On Pricing Basket Credit Default Swaps

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Portfolio Default Risk Models

- Modeling portfolio default risk is a key topic in credit risk management.
- Structural firm value approach (Black and Scholes (1973) and Merton (1974)) and reduced form intensity-based approach (Jarrow and Turnbull (1995) and Madan and Unal (1998)).
- Intensity approach widely used in modeling portfolio default risk.
- Bottom-up models and top-down models.
- Bottom-up models focus on modeling default intensities of individual reference entities and their aggregation to form a portfolio default intensity. (Duffie and Garleanu (2001), Jarrow and Yu (2001), Schönbucher and Schubert (2001), Giesecke and Goldberg (2004), etc.)
- Major applications are valuation of portfolio credit derivatives such as CDOs and and basket CDSs.

Basket Credit Default Swaps

- The kth to default basket CDS is a popular type of multi-name credit derivatives.
- Protection buyer of a kth to default basket CDS pays periodic premiums to protection seller according to some pre-determined swap rates until occurrence of kth default in a reference pool or maturity.
- Protection seller of kth to default basket CDS pays to protection buyer amount of loss due to kth default in the pool when it occurs.
- Key to valuing these derivatives is to know portfolio loss distribution.

Literatures

- Different approaches have been proposed in literature for evaluating kth default basket CDS under intensity-based default **contagion** model.
- Herbertsson & Rootze (2006) introduce a matrix-analytic approach to value kth to default basket CDS.
- Yu (2007) adopts total hazard construction method to generate default times with a broad class of correlation structure.
- Zheng & Jiang (2009) use total hazard construction to derive joint distribution of default times and an analytical formula for basket CDS rates in a homogeneous case.

Model Setup

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space, where we assume P is a risk-neutral martingale measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying usual conditions.

- We consider a portfolio with n credit entities. For each $i = 1, 2, \cdots, n$, let τ_i be default time of name i. Write $N_i(t) = 1_{\{\tau_i \leq t\}}$ for a single jump process associated with default time τ_i, and $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous, P-completed, natural filtration generated by N^i.

- Suppose $\{X_t\}_{t\geq 0}$ is state process, which represents common factor process for joint defaults. Write $\{\mathcal{F}_t^X\}_{t\geq 0}$ for right-continuous, P-completed, natural filtration generated by process $\{X_t\}_{t\geq 0}$.

- For each $t \geq 0$, write

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \ldots \vee \mathcal{F}_t^n \; .$$

  Here $\mathcal{F}_t$ represents minimal σ-algebra containing information about processes X and $\{N_i\}_{i=1}^n$ up to and including time t.

- We assume that for each $i = 1, 2, \cdots, n$, N_i possesses a nonnegative, $\{\mathcal{F}_t\}_{t\geq 0}$-predictable, intensity process λ_i satisfying

$$E\left(\int_0^t \lambda_i(s)ds\right) < \infty, \quad t \geq 0,$$

such that compensated process:

$$M_i(t) := N_i(t) - \int_0^{t\wedge\tau_i} \lambda_i(s)ds \ , \quad t \geq 0 \ ,$$

is an $(\{\mathcal{F}_t\}_{t\geq 0}, P)$-martingale.

- We further assume that stochastic process X is "exogenous", i.e., conditional on whole path of X, λ_i are $\{\vee_i \mathcal{F}_t^i\}_{t\geq 0}$-predictable.

- To model interaction we consider following intensity rate process:

$$\lambda_i(t) = a_i(t) + \sum_{j\neq i} b_{ij}(t) e^{-d_{ij}(t-\tau_j)} 1_{\{\tau_j \leq t\}}, \quad i = 1, 2, \ldots, n, \quad (1)$$

where $a_i(t)$ and $b_{ij}(t)$ are $\mathcal{F}^X$-adapted processes, and d_{ij} are nonnegative constants representing rates of decay.

Homogeneous Case

- Contagion intensity process:

$$\lambda_i(t) = a\left(1 + \sum_{j\neq i} c e^{-d(t-\tau_j)} 1_{\{\tau_j \le t\}}\right), \quad i = 1, 2, \ldots, n \ , \tag{2}$$

where a positive and c, d nonnegative constants.

- Suppose, for each $i = 1, 2, \cdots, n$, τ^i denotes ith default time.
- Let $\lambda^{k+1}(t)$ be $(k+1)$th default rate at time t that triggers τ^{k+1}
- Then $\lambda^{k+1}(t)$ will be sum of default rates of surviving entities after τ^k:

$$\lambda^{k+1}(t) = a(1 + \sum_{i=1}^{k} c e^{-d(t-\tau^i)})(n-k),$$

where $\tau^k < t \le \tau^{k+1}$.

- We have

$$P(\tau^{k+1} > t \mid \tau^i, i = 1, \ldots, k) = \exp\{-\int_{\tau^k}^t \lambda^{k+1}(s) ds\},$$

where $t \geq \tau^k$, which implies,

$$f_{\tau^{k+1}|\tau^i, i=1,\ldots,k}(t) = a(1+\sum_{i=1}^{k} ce^{-d(t-\tau^i)})(n-k) \exp\{-\int_{\tau^k}^t a(1+\sum_{i=1}^{k} ce^{-d(s-\tau^i)})(n-k)ds\}, \tag{3}$$

where $f_{\tau^1}(t) = nae^{-nat}$.

- One can apply (3) to derive joint pdf of $\tau^1, \tau^2, \ldots, \tau^{k+1}$ by using recursion:

$$f_{\tau^1,\tau^2,\ldots,\tau^{k+1}}(t_1, t_2, \ldots, t_{k+1}) = f_{\tau^{k+1}|\tau^i=t_i, i=1,\ldots,k}(t_{k+1}) f_{\tau^1,\tau^2,\ldots,\tau^k}(t_1, t_2, \ldots, t_k)\,,$$

where $t_1 < t_2 < \ldots < t_{k+1}$.

- Unconditional density function of τ^k is given by integral:

$$f_{\tau^k}(t) = \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_2} f_{\tau^1,\tau^2,\ldots,\tau^{k-1},\tau^k}(t_1, t_2, \ldots, t_{k-1}, t) dt_1 \cdots dt_{k-2} dt_{k-1} \tag{4}$$

Example 1 *If $n = 2$, then joint density function of τ^1 and τ^2 is:*

$$f_{\tau^1,\tau^2}(t_1, t_2) = \begin{cases} 2a^2(1 + ce^{-d(t_2-t_1)}) \exp\{-a(t_1 + t_2) + \frac{ac}{d}(e^{-d(t_2-t_1)} - 1)\}, & t_1 \leq t_2 \\ 0, & t_1 > t_2. \end{cases}$$

Unconditional density function of τ^1 is given by

$$f_{\tau^1}(t) = 2ae^{-2at},$$

and unconditional density function of τ^2 is:

$$f_{\tau^2}(t) = 2a^2 \int_0^t (1 + ce^{-d(t-t_1)}) \exp\{-a(t + t_1) + \frac{ac}{d}(e^{-d(t-t_1)} - 1)\}dt_1.$$

Proposition 1 *Suppose there are n entities in our portfolio, where contagion intensity process follows*

$$\lambda_i(t) = a\left(1 + \sum_{j \neq i} c1_{\{\tau_j \leq t\}}\right), \quad i = 1, 2, \ldots, n \ . \tag{5}$$

Then unconditional density functions of τ^k, $k = 1, 2, \ldots, n$, satisfy following recursive formula:

$$f_{\tau^{k+1}}(t) \ = \ a(1 + kc)(n - k) \int_0^t f_{\tau^k}(u)e^{-a(1+kc)(n-k)(t-u)}du, \tag{6}$$

where initial condition is:

$$f_{\tau^1}(t) = nae^{-nat}.$$

Corollary 1 *Assume that $c \neq 1/i$ for $i = 1, 2, \ldots, n-1$. Then unconditional density function of τ^k is given by:*

$$f_{\tau^k}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} a e^{-\beta_j at}, \tag{7}$$

where coefficients are given by:

$$\begin{cases} \alpha_{k+1,j} = \begin{cases} \dfrac{\alpha_{k,j}\beta_k}{\beta_k - \beta_j}, & j = 0, 1, \ldots, k-1 \\ -\sum\limits_{u=0}^{k-1} \dfrac{\alpha_{k,u}\beta_k}{\beta_k - \beta_u}, & j = k \end{cases} \\ \beta_j = (n-j)(1+jc) \end{cases}$$

and $\alpha_{1,0} = n$.

Stochastic Intensity

Proposition 2 *Suppose contagion stochastic intensity process follows (2) with constant intensity rate a replaced by an "exogenous" stochastic process* $X(t)$*. Then unconditional density functions of* τ^k*,* $k = 1, 2, \ldots, n$*, given realization of* $(X(s))_{0\le s<\infty}$*, satisfy following recursive formula:*

$$f_{\tau^{k+1}|(X(s))_{0\le s<\infty}}(t) = (n-k)(1+kc)X(t)\int_0^t f_{\tau^k|(X(s))_{0\le s<\infty}}(u)e^{-(n-k)(1+kc)\int_u^t X(s)ds}du, \tag{8}$$

for $k = 1, 2, \ldots, n-1$*, where* $f_{\tau^1|(X(s))_{0\le s<\infty}}(t) = nX(t)e^{-n\int_0^t X(u)du}$.

Corollary 2 *Assume that* $c \neq 1/i$ *for* $i = 1, 2, \ldots, n-1$*. Then density function of* τ^k*, given* $\mathcal{F}_\infty^X$ *is given by*

$$f_{\tau^k|(X(s))_{0\le s<\infty}}(t) = \sum_{j=0}^{k-1}\alpha_{k,j}X(t)e^{-\beta_j\int_0^t X(u)du}, \tag{9}$$

where $\alpha_{k,j}$ *and* β_j *are given in Corollary 1.*

Multi-state Stochastic Intensity Process

- Assume stochastic intensity process $X(t)$ alternates between x_1 and x_2, so that

$$X(t) = \begin{cases} x_1, & \text{when exogenous state lies in 1} \\ x_2, & \text{when exogenous state lies in 2.} \end{cases}$$

- Let η_i denote rate of leaving state i and π_i random time to leave state i, where π_i is an exponential random variable, i.e.,

$$P(\pi_i > t) = e^{-\eta_i t}, i = 1, 2.$$

- In this case, we consider a two-state, Markov regime-switching, intensity-based model for portfolio default risk.
- Let $T_i(t)$ be total time between 0 and t during which $X(s) = x_1$, starting from $X(0) = x_i$.
- We then draw exponential random variables ξ_1 with intensity η_1 and ξ_2 with intensity η_2 independent of X.

- Therefore

$$T_1(t) \hat{=} \begin{cases} \xi_1 + T_2(t - \xi_1), & \xi_1 \le t \\ t, & \xi_1 > t \end{cases}, \quad T_2(t) \hat{=} \begin{cases} T_1(t - \xi_2), & \xi_2 \le t \\ 0, & \xi_2 > t \end{cases}, \tag{10}$$

  where "$\hat{=}$" means "equals in distribution".

- Let

$$\psi_i(l, t) = E(e^{-lT_i(t)}), \ i = 1, 2, \ l \in \mathcal{R}.$$

  By (10), we have

$$\begin{cases} \psi_1(l, t) = \int_0^t \eta_1 e^{-(\eta_1 + l)u} \psi_2(l, t - u) du + e^{-(\eta_1 + l)t}, \\ \psi_2(l, t) = \int_0^t \eta_2 e^{-\eta_2 u} \psi_1(l, t - u) du + e^{-\eta_2 t}. \end{cases}$$

- Taking Laplace transform on both sides gives:

$$\begin{cases} \mathcal{L}(\psi_1(l, t))(\cdot, s) = \frac{1}{l+s+\eta_1} + \frac{\eta_1}{l+s+\eta_1} \mathcal{L}(\psi_2(l, t))(\cdot, s), \\ \mathcal{L}(\psi_2(l, t))(\cdot, s) = \frac{1}{s+\eta_2} + \frac{\eta_2}{s+\eta_2} \mathcal{L}(\psi_1(l, t))(\cdot, s). \end{cases}$$

- Therefore,

$$\mathcal{L}(\psi_1(l, t))(\cdot, s) = \frac{s + \eta_1 + \eta_2}{s^2 + s\eta_1 + s\eta_2 + sl + l\eta_2}.$$

- By taking inverse Laplace transform of above equation, we have

$$\psi_1(l,t) = \begin{cases} e^{-\alpha t}[\cos(\sqrt{\omega}t) + \frac{\beta}{\sqrt{\omega}}\sin(\sqrt{\omega}t)], & \omega > 0 \\ e^{-\alpha t}(1+\beta t), & \omega = 0 \\ e^{-\alpha t}[\cosh(\sqrt{-\omega}t) + \frac{\beta}{\sqrt{-\omega}}\sinh(\sqrt{-\omega}t)], & \omega < 0. \end{cases}$$

Here

$$\alpha = \frac{\eta_1 + \eta_2 + l}{2}, \ \beta = \frac{\eta_1 + \eta_2 - l}{2}, \ \omega = l\eta_2 - \alpha^2.$$

- Similarly we have

$$\mathcal{L}(\psi_2(l,t))(\cdot, s) = \frac{s + \eta_1 + \eta_2 + l}{s^2 + s\eta_1 + s\eta_2 + sl + l\eta_2},$$

and

$$\psi_2(l,t) = \begin{cases} e^{-\alpha t}[\cos(\sqrt{\omega}t) + \frac{\alpha}{\sqrt{\omega}}\sin(\sqrt{\omega}t)], & \omega > 0 \\ e^{-\alpha t}(1+\alpha t), & \omega = 0 \\ e^{-\alpha t}[\cosh(\sqrt{-\omega}t) + \frac{\alpha}{\sqrt{-\omega}}\sinh(\sqrt{-\omega}t)], & \omega < 0 \end{cases}$$

- We proceed with derivation of unconditional density function of τ^k.

Given $X(0) = x_i$,

$E(e^{-L\int_0^t X(s)ds}) = E(e^{-L(x_1 T_i(t) + x_2(t - T_i(t)))}) = e^{-Lx_2 t}\psi_i(L(x_1 - x_2), t),\ L > 0$.

Then

$$E(X(t)e^{-L\int_0^t X(s)ds}) = -\frac{1}{L}\frac{d(e^{-Lx_2 t}\psi_i(L(x_1 - x_2), t))}{dt}.$$

- Combining results in Corollary 2, unconditional density function of τ^k can be obtained: when $X(0) = x_i$,

$$f_{\tau^k}(t) = -\sum_{j=0}^{k-1}\frac{\alpha_{k,j}}{\beta_j}\frac{d(e^{-\beta_j x_2 t}\psi_i(\beta_j(x_1 - x_2), t))}{dt}. \qquad (11)$$

Heterogeneous Case

- For simplicity of discussion, we consider a two-group of entities.
- First group(G_1) consists n_1 obligors and $\lambda_i(t)$ denotes default rate of name i in G_1 at time t, while second group(G_2) consists n_2 obligors and $\tilde{\lambda}_i(t)$ denotes default rate of name i in G_2 at time t.
- Interacting intensity process of two-group case is assumed as follows:

$$\begin{cases} \lambda_i(t) = a\left(1 + b\sum_{j\neq i} 1_{\{\tau_j \leq t\}} + c\sum_{j} 1_{\{\tilde{\tau}_j \leq t\}}\right), \\ \tilde{\lambda}_i(t) = \tilde{a}\left(1 + \tilde{b}\sum_{j} 1_{\{\tau_j \leq t\}} + \tilde{c}\sum_{j\neq i} 1_{\{\tilde{\tau}_j \leq t\}}\right), \end{cases} \tag{12}$$

  where τ_j and $\tilde{\tau}_j$ denote default time of name j in G_1 and G_2, respectively, $a, \tilde{a}$ are positive constants and $b, c, \tilde{b}, \tilde{c}$ are nonnegative constants.
- Let N^i be number of defaults in G_1 right after ith default of our portfolio, where N^0 is assigned to be 0.

Proposition 3 *Suppose our portfolio has two groups of entities G_1 and G_2, where G_1 and G_2 consist of n_1 and n_2 obligors respectively, and $n = n_1 + n_2$. For each $t \geq 0$, let $\lambda_i(t)$ and $\tilde{\lambda}_i(t)$ denote default rates of name i in G_1 and G_2 at time t, respectively. These default rates follow (12). For each $i = 1, 2, \cdots, n$, let N^i denote number of defaults in G_1 right after ith default of our portfolio. Then recursive formula of joint distribution of τ^k and N^k is given by:*

$$\begin{aligned} f_{\tau^{k+1},N^{k+1}}(t, m+1) \;=\; & \tilde{\zeta}_{k,m+1} \int_0^t f_{\tau^k,N^k}(u, m+1) e^{-(\zeta_{k,m+1}+\tilde{\zeta}_{k,m+1})(t-u)} du \\ & + \zeta_{k,m} \int_0^t f_{\tau^k,N^k}(u, m) e^{-(\zeta_{k,m}+\tilde{\zeta}_{k,m})(t-u)} du, \end{aligned} \tag{13}$$

where

$$\begin{cases} f_{\tau^k,N^k}(t, m) = -\dfrac{dP(\tau^k > t, N^k = m)}{dt}, \\ \zeta_{k,m} = a(n_1 - m)[1 + bm + c(k - m)], \\ \tilde{\zeta}_{k,m} = \tilde{a}(n_2 - (k - m))[1 + \tilde{b}m + \tilde{c}(k - m)], \end{cases}$$

and $f_{\tau^1,N^1}(t, 1) = n_1 a e^{-(n_1 a + n_2 \tilde{a})t}$, $f_{\tau^1,N^1}(t, 0) = n_2 \tilde{a} e^{-(n_1 a + n_2 \tilde{a})t}$, and $f_{\tau^1,N^1}(t, m) = 0$ for $m \neq 0, 1$.

Corollary 3 *Assume that $\beta_{k,m} \neq \beta_{i,j}$ for $1 \leq k \leq n, \max\{0, k-n_2\} \leq m \leq \min\{k, n_1\}$ and $i = 0, \ldots, k-1, j = 0, \ldots, m$. Then joint density function of τ^k and N^k is given by:*

$$f_{\tau^k, N^k}(t, m) = \sum_{i=0}^{k-1} \sum_{j=0}^{m} \alpha_{k,m,i,j} e^{-\beta_{i,j} t}, \tag{14}$$

where coefficients are given by recursive formula:

$$\begin{cases} \alpha_{k+1,m+1,i,j} = \begin{cases} \dfrac{\alpha_{k,m+1,i,j}\tilde{\zeta}_{k,m+1}}{\beta_{k,m+1} - \beta_{i,j}} + \dfrac{\alpha_{k,m,i,j}\zeta_{k,m}}{\beta_{k,m} - \beta_{i,j}}, & i = 0, 1, \ldots, k-1, j = 0, \ldots, m \\ \dfrac{\alpha_{k,m+1,i,j}\tilde{\zeta}_{k,m+1}}{\beta_{k,m+1} - \beta_{i,j}}, & i = 0, 1, \ldots, k-1, j = m+1 \\ -\sum\limits_{u=0}^{k-1} \sum\limits_{v=0}^{m+1} \dfrac{\alpha_{k,m+1,u,v}\tilde{\zeta}_{k,m+1}}{\beta_{k,m+1} - \beta_{u,v}}, & i = k, j = m+1 \\ -\sum\limits_{u=0}^{k-1} \sum\limits_{v=0}^{m} \dfrac{\alpha_{k,m,u,v}\zeta_{k,m}}{\beta_{k,m} - \beta_{u,v}}, & i = k, j = m \\ 0, & \text{otherwise} \end{cases} \\ \beta_{i,j} = \zeta_{i,j} + \tilde{\zeta}_{i,j} \end{cases}$$

and boundary conditions are as follows:

$$\alpha_{1,0,0,0} = n_2\tilde{a},\ \ \alpha_{1,1,0,0} = n_1 a,\ \ \alpha_{1,1,0,1} = 0,\ \ \alpha_{1,m,i,j} = 0, m \neq 0, 1,$$

and

$$\alpha_{k+1,0,i,0} = \begin{cases} \dfrac{\tilde{\zeta}_{k,0}\alpha_{k,0,i,0}}{\beta_{k,0} - \beta_{i,0}}, & i = 0. \dots, k-1 \\ -\sum\limits_{u=0}^{k-1} \dfrac{\tilde{\zeta}_{k,0}\alpha_{k,0,u,0}}{\beta_{k,0} - \beta_{u,0}}, & i = k. \end{cases}$$

As a result, unconditional density function of τ^k is given by

$$f_{\tau^k}(t) = \sum_{m=\max\{0,k-n_2\}}^{\min\{k,n_1\}} f_{\tau^k,N^k}(t,m).$$

Evaluation of Basket CDS Rates

- Consider a kth to default basket CDS with maturity T.
- Assume S_k is kth swap rate, and R is recovery rate and r is annualized riskless interest rate.
- Protection buyer A pays a periodic fee $S_k\Delta_i$ to protection seller B at time t_i, $i = 1, 2, \ldots, N$, where $0 = t_0 < t_1 < \ldots < t_N = T$ and $\Delta_i = t_i - t_{i-1}$.
- If kth default happens in interval $[t_j, t_{j+1}]$, A will also pay B accrued default premium up to τ^k.
- On the other hand, if $\tau^k \leq T$, B will pay A loss occurred at τ^k, that is, $1 - R$.
- Swap rate S_k is given by

$$S_k = \frac{(1-R)E(e^{-r\tau^k}1_{\{\tau^k \leq T\}})}{\Sigma_{i=1}^N E(\Delta_i e^{-rt_i}1_{\{\tau^k > t_i\}} + (\tau^k - t_{i-1})e^{-r\tau^k}1_{\{t_{i-1} < \tau^k \leq t_i\}})}. \tag{15}$$

Test 1 (Exponential Decay Case)

Basket CDS rates with exponential decay intensity-based default model $(n = 2, k = 2, T = 3, \Delta = 0.5, R = 0.5, r = 0.05)$

	c	0.2	1	5
a	d			
0.1	0.001	0.0134	0.0211	0.0479
	0.01	0.0134	0.0210	0.0477
	0.1	0.0132	0.0203	0.0459
	1	0.0123	0.0160	0.0322
	10	0.0115	0.0120	0.0147
	100	0.0114	0.0114	0.0117
1	0.001	0.3654	0.4961	0.7529
	0.01	0.3651	0.4955	0.7526
	0.1	0.3626	0.4898	0.7502
	1	0.3464	0.4390	0.7184
	10	0.3262	0.3447	0.4392
	100	0.3222	0.3242	0.3342

Test 2 (Markov Regime-Switching Case)

Basket CDS rates in two-state stochastic intensity process case ($n = 10, T = 3, \Delta = 0.5, R = 0.5, r = 0.05, c = 3, X(0) = x_1$.)
Condition 1: $x_1 = 1, x_2 = 1, \eta_1 = 1, \eta_2 = 1$,
Condition 2: $x_1 = 1, x_2 = 2, \eta_1 = 1, \eta_2 = 1$,
Condition 3: $x_1 = 1, x_2 = 2, \eta_1 = 1, \eta_2 = 2$,
Condition 4: $x_1 = 1, x_2 = 2, \eta_1 = 2, \eta_2 = 1$.

k	Condition 1	Condition 2	Condition 3	Condition 4
1	5.0242	5.2507	5.2409	5.4575
2	3.9288	4.1170	4.1087	4.2891
3	3.4456	3.6184	3.6106	3.7766
4	3.1369	3.3005	3.2930	3.4503
5	2.9035	3.0605	3.0532	3.2043
6	2.7070	2.8588	2.8516	2.9979
7	2.5270	2.6743	2.6672	2.8093
8	2.3473	2.4904	2.4833	2.6214
9	2.1459	2.2847	2.2775	2.4114
10	1.8608	1.9945	1.9870	2.1159

Test 3 (Heterogeneous Case)

Basket CDS rates in heterogeneous case with 2 groups ($n_1 = 5, n_2 = 5, T = 3, \Delta = 0.5, R = 0.5, r = 0.05, a = \tilde{a} = 1.$)
Condition 1: $b = \tilde{b} = c = \tilde{c} = 3$, Condition 2: $b = \tilde{c} = 3, \tilde{b} = c = 0.3$,
Condition 3: $b = \tilde{b} = c = \tilde{c} = 0.3$, Condition 4: $b = \tilde{b} = 3, c = \tilde{c} = 0.3$.
Analytic approach takes less than one second to compute all swap rates of one column with MATLAB on a computer with an Intel 3.2 GHz CPU, while Monte Carlo approach takes more than 5 minutes to run 100,000 simulations.

k	Condition	1	Condition	2	Condition	3	Condition	4
	AP	MC	AP	MC	AP	MC	AP	MC
1	5.0242	5.0265	5.0242	5.0352	5.0242	5.0205	5.0242	5.0463
2	3.9288	3.9352	3.4752	3.4692	2.7073	2.7167	3.2065	3.2167
3	3.4456	3.4510	2.8287	2.8245	1.9036	1.9123	2.5866	2.5922
4	3.1369	3.1417	2.4246	2.4209	1.4799	1.4860	2.2543	2.2567
5	2.9035	2.9062	2.1161	2.1135	1.2081	1.2095	2.0302	2.0333
6	2.7070	2.7068	1.8376	1.8366	1.0112	1.0116	1.8554	1.8549
7	2.5270	2.5270	1.6445	1.6392	0.8550	0.8535	1.7036	1.7013
8	2.3473	2.3477	1.4821	1.4757	0.7203	0.7205	1.5582	1.5545
9	2.1459	2.1440	1.3215	1.3171	0.5921	0.5920	1.4015	1.3985
10	1.8608	1.8625	1.1169	1.1096	0.4451	0.4448	1.1889	1.1851

Sensitivity Study

To study sensitivities of swap rates, we first find recursively derivatives of pdf in homogeneous case with intensity (5). Note that

$$\frac{\partial f_{\tau^k}(t)}{\partial a} = \sum_{j=0}^{k-1} \alpha_{k,j}(1-\beta_j at)e^{-\beta_j at},$$

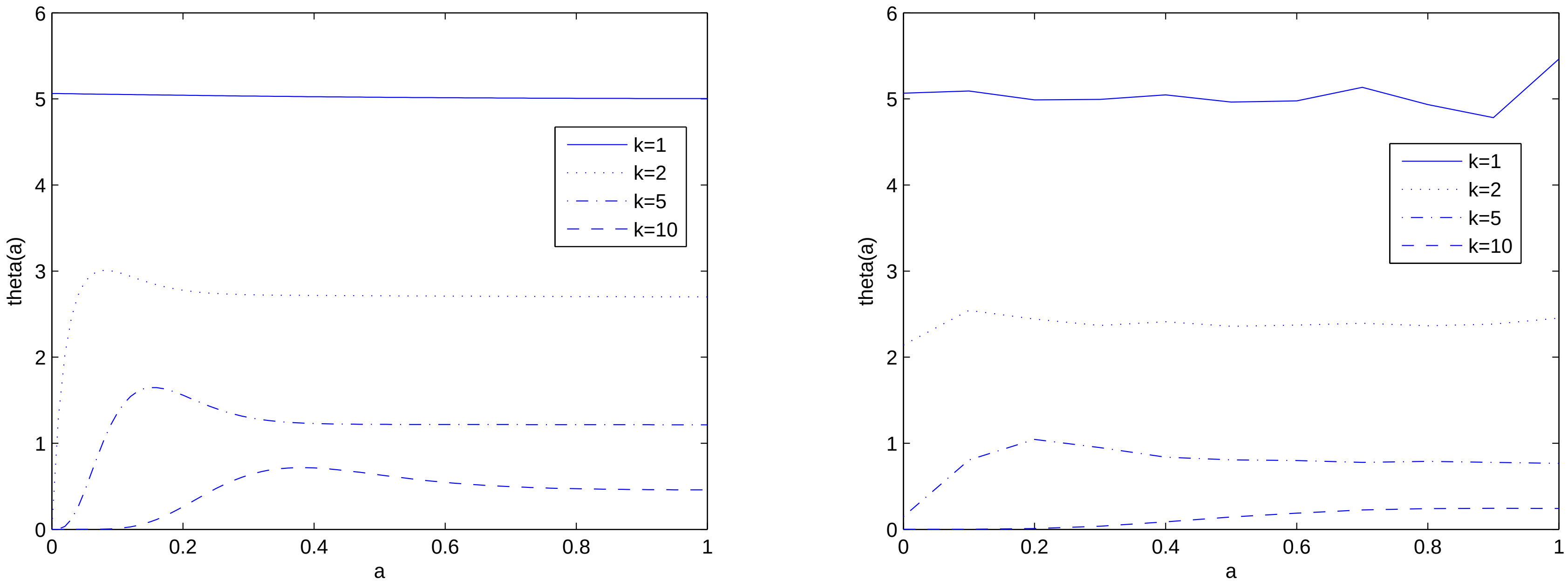
and

$$\frac{\partial f_{\tau^k}(t)}{\partial c} = \sum_{j=0}^{k-1}(\alpha'_{k,j} - \alpha_{k,j}(n-j)jat)ae^{-\beta_j at},$$

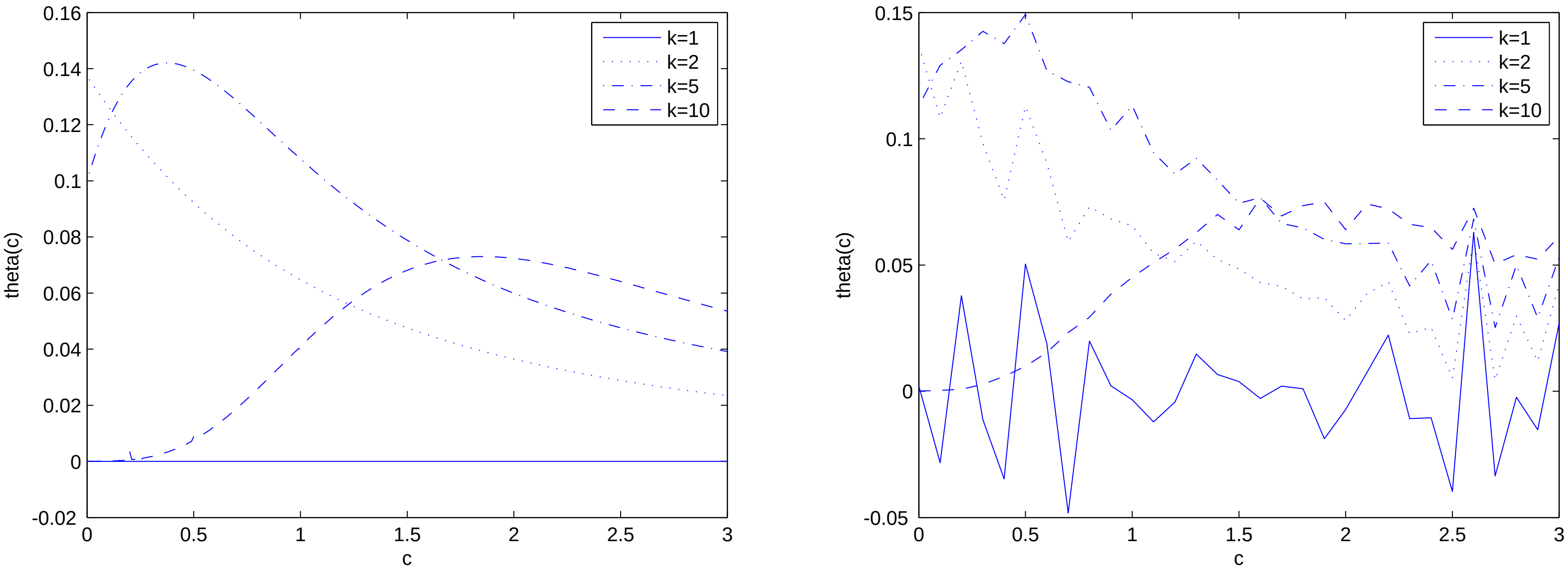
where

$$\begin{cases} \alpha'_{k+1,j} = \begin{cases} \alpha_{k,j}\gamma'_{k,j} + \alpha'_{k,j}\gamma_{k,j}, & j = 0,1,\ldots,k-1 \\ -\sum_{u=0}^{k-1} \alpha_{k,u}\gamma'_{k,u} + \alpha'_{k,u}\gamma_{k,u}, & j = k \end{cases} \\ \gamma_{k,j} = \dfrac{\beta_k}{\beta_k - \beta_j}, \\ \gamma'_{k,j} = \dfrac{\partial \gamma_{k,j}}{\partial c}, \end{cases}$$

and $\alpha'_{1,0} = 0$, $\alpha_{k,j}$ and β_j are given in Corollary 1.



Derivatives of swap rates S_k with respect to a [left:AP, right:MC] (n=10, c=0.3)



Derivatives of swap rates S_k with respect to c [left:AP, right:MC] (n=10, a=0.1)

Summary

- We propose a simple and efficient method to derive kth default time distribution under interacting intensity default contagion model.
- Our method gives recursive formulas for order default time distribution and further derive analytic solutions in a group of homogeneous entities and also two group of heterogeneous entities.
- In homogeneous case, we further include exponential decay, in which we give joint distribution of order default time, and further obtain kth default time distribution.
- We also work on stochastic intensity process in a homogeneous situation, an give pricing formula under a two-state, Markov regime-switching stochastic intensity model.
- In addition, our proposed method can study sensitivities of swap rates to a change in underlying parameters easily when compare to simulation method.