

Geodesic Methods for Image Segmentation

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Some of this work has been in collaboration with
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Geodesic Minimal Paths

- Minimal paths, Eikonal Equation , Fast Marching and Front Propagation
- 3D Fast Marching, some examples
- Fast Marching on a surface and adaptive Remeshing
- Anisotropic Fast Marching
- Closed Contour as a set of minimal paths. Perceptual Grouping. Key points method
- Geodesic Voting and tree structure segmentation
- Adding iteratively Key points for geodesic meshing
- Surface between two curves as a network of paths
- Path Network and Transport Equation
- Application to Virtual Endoscopy
- Segmentation by Fast Marching : Freezing, Dual fronts

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Active Contours limitation

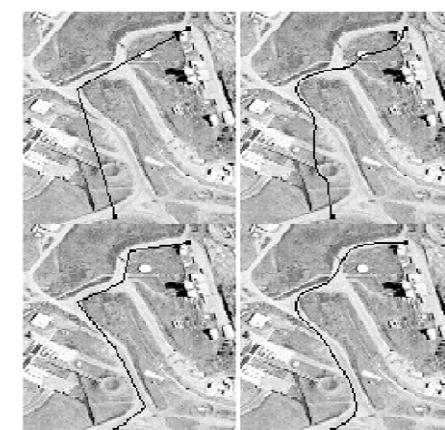


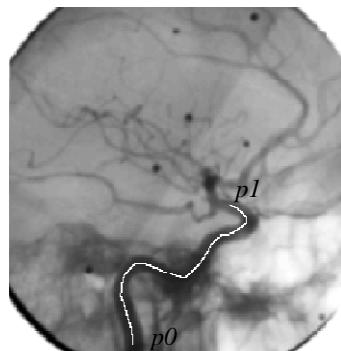
FIG. 10: Snake Initialization classical snakes need a very close initialization to avoid a Local Minimum.

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Paths of minimal energy



Looking for a path along which a feature Potential $P(x,y)$ is minimal

example: a vessel
dark structure
 P = gray level

Input : Start point $p0=(x0,y0)$

End point $p1=(x,y)$

Image

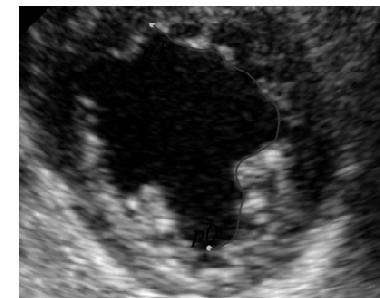
Output: Minimal Path

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Paths of minimal energy



Looking for a path along which a feature Potential $P(x,y)$ is minimal

example: cardiac ventricle
contour
 P = gradient based

Input : Start point $p0=(x0,y0)$

End point $p1=(x,y)$

Image

Output: Minimal Path

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Minimal Paths: Eikonal Equation

simplified formulation for active contour model energy

$$E(C) = \int_0^L \{w + P(C(s))\} ds = \int_0^L \tilde{P}(C(s)) ds$$

Potential $P > 0$ takes lower values near interesting features :
on contours, dark structures, ...
 w is a regularization parameter

STEP 1 : search for the surface of minimal action U of $p0$ as the minimal energy integrated along a path between start point $p0$ and any point p in the image

Start point $C(0)=p0$,

$$U_{p0}(p) = \inf_{C(0)=p0; C(L)=p} E(C) = \inf_{C(0)=p0; C(L)=p} \int_0^L \tilde{P}(C(s)) ds$$

STEP 2: Back-propagation from the end point $p1$ to the start point $p0$:

Simple Gradient Descent along U_{p0}

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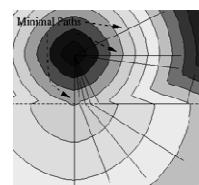
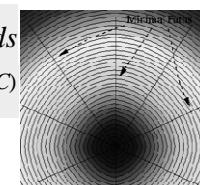
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Minimal paths - 2D synthetic examples

$$E(C) = \int_0^L \tilde{P}(C(s)) ds$$

$$U_{p0}(p) = \inf_{C(0)=p0; C(L)=p} E(C)$$

$P=c$



P1
slower
 $P2 < P1$
faster

Examples of shortest paths on univalued or bivalued potential

Fermat Principle in Geometric Optics :
Path followed by light minimizes time

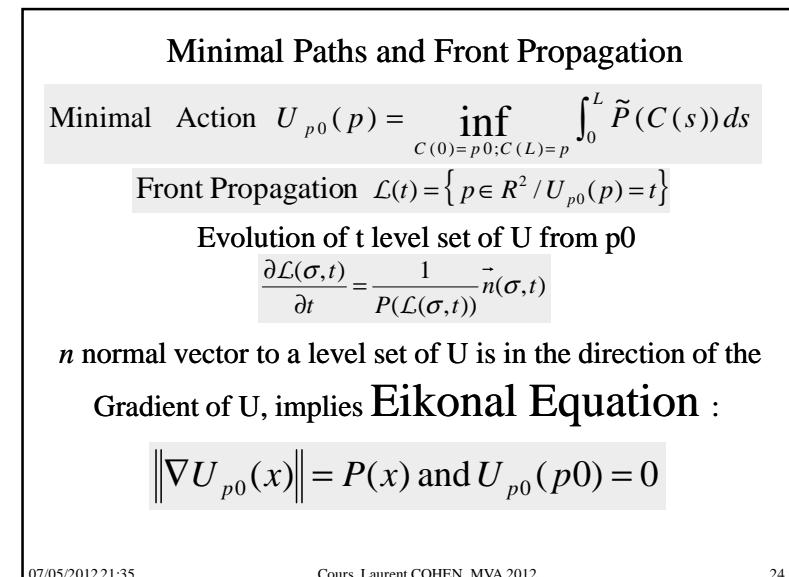
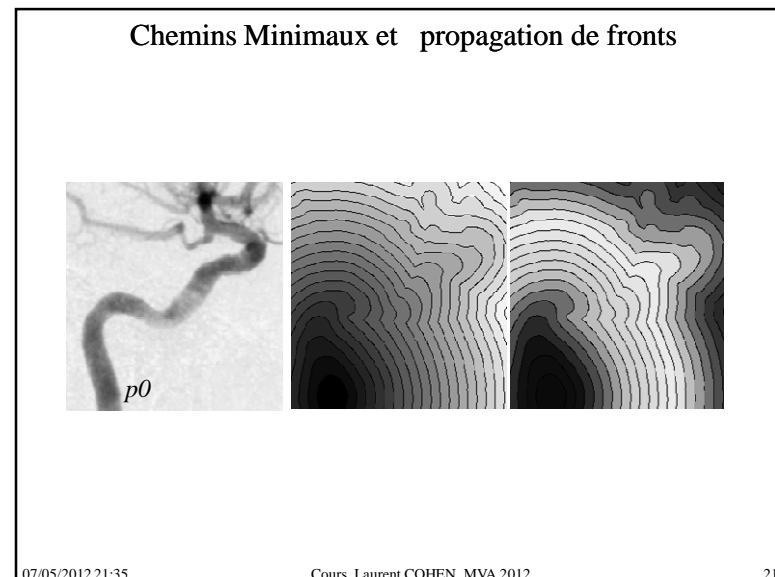
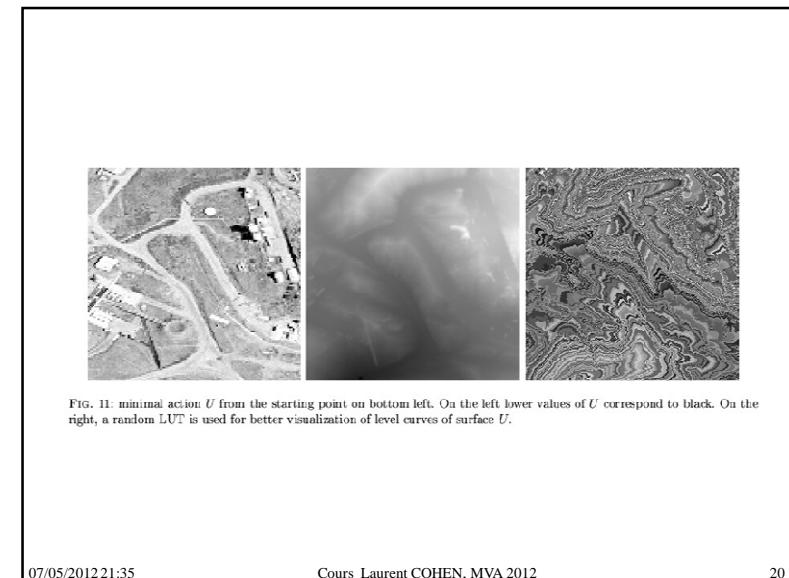
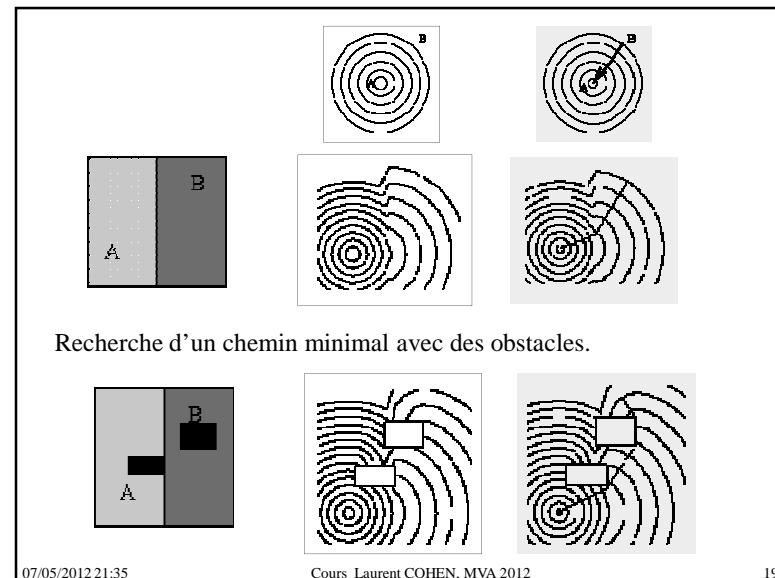
$$T = \frac{1}{c} \int_{p0}^{p1} n ds \quad \text{where } n > 1 \text{ is refraction index } v = c/n$$

Snell-Descartes 'law

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Minimal Paths: Eikonal Equation

STEP 1 : minimal action U of p_0 as the minimal energy integrated along a path between start point p_0 and any point p in the image

Startpoint $C(0)=p_0$,

$$U_{p_0}(p) = \inf_{C(0)=p_0; C(L)=p} E(C) = \inf_{C(0)=p_0; C(L)=p} \int_0^L P(C(s)) ds$$

Solution of Eikonal equation:

$$\|\nabla U_{p_0}(x)\| = P(x) \text{ and } U_{p_0}(p_0) = 0$$

Example P=1, U Euclidean distance to p0

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Minimal Paths: back propagation

$$E(C) = \int_0^L P(C(s)) ds$$

STEP 2: Back-propagation from the end point p_2 to the start point p_1 :

Simple Gradient Descent along U_{p_1}

$$\frac{dC}{ds}(s) = -\nabla U_{p_1}(C(s)) \text{ with } C(0) = p_2.$$

Theorem 1: (Euler Lagrange of E) Any curve C which is a local minimum of energy E is a solution of

$$\nabla \mathcal{P}(C) \cdot \vec{n} = \mathcal{P}(C)_\kappa$$

Definition 2 (Critical curves) We say that C is a critical curve of the energy E if C is a solution of the Euler-Lagrange equation (5).

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Minimal Paths: back propagation

Define C_1 by back propagation and $C(s)$ by reverse:

$$C_1(0) = p_1, C_1(L) = p_0, C_1'(s) = -\frac{\nabla U_{p_0}}{\|\nabla U_{p_0}\|}, C(s) = C_1(L-s)$$

$$U_{p_0}(p_1) = \min_{C(0)=p_1; C(L)=p_0} \int_0^L P(\gamma(s)) ds$$

$$U_{p_0}(p_1) = U_{p_0}(p_1) - U_{p_0}(p_0) = U_{p_0}(C(L)) - U_{p_0}(C(0))$$

$$U_{p_0}(p_1) = \int_0^L [U_{p_0}(C(s))]' ds = \int_0^L \nabla U_{p_0}(C(s)). C'(s) ds$$

$$U_{p_0}(p_1) = \int_0^L \nabla U_{p_0}(C(s)). \frac{\nabla U_{p_0}}{\|\nabla U_{p_0}\|} ds$$

$$U_{p_0}(p_1) = \int_0^L \|\nabla U_{p_0}(C(s))\| ds = \int_0^L P(C(s)) ds$$

Thus C reaches the minimum of the energy,
it is a minimal path.

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Equation Eikonale- Approche Séquentielle

$$\mathcal{U}(p_0) = 0$$

\mathcal{U} steady state of

$$\frac{\partial \mathcal{U}}{\partial \tau} = \tilde{P} - \|\nabla \mathcal{U}\|,$$

$\mathcal{U} = \mathcal{U}_\infty$ satisfies

$$\|\nabla \mathcal{U}\| = \tilde{P},$$

Iterative Sequential Scheme : $U_{i,j}$ given by

$$(\max\{u - U_{i-1,j}, u - U_{i+1,j}, 0\})^2 + (\max\{u - U_{i,j-1}, u - U_{i,j+1}, 0\})^2 - I_{i,j}^2,$$

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FAST MARCHING in 2D:

very efficient algorithm $O(N \log N)$ for Eikonal Equation

Introduced by Sethian / Tsitsiklis

Numerical approximation of $U(x_{ij})$ as the solution to the discretized problem with upwind finite difference scheme

$$\|\nabla U\| = \tilde{P}$$

$$\max(u - U(x_{i-1,j}), u - U(x_{i+1,j}), 0)^2 + \max(u - U(x_{i,j-1}), u - U(x_{i,j+1}), 0)^2 = h^2 \tilde{P}(x_{i,j})^2$$

This 2nd order equation induces that :

action U at $\{i,j\}$ depends only of the neighbors that have lower actions.

Fast marching introduces order in the selection of the grid points for solving this numerical scheme.

Starting from the initial point p_0 with $U = 0$,
the action computed at each point visited can only grow.

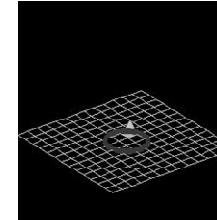
Level sets of U can be seen as a Front propagation outwards.

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Fast Marching



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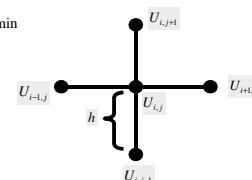
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Implementation: 2 Steps for building γ_{\min}

- Step 1: Solve the eikonal equation

$$\|\nabla U_{p_0}(x)\| = P(x) \text{ and } U_{p_0}(p_0) = 0$$

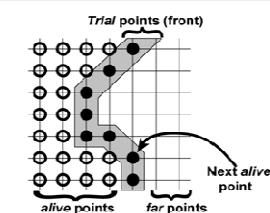
\Downarrow Upwind scheme



$$\max\left(\frac{U_{i,j} - U_{i-1,j}}{h}, \frac{U_{i,j} - U_{i+1,j}}{h}, 0\right)^2 + \max\left(\frac{U_{i,j} - U_{i,j-1}}{h}, \frac{U_{i,j} - U_{i,j+1}}{h}, 0\right)^2 = P_{ij}^2$$

Fast algorithm to compute the **action map** on the discretization grid :

- Sethian** Fast Marching: $N \log(N)$ complexity, first order.
- Kim** Group Marching: N complexity, first order.
- Sweeping (iterative) methods



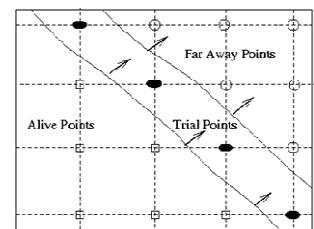
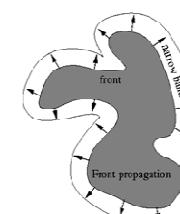
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Algorithme du Fast Marching

$$(\max\{u - U_{i-1,j}, u - U_{i+1,j}, 0\})^2 + (\max\{u - U_{i,j-1}, u - U_{i,j+1}, 0\})^2 = \tilde{P}_{ij}^2$$



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Algorithm for 2D Fast Marching

- Definitions :
 - Alive set : all grid points at which the action value \mathcal{U} has been reached and will not be changed;
 - Trial set : next grid points (4-connectivity neighbors) to be examined. An estimate $\tilde{\mathcal{U}}$ of \mathcal{U} has been computed using Equation (4) from alive points only (i.e. from \mathcal{U}) :

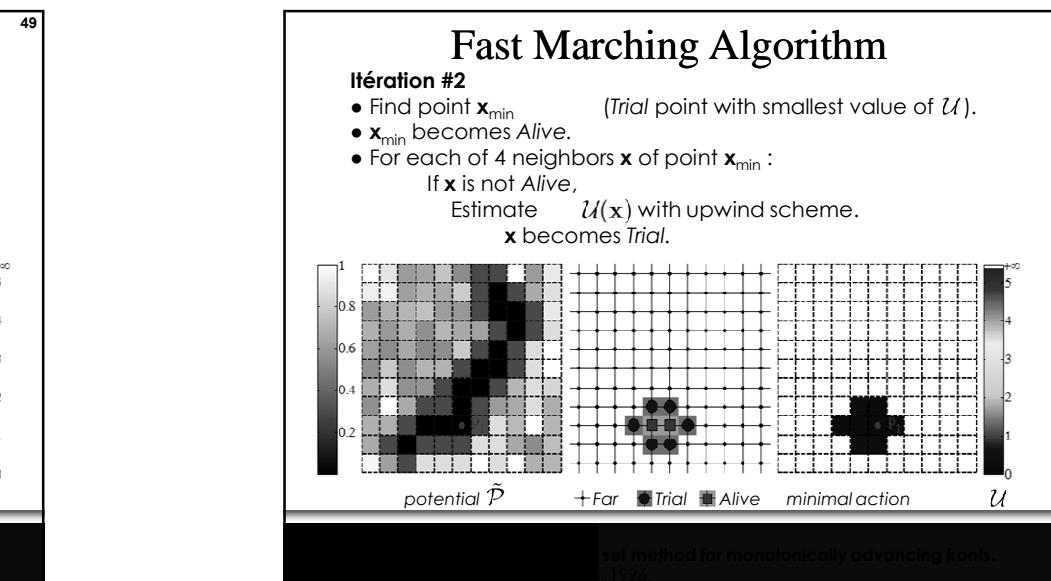
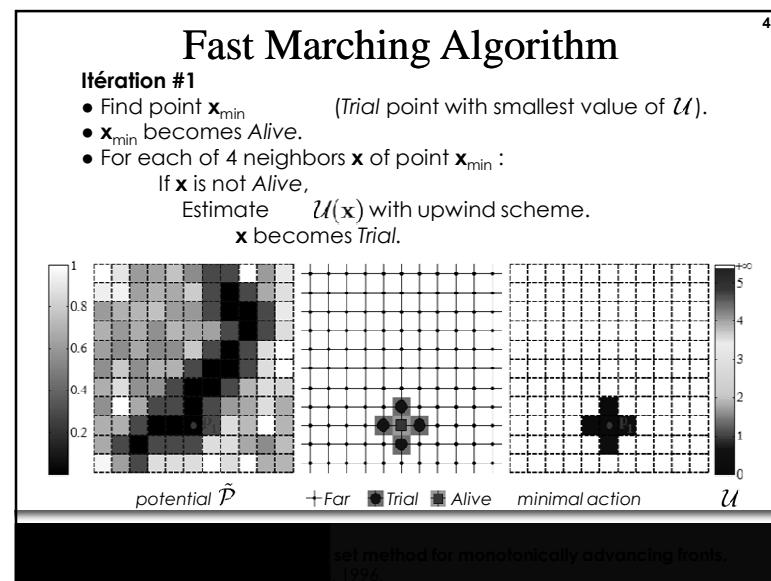
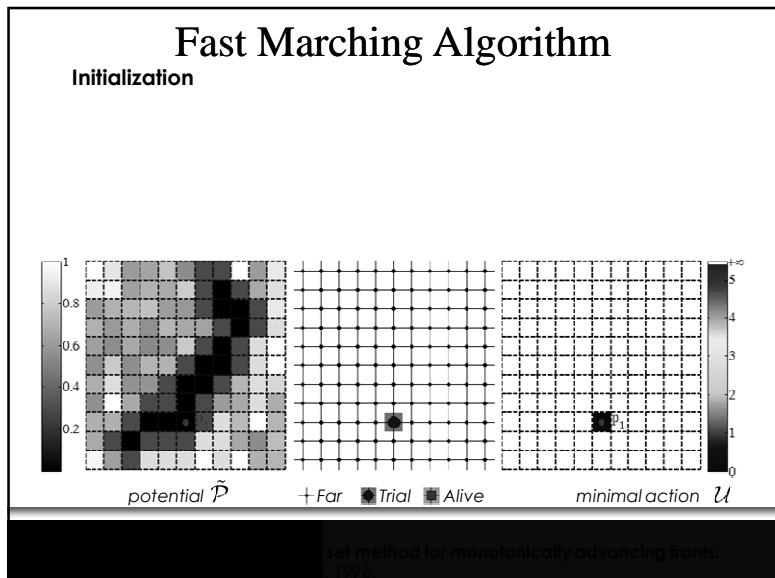
$$(\max\{u - \mathcal{U}_{i-1,j}, u - \mathcal{U}_{i+1,j}, 0\})^2 + (\max\{u - \mathcal{U}_{i,j-1}, u - \mathcal{U}_{i,j+1}, 0\})^2 = \tilde{\mathcal{P}}_{i,j}^2 \quad (4)$$

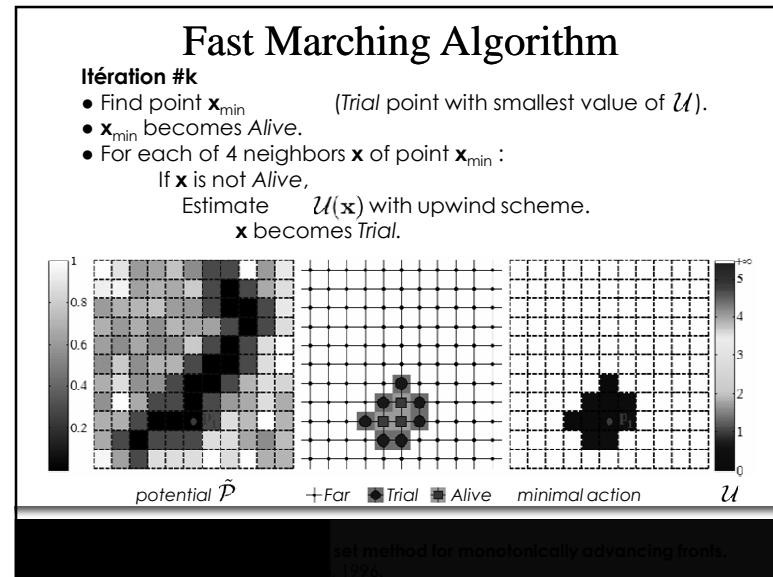
- Far set : all other grid points, there is not yet an estimate for \mathcal{U} ;
- Initialization :
 - Alive set is confined to the starting point p_0 , with $\mathcal{U}(p_0) = 0$;
 - Trial is confined to the four neighbors p of p_0 with initial value $\mathcal{U}(p) = \tilde{\mathcal{P}}(p)$ ($\mathcal{U}(p) = \infty$);
 - Far is the set of all other grid points with $\mathcal{U} = U = \infty$;
- Loop :
 - Let $p = (i_{min}, j_{min})$ be the Trial point with the smallest action \mathcal{U} ;
 - Move it from the Trial to the Alive set (i.e. $\mathcal{U}(p) = \mathcal{U}_{i_{min}, j_{min}}$ is frozen);
 - For each neighbor (i, j) (4-connectivity in 2D) of (i_{min}, j_{min}) :
 - If (i, j) is Far, add it to the Trial set and compute $\mathcal{U}_{i,j}$ using Table 2;
 - If (i, j) is Trial, update the action $\mathcal{U}_{i,j}$ using Eqn. (4) and Table 2.

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TAB. 1: Fast Marching algorithm
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**Algorithm for 2D Up-Wind Scheme**

Notice that for solving Equation (4), only alive points (\mathcal{U}) are considered. Considering the neighbors of grid point (i, j) in 4-connectivity. We note $\{A_1, A_2\}$ and $\{B_1, B_2\}$ the two couples of opposite neighbors such that we get the ordering $\mathcal{U}(A_1) \leq \mathcal{U}(A_2)$, $\mathcal{U}(B_1) \leq \mathcal{U}(B_2)$, and $\mathcal{U}(A_1) \leq \mathcal{U}(B_1)$. Considering that we have $u \geq \mathcal{U}(B_1) \geq \mathcal{U}(A_1)$, the equation derived is

$$(u - \mathcal{U}(A_1))^2 + (u - \mathcal{U}(B_1))^2 = \tilde{P}_{i,j}^2 \quad (5)$$

1. Computing the discriminant Δ of Equation (5) we have two possibilities

- If $\Delta \geq 0$, u should be the largest solution of Equation (5) ;
 - If the hypothesis $u > \mathcal{U}(B_1)$ is wrong, go to 2;
 - If this value is larger than $\mathcal{U}(B_1)$, this is the solution;
 - If $\Delta < 0$, B_1 has an action too large to influence the solution. It means that $u > \mathcal{U}(B_1)$ is false. Go to 2;

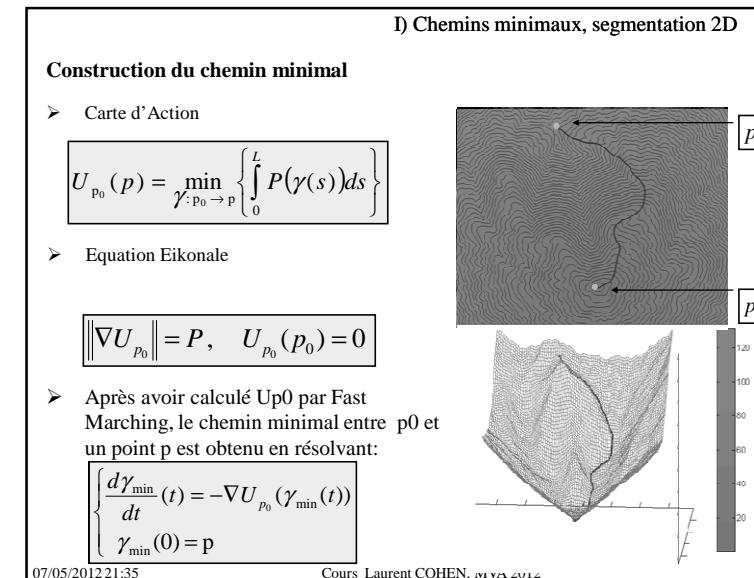
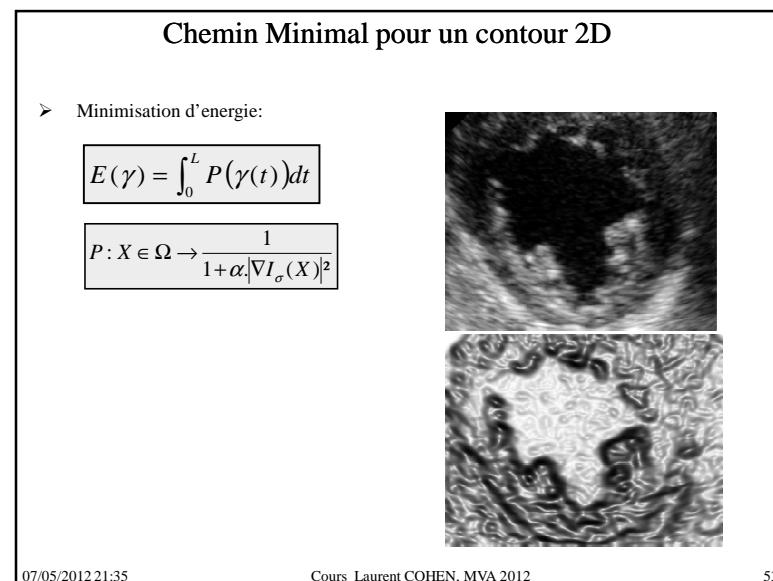
Simple calculus can replace case 1 by the test :

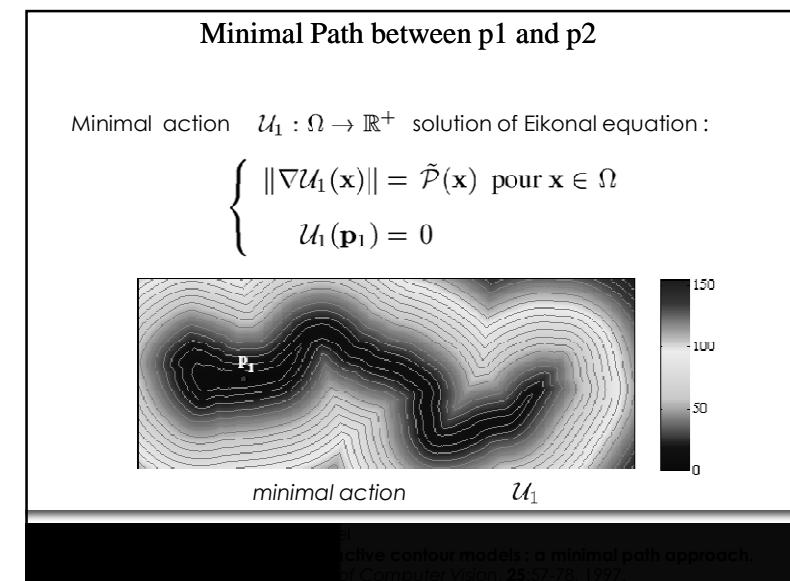
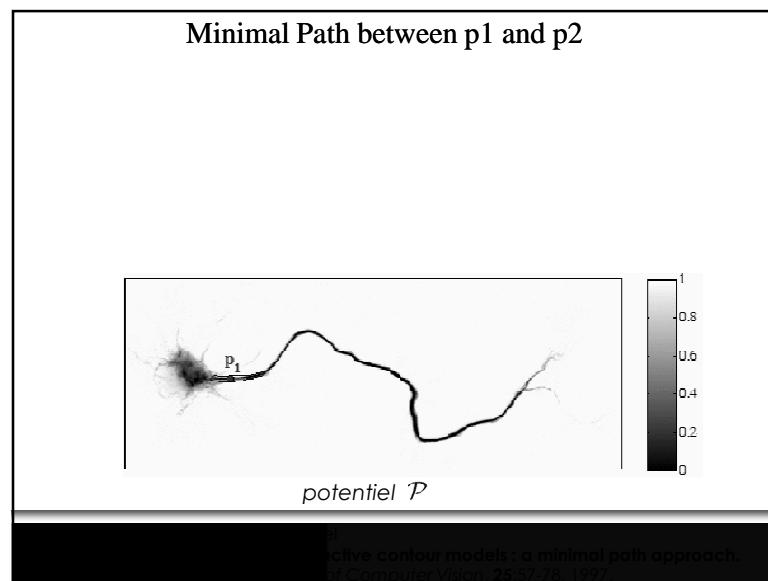
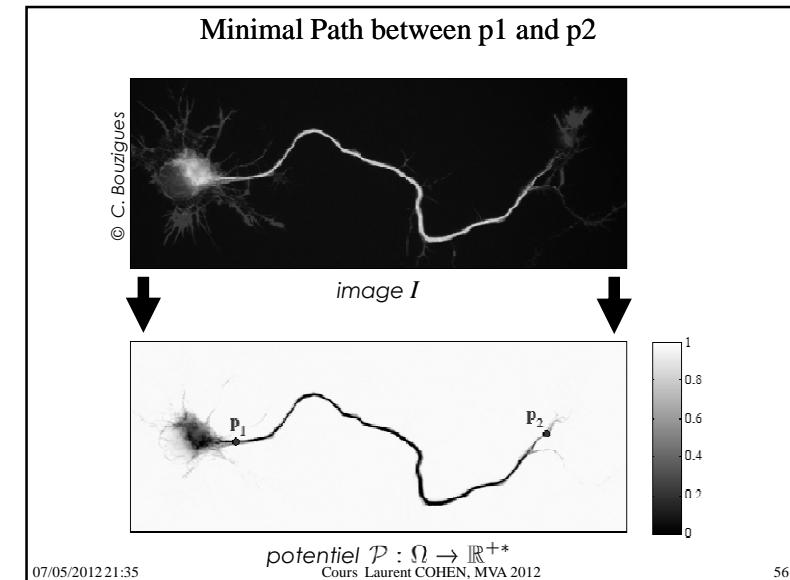
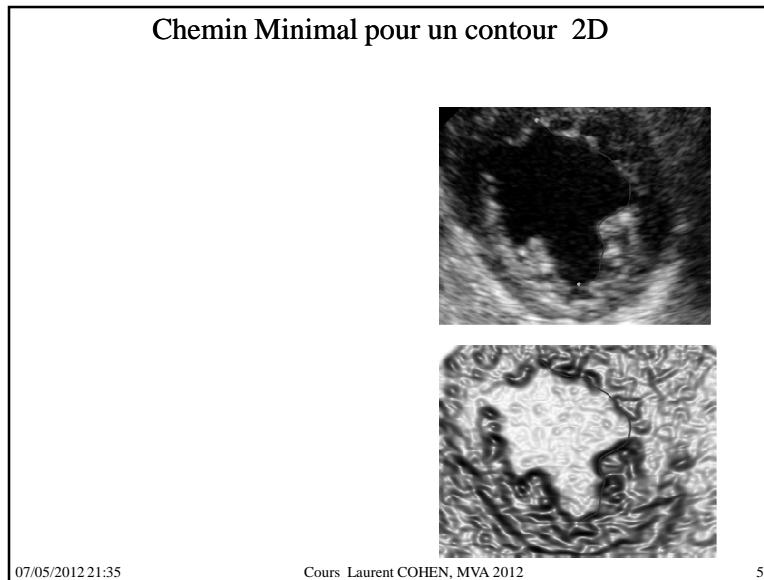
1bis. If $\tilde{P}_{i,j} > \mathcal{U}(B_1) - \mathcal{U}(A_1)$,
 $u = \frac{\mathcal{U}(B_1) + \mathcal{U}(A_1) + \sqrt{2\tilde{P}_{i,j}^2 - (\mathcal{U}(B_1) - \mathcal{U}(A_1))^2}}{2}$ is the largest solution of Equation (5),
else go to 2;

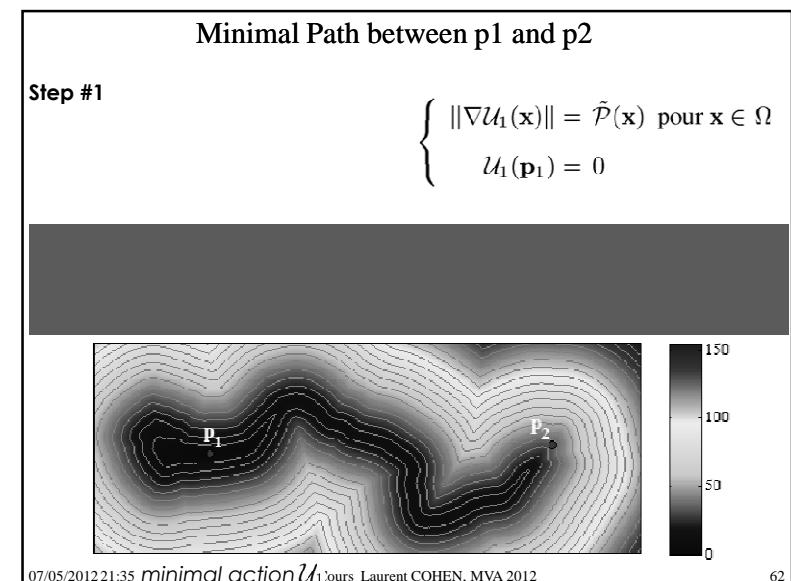
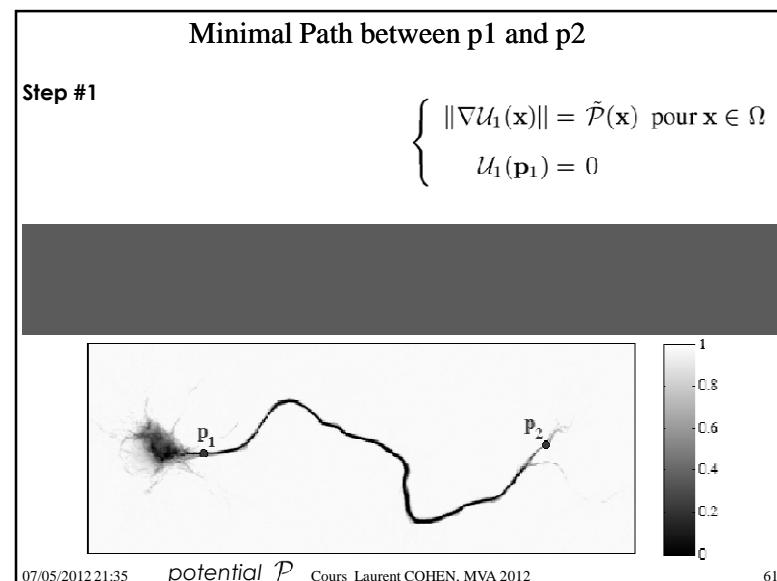
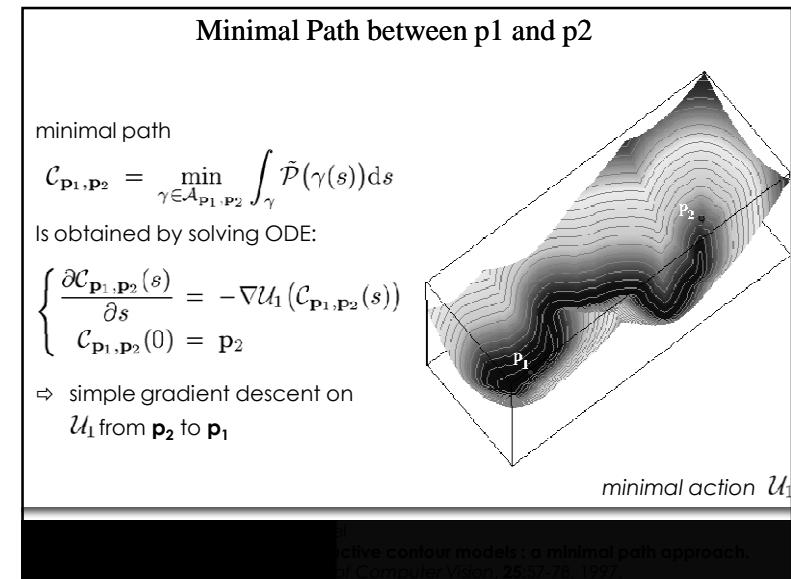
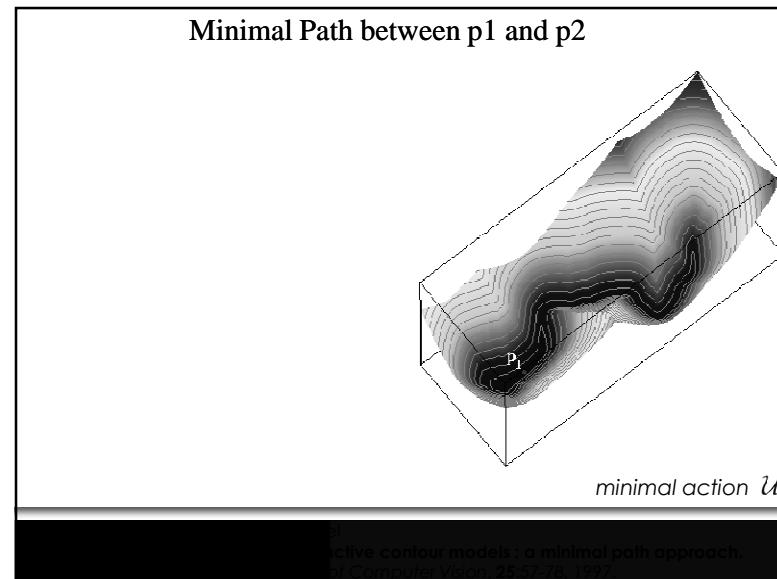
2. Considering that we have $u < \mathcal{U}(B_1)$ and $u \geq \mathcal{U}(A_1)$, we finally have $u = \mathcal{U}(A_1) + \tilde{P}_{i,j}$.

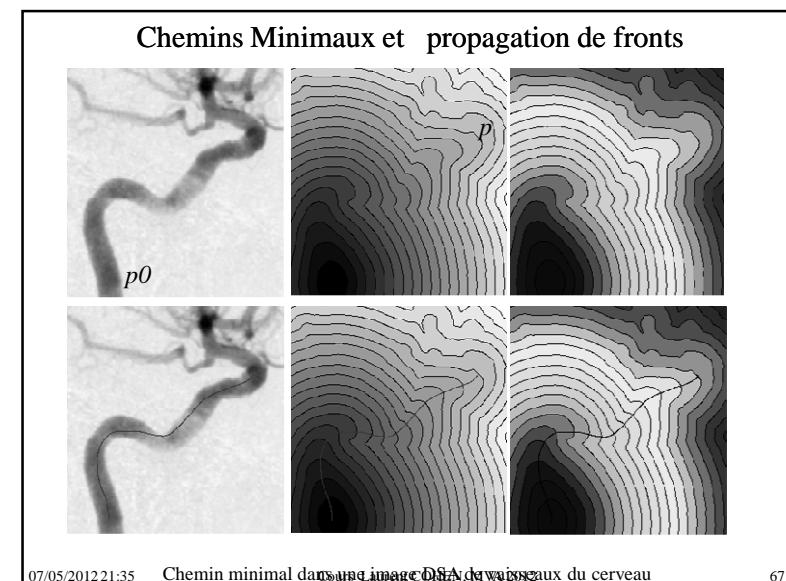
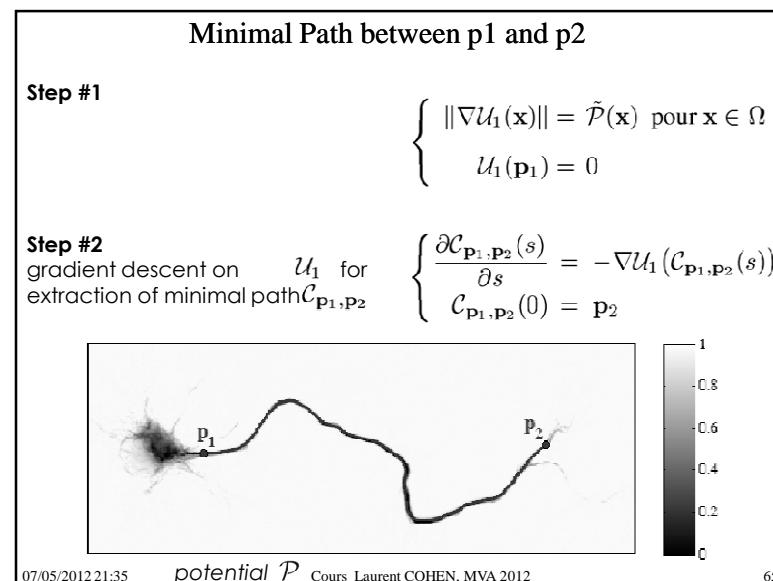
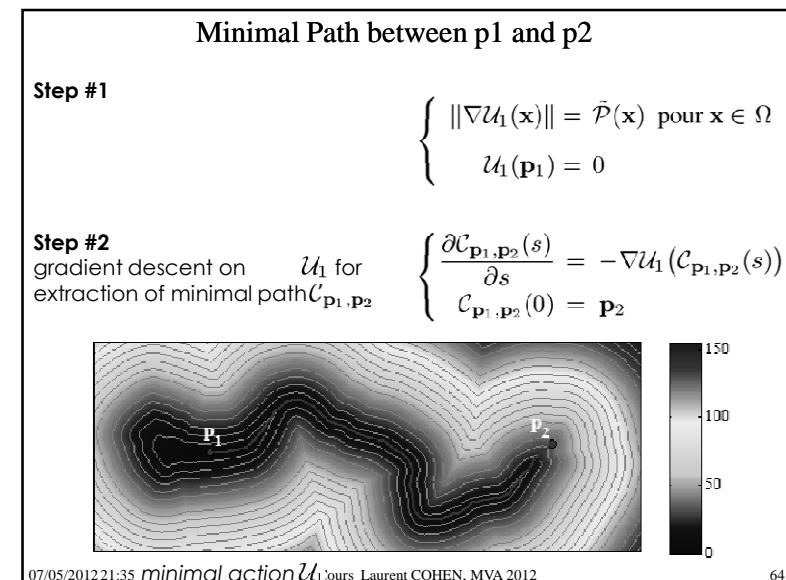
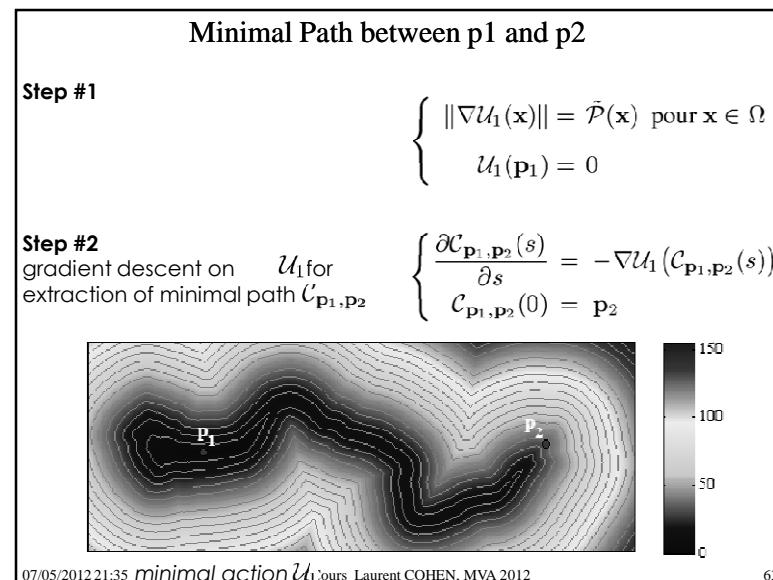
TAB. 2: Solving locally the upwind scheme
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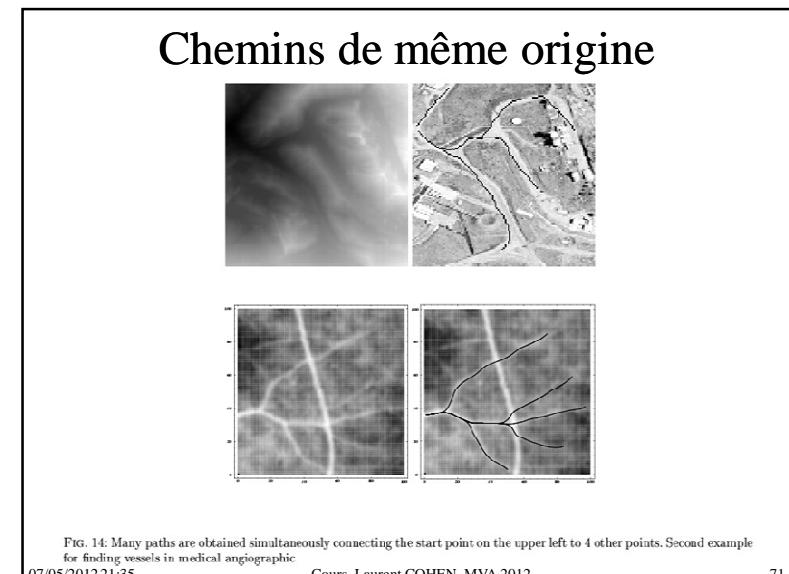
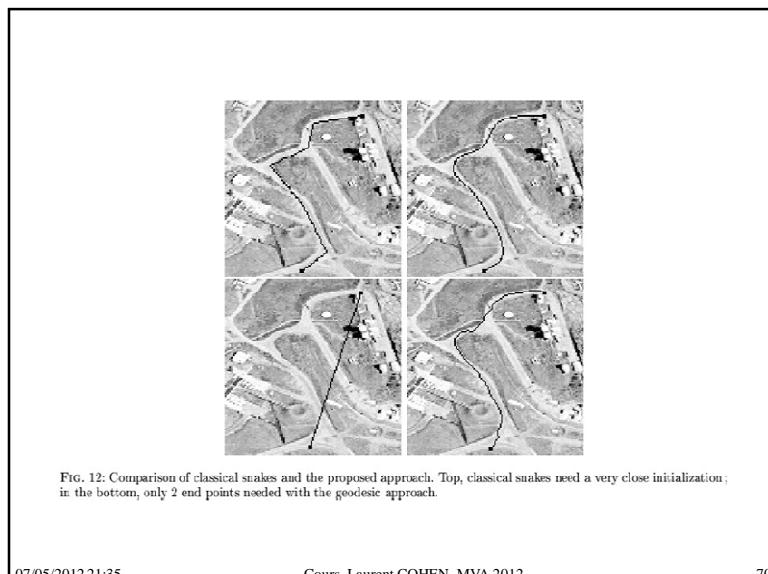
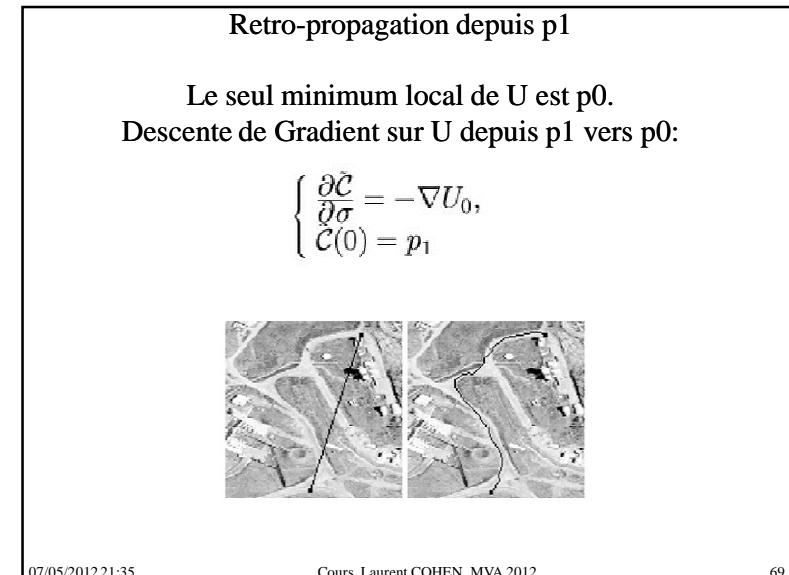
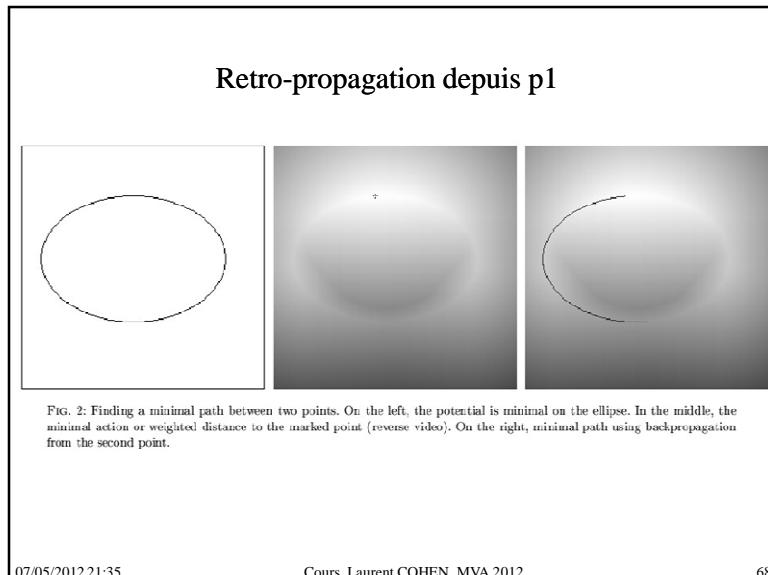
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Graph Search algorithms
Dynamic Programming

Minimization of $\int_{\Omega} \tilde{P}(\mathcal{C})ds$ (19)

- A^* algorithm : Dijkstra 1959
 - distance image initialized with value ∞ ,
 - expands to a neighbor pixel a previously obtained minimal path ending at the vertex with smallest current cost value.
 - 1 iteration per pixel and a search for the best pixel to update : $O(N \log N)$.
 - Similar to Fast marching but **not consistent**.
- F^* algorithm : Fischler, Tenenbaum, Wolf 1981.
 - same but sequential;
 - image scanned iteratively top to bottom, row by row, left to right followed by right to left, and then bottom to top.
 - similar in spirit to shape from shading but **not consistent**.

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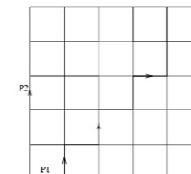
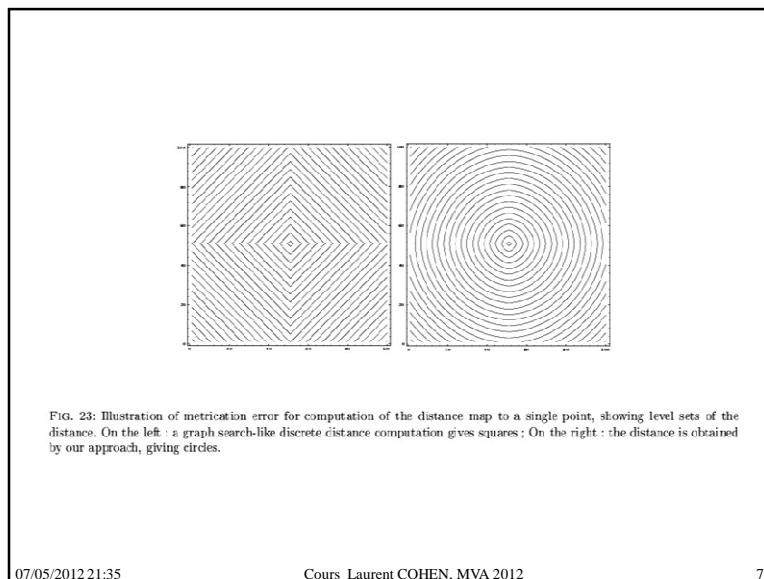
- Metrification error -

FIG. 22: An L^1 norm cause the shortest path to suffer from errors of up to 41%. In this case both P_1 and P_2 are optimal, and will stay optimal no matter how much we refine the (4-neighboring) grid.

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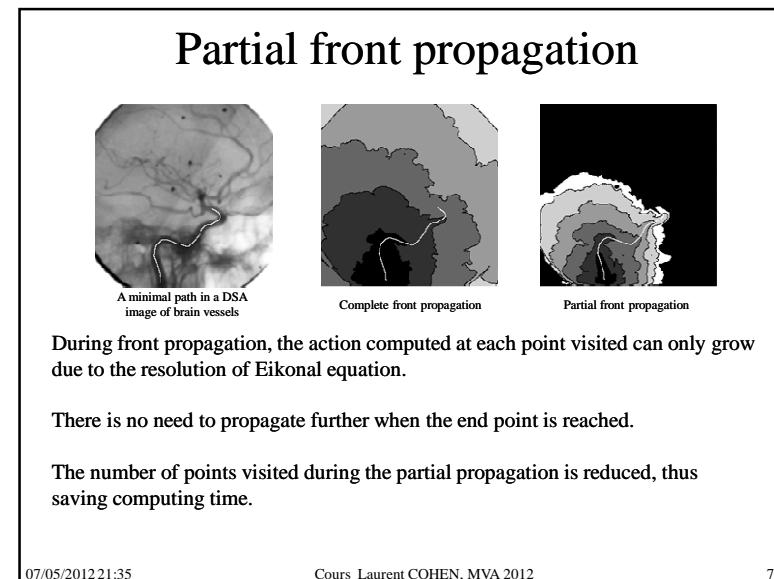
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Chemin minimal

On cherche un chemin reliant les deux points jaunes et suivant les zones les plus foncées.

Reference:
L. D. Cohen and R. Kimmel
Global minimum for active contour models : a minimal path approach.
International Journal on Computer Vision, **24**:57-78, 1997.

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Propagation par Fast Marching du point de départ au point d'arrivée

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Minimal paths for 2D segmentation

- ▶ $P(x) = 1 \implies$ droite (plus court chemin euclidien)

Chemin Carte de distance

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Minimal paths for 2D segmentation

► $P(x) = w + (I(x) - I(x_0))^2 \iff$ chemin d'intensité homogène



Chemin



Carte de distance

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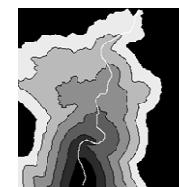
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propagation simultanée de fronts



Chemin minimal image DSA



propagation partielle du front



propagation simultanée

Propagation simultanée depuis les deux extrémités.

Point de Collision des 2 fronts: point selle de U se trouvant sur le chemin minimal a mi chemin en énergie.

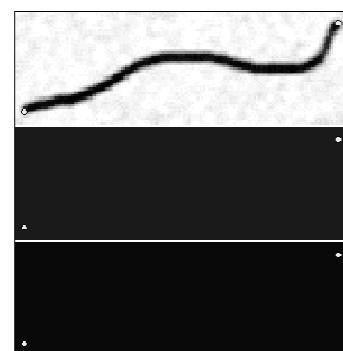
Réduction du nombre de points visités importante:
par exemple pour $P=1$, temps divisé par 2 en 2D et 4 en 3D

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Propagation de fronts simultanées aux deux extrémités



Reference:

T. Deschamps and L. D. Cohen
Minimal paths in 3D images and application to virtual endoscopy.
Proceedings ECCV'00, Dublin, Ireland, 2000.

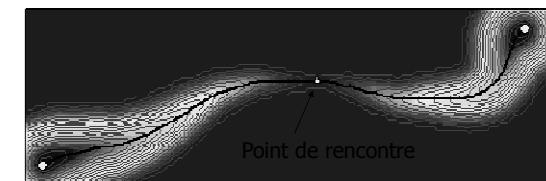
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Chemin minimal

Propagation de fronts simultanée jusqu'à leur rencontre



Reference:

T. Deschamps and L. D. Cohen
Minimal paths in 3D images and application to virtual endoscopy.
Proceedings ECCV'00, Dublin, Ireland, 2000.

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Action Minimale à un ensemble

STEP 1 : minimal action U of S as the minimal energy integrated along a path between start point p_0 in S and any point p in the image

Start point $C(0) = p_0 \in S$:

$$U_S(p) = \inf_{C(0) \in S; C(L) = p} E(C) = \inf_{p_0 \in S} U_{p_0}(p)$$

Solution of Eikonal equation:

$\|\nabla U_S(x)\| = P(x)$ and $U_S(p_0) = 0$ for all $p_0 \in S$

$$Z_{p_0} = \{x / \forall q \in S, U_q(x) \geq U_{p_0}(x)\}$$

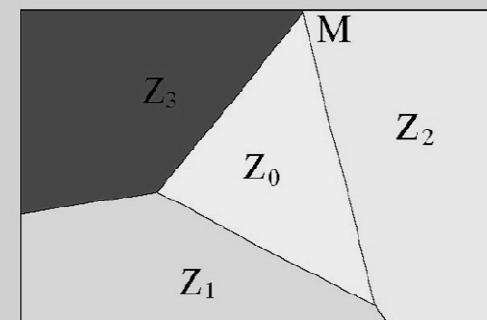
$$\forall x \in Z_{p_0}, \quad U_S(x) = U_{p_0}(x)$$

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Segmentation de Voronoi Voronoi Tessellation



$$\Pi(\psi, S) = \{Z(x_i, \psi)\}_{x_i \in S} \cup \{M(\psi, S)\}$$

[Dirichlet, 1850], [Voronoi, 1909]

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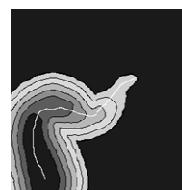
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Propagation avec une seule extrémité



Chemin minimal image DSA



Action minimale



Longueur du chemin minimal

Parfois il est difficile de donner une seconde extrémité (3D)

Calcul simultané de la longueur du chemin minimal en chaque point
Coût en calcul faible : inclus dans le fast marching avec $P = 1$.

Seconde extrémité : le premier point du front atteignant une longueur donnée.

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