eigen-decomposition:

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Local anisotropy of the metric:

$$\alpha(x) = \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{\sqrt{(a-b)^2 + 4c^2}}{a+b} \in [0,1] \quad \text{for} \quad H(x) = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$



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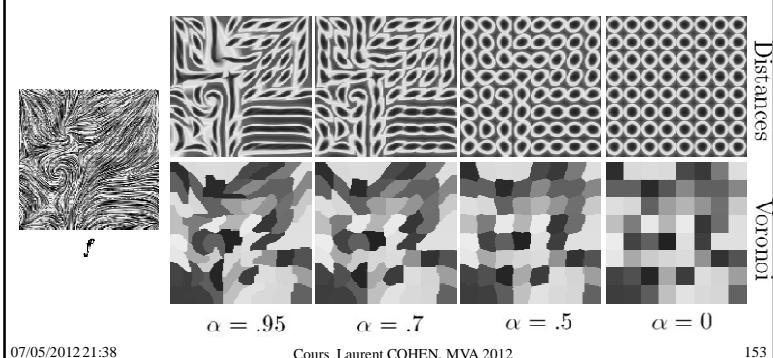
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Anisotropic Voronoi Segmentation

Voronoi segmentation:

$$\Omega = C_0 \bigcup_{x_i \in \mathcal{S}} \mathcal{C}_i \quad \text{where} \quad \mathcal{C}_i = \{x \in \Omega \setminus \forall j \neq i, d(x_i, x) \leq d(x_j, x)\}$$

Outer cell: $\mathcal{C}_0 = \text{Closure}(\Omega^\circ)$.



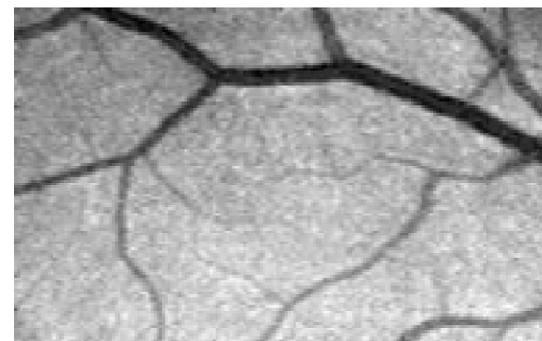
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3D Minimal Paths for tubular shapes in 2D

2D in space , 1D for radius of vessel

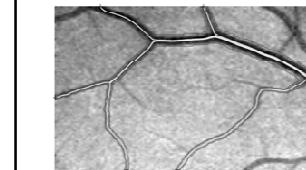
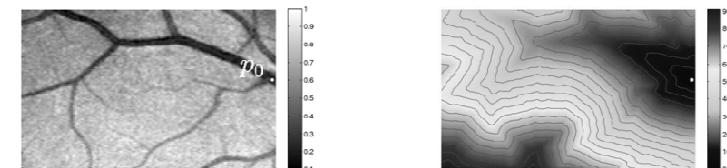


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3D Minimal Paths for tubular shapes in 2D Motivation



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Orientation dependent Energy

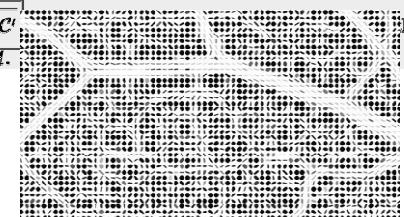
Minimal paths method : looking for a path minimizing the energy

$$E(\mathcal{C}) = \int_0^L P(\mathcal{C}(s)) ds$$

Since the tubular structures have directions, we should consider the orientation:

$$E(\mathcal{C}) = \int_0^L P(\mathcal{C}(s), \mathcal{C}'(s)) ds$$

where $P(\mathcal{C}, \mathcal{C}') = \sqrt{\mathcal{C}^T \mathcal{M}(\mathcal{C}) \mathcal{C}'}$
way \mathcal{C} , relative to a metric \mathcal{M} .



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3D Minimal Path for tubular shapes in 2D

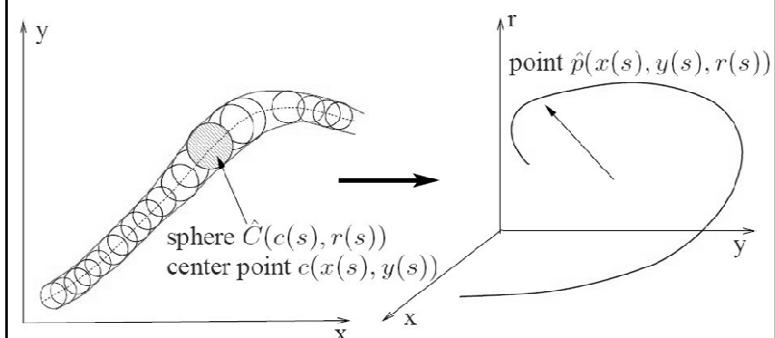


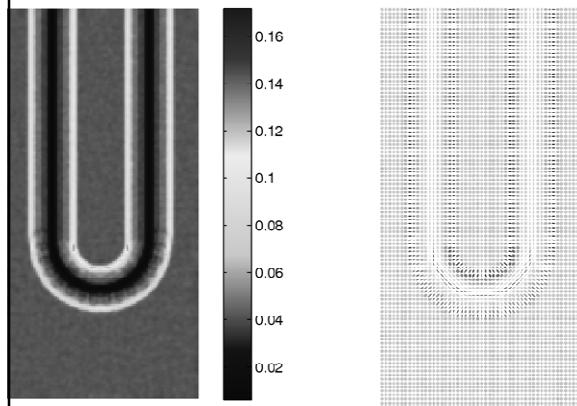
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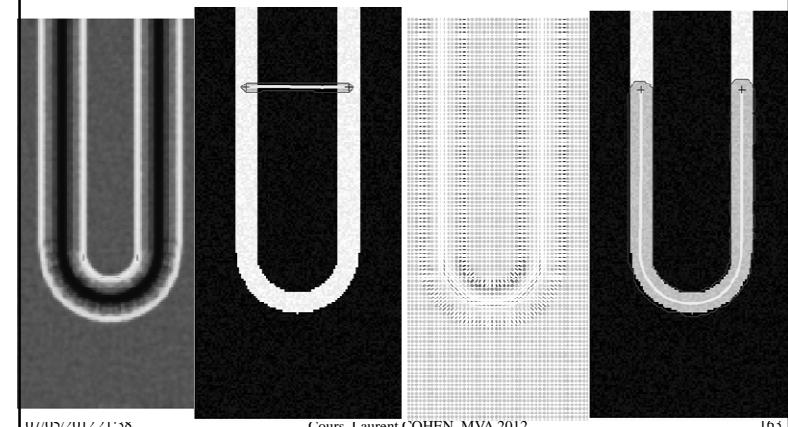
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Orientation dependent Energy



Orientation dependent Energy

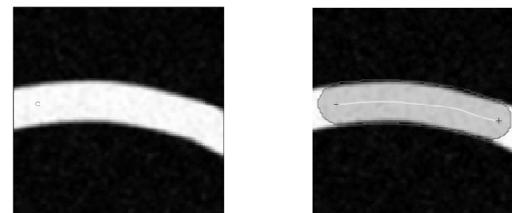


Examples of 3D Minimal Paths for tubular shapes in 2D

Anisotropic Fast Marching algorithm to solve

$$\|\nabla \mathcal{U}(x)\|_{\mathcal{M}^{-1}} = \sqrt{\nabla \mathcal{U}(x)^T \mathcal{M}^{-1}(x) \nabla \mathcal{U}(x)} = 1 \text{ and } \mathcal{U}_{p_0}(p_0) = 0$$

and back-propagation $\mathcal{C}' \propto \mathcal{M}^{-1} \nabla \mathcal{U}$



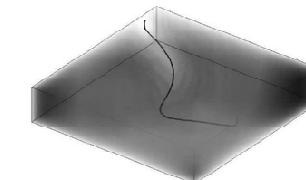
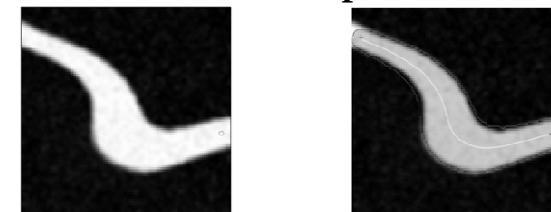
Tubular anisotropy for 3D vessels segmentation. Fethallah Benmansour and Laurent D. Cohen. Preprint, 2009.

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Examples of 3D Minimal Paths for tubular shapes in 2D



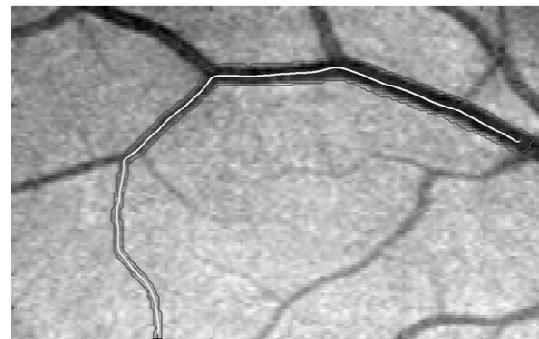
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Examples of 3D Minimal Paths for tubular shapes in 2D

2D in space , 1D for radius of vessel



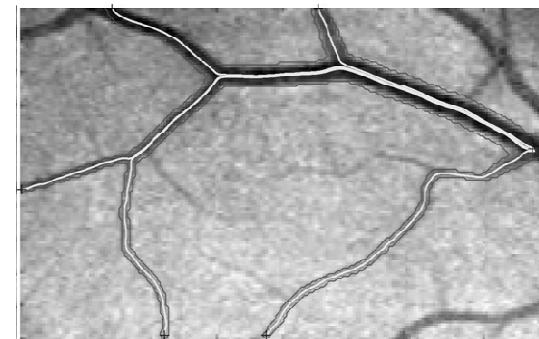
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Examples of 3D Minimal Paths for tubular shapes in 2D

2D in space , 1D for radius of vessel



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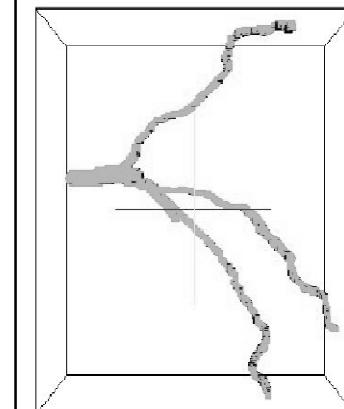


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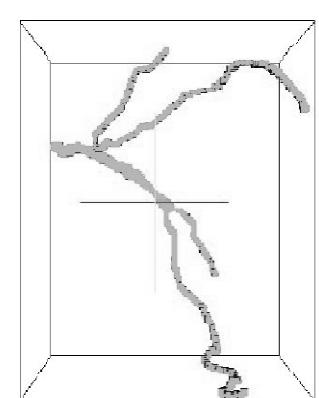
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Examples of 4D Minimal Paths for tubular shapes in 3D



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Examples of 4D Minimal Paths for tubular shapes in 3D



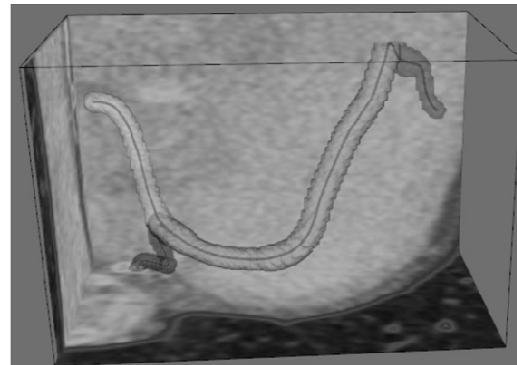
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Examples of 4D Minimal Paths for tubular shapes in 3D

3D in space , 1D for radius of vessel



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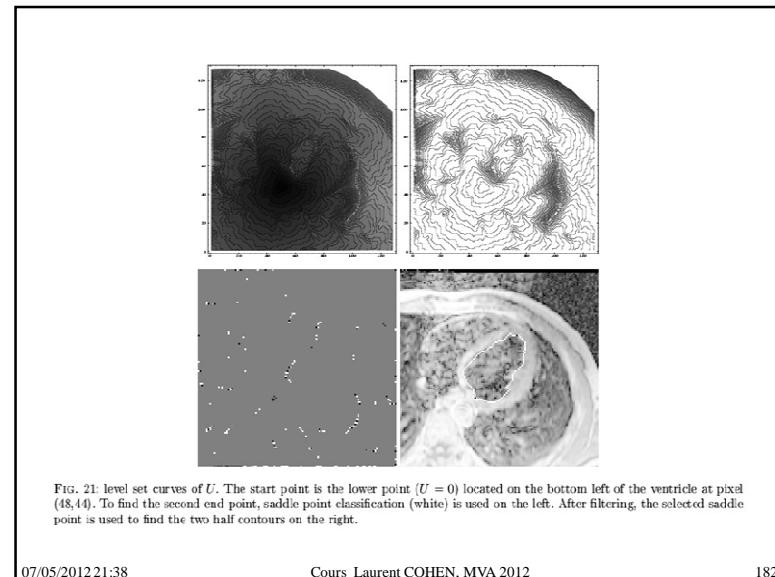
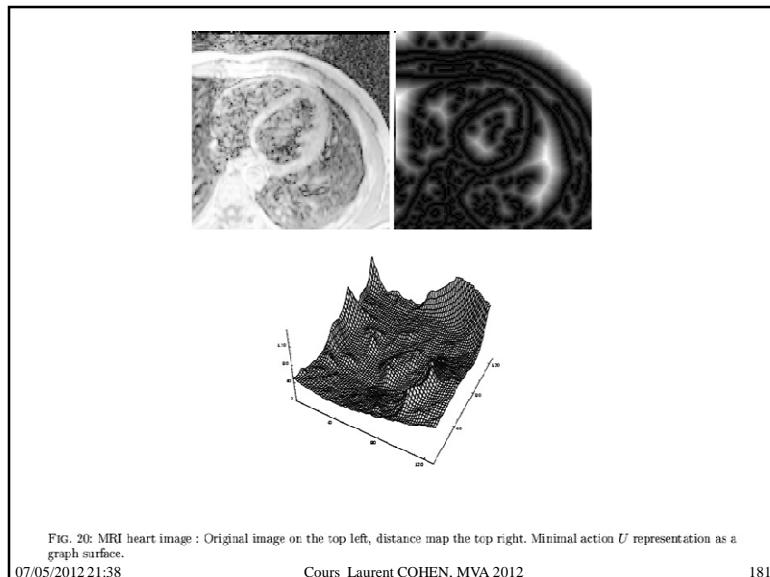
Closed Boundary Extraction from a Single Point

- Minimal Action from p_0
- Search for Saddle point of U
 - Number of level crossings
 - Gaussian Curvature test
- Saddle points filtering
- Two minimal paths from the selected point

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Trouver un Ensemble de Chemins Minimaux

- **Chemin Minimal entre 2 extrémités p1 et p2.**
- **Problème 1:** Étant donné un ensemble de points, trouver un ensemble de chemins minimaux les reliant: basé sur les points selle de U .
- **Problème 2:** Étant donné un ensemble de régions, trouver un ensemble de chemins minimaux les reliant, basé sur la même technique
- **Application à des images médicales**

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Chemins minimaux pour compléter des contours?



FIG. 1: Examples of incomplete contours

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Chemins à partir de Sources multiples pk

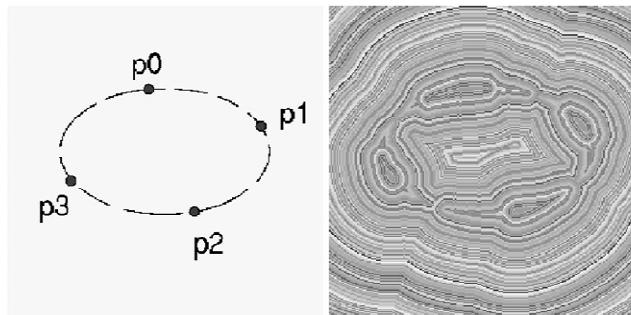


Figure 3: Ellipse example with four points. On the left the incomplete ellipse as potential and four given points; on the right the minimal action map (random LUT to show the level sets) from these points.

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Chemins à partir de Sources multiples pk

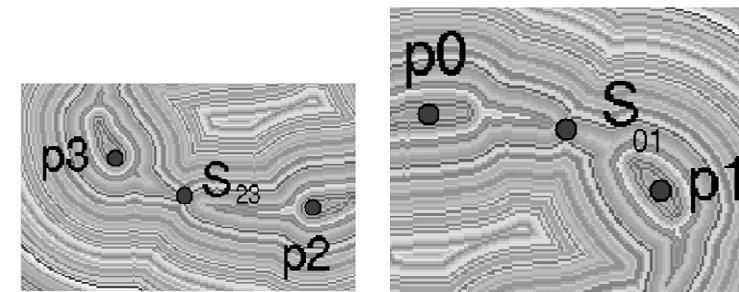


Figure 4: Zoom on *saddle points* between two key points.

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Chemins à partir de Sources multiples pk

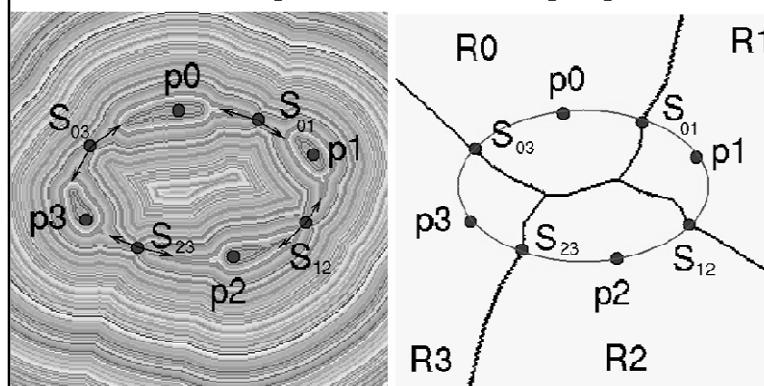


Figure 5: Ellipse example with four points. On the left the saddle points are found and backpropagation is made from them to each of the two points from where the point comes; on the right, the minimal paths and the Voronoi diagram obtained.

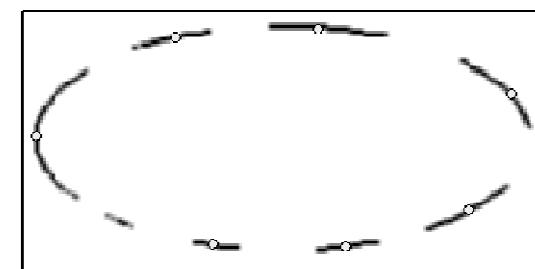
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Groupement perceptuel à partir d'un ensemble de chemins minimaux

Il faut compléter les contours noirs pour former automatiquement l'ellipse entière

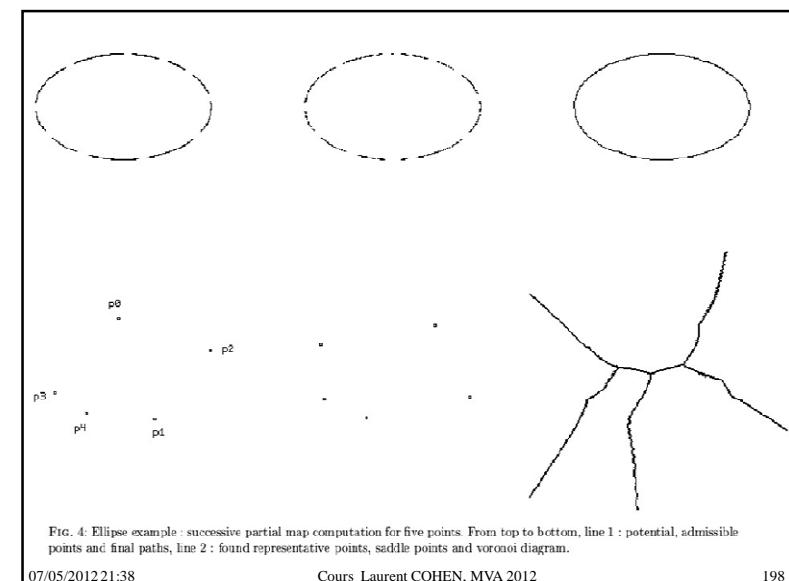
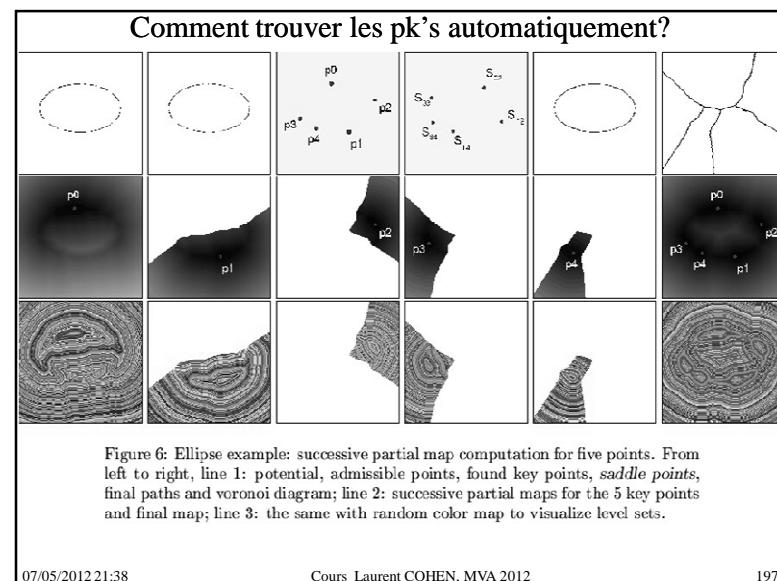
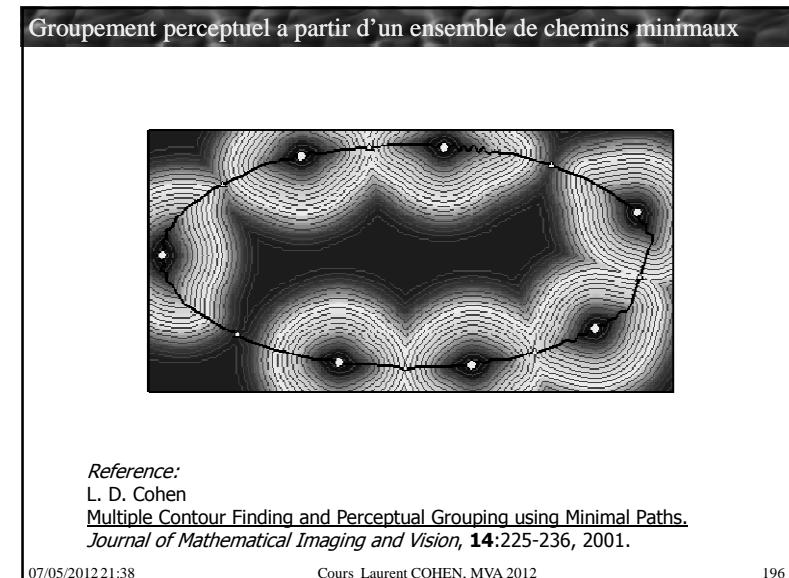
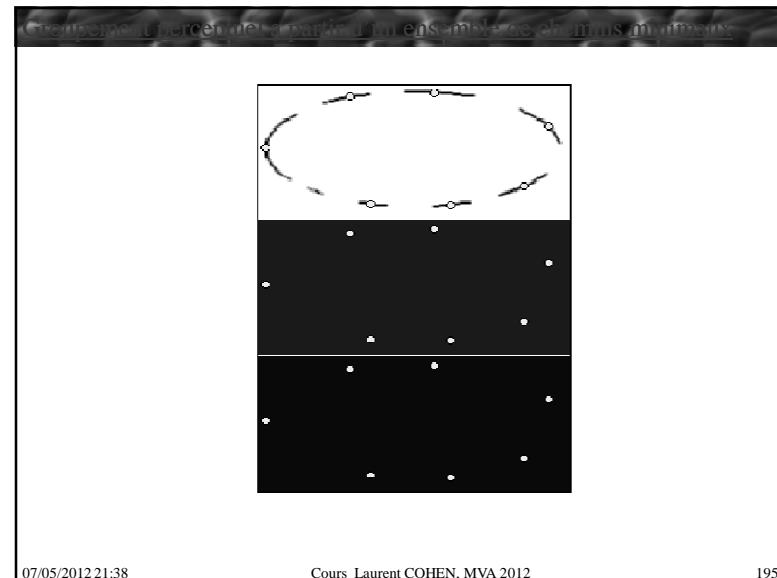


Reference:
L. D. Cohen
[Multiple Contour Finding and Perceptual Grouping using Minimal Paths](#).
Journal of Mathematical Imaging and Vision, **14**:225-236, 2001.

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Chemins à partir de Sources multiples :
Détermination automatique des points source pk

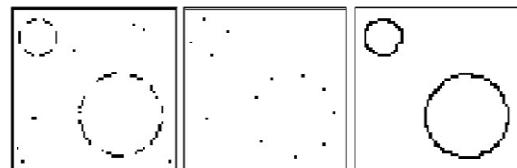


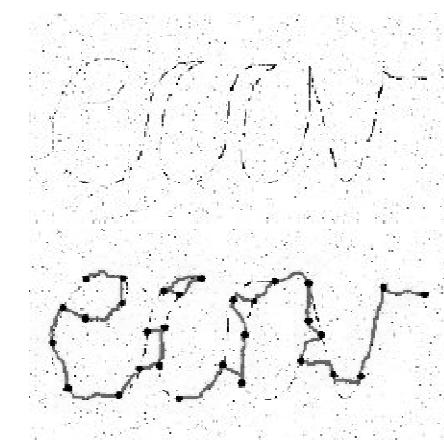
Figure 8: Two circles; From left to right: potential, key points and final paths.

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Perceptual Grouping using Minimal Paths



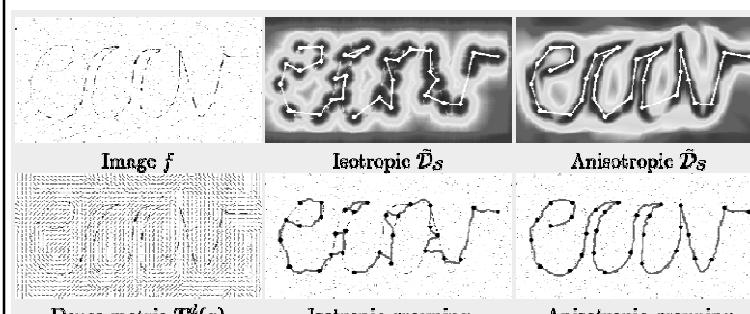
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Perceptual Grouping using Minimal Paths

Using the orientation with anisotropic geodesics



[Anisotropic Geodesics for Perceptual Grouping and Domain Meshing](#). Sébastien Bougleux and Gabriel Peyré and Laurent D. Cohen. Proc. tenth European Conference on Computer Vision (ECCV'08), Marseille, France, October 12-18, 2008.

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