Log-Robust Portfolio Management

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Outline

Motivation – The LogNormal Model

- Black and Scholes (1973).

- If there is no correlation, random stock price of asset i at time T, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T} Z_i.$$

  where Z_i obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0,1)$, and:

  T : the length of the time horizon,
  $S_i(0)$: the initial (known) value of stock i,
  μ_i : the drift of the process for stock i,
  σ_i : the infinitesimal standard deviation of the process for stock i,

- Widely used in industry, especially for option pricing.

Motivation (Cont'd)

- Other distributions have been investigated by:
  - Fama (1965),
  - Blattberg and Gonedes (1974),
  - Kon (1984),
  - Jansen and deVries (1991),
  - Richardson and Smith (1993),
  - Cont (2001).
- In real life, the distribution of stock prices have fat tails (Jansen and deVries (1991), Cont (2001))

Motivation (Cont'd)

- Jansen and deVries (1991) states:

  " Numerous articles have investigated the distribution of share prices, and find that the returns are **fat-tailed.** Nevertheless, there is still controversy about the amount of probability mass in the tails, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven **hard to resolve.**"

- The Gaussian distribution in the Log-Normal model leads the manager to take more risk than he is willing to accept.

Motivation (Cont'd)

- Numerous studies suggest that the continuously compounded rates of return are indeed the true drivers of uncertainty.
- There does not seem to be one good distribution for these rates of return.
- Managers want to protect their portfolio from adverse events.
- This makes **robust optimization** particularly well-suited for the problem at hand.

Robust Optimization

Robust Optimization:

- Models random variables as uncertain parameters belonging to known intervals.
- Optimizes the worst-case objective.
- All (independent) random variables are not going to reach their worst case simultaneously! They tend to cancel each other out. (Law of large numbers.)
- Key to the performance of the approach is to take the worst case over a "reasonable uncertainty set."
- Tractability of max-min approach depends on the ability to rewrite the problem as one big maximization problem using strong duality.
- Setting of choice: objective **linear** in the uncertainty.

Robust Optimization (Cont'd)

- Theory of Robust Optimization:
  - Ben-Tal and Nemirovski (1999),
  - Bertsimas and Sim (2004).
- Applications to Finance:
  - Bertsimas and Pachamanova (2008).
  - Fabozzi et. al. (2007).
  - Pachamanova (2006).
  - Erdogan et. al. (2004).
  - Goldfarb and Iyengar (2003).

Robust Optimization (Cont'd)

- All the researchers who have applied robust optimization to portfolio management before us have modeled the **returns** $S_i(T)$ as the uncertain parameters.
- This matters because of the nonlinearity (exponential term) in the asset price equation.
- To the best of our knowledge, we are the first ones to apply robust optimization to the true drivers of uncertainty.

Contributions

- We incorporate randomness on the continuously compounded rates of return using range forecasts and a budget of uncertainty.
- We maximize the worst-case portfolio value at the end of the time horizon in a one-period setting.
- For the model without short-sales, we derive a tractable robust formulation, specifically, a linear programming problem, with only a moderate increase in the number of constraints and decision variables.
- For the model with short-sales and independent assets, we devise an exact algorithm that involves solving a series of LP problems and of convex problems of one variable.
- For the model with short-sales and correlated assets, we study some heuristics.
- We gain insights into the worst-case scaled deviations and the structures of the optimal strategies.

We use the following notation:

n :	the number of stocks,
T :	the length of the time horizon,
$S_i(0)$:	the initial (known) value of stock i,
$S_i(T)$:	the (random) value of stock i at time T,
w_0 :	the initial wealth of the investor,
μ_i :	the drift of the process for stock i,
σ_i :	the infinitesimal standard deviation of the process for stock i,
x_i :	the amount of money invested in stock i.

Problem Formulation

- Assumptions:
  - Short sales are not allowed.
  - All stock prices are independent.
- In the traditional Log-Normal model, the random stock price i at time T, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T} Z_i.$$

- Z_i obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0, 1)$.

Problem Formulation (Cont'd)

- We model Z_i as uncertain parameters with nominal value of zero and known support$[-c, c]$ for all i.

$$Z_i = c\,\tilde{z}_i,$$

- $\tilde{z}_i \in [-1, 1]$ represents the *scaled deviation* of Z_i from its mean, which is zero.

- Budget of uncertainty constraint:

$$\sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma,$$

Problem Formulation (Cont'd)

The robust portfolio management problem can be formulated as a maximization of the worst-case portfolio wealth:

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i \sqrt{T} c \tilde{z}_i\right] \\
& \quad \text{s.t.} \ \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\
& \quad\quad\quad |\tilde{z}_i| \leq 1 \ \forall i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0. \\
& x_i \geq 0 \ \forall i.
\end{aligned}
$$

The problem is **linear** in the asset allocation and nonlinear but **convex** in the scaled deviations.

Tractable Reformulation

Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem is: $w_0 \exp(F(\Gamma))$, *where* F *is the function defined by:*

$$
\begin{aligned}
F(\Gamma) = \max_{\eta,\chi,\xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sigma_i \sqrt{T}\, c\, \chi_i \geq 0, \ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i.
\end{aligned}
$$

(ii) The optimal amount of money invested at time 0 *in stock* i *is* $\chi_i\, w_0$, *for all* i.

Structure of the optimal allocation (Cont'd)

Theorem

Assume assets are ordered in decreasing order of the stock returns without uncertainty $k_i = \exp((\mu_i - \sigma_i^2/2)T)$ (i.e., $k_1 > \cdots > k_n$).
There exists an index j such that the optimal asset allocation is given by:

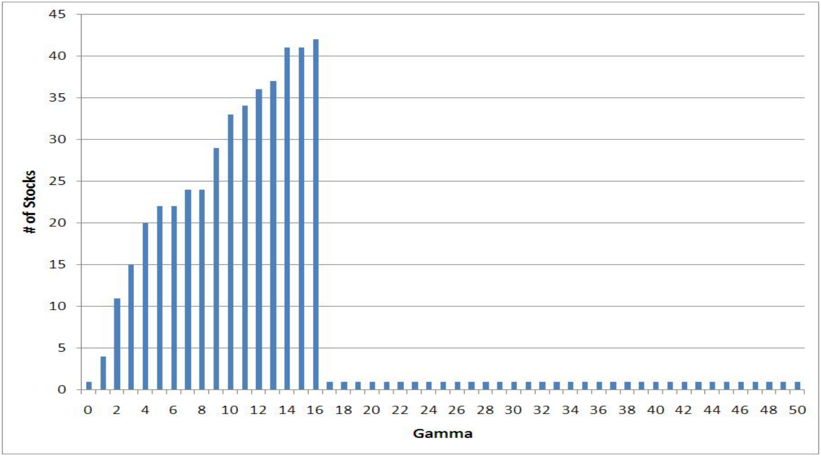
$$x_i = \begin{cases} \dfrac{1/\sigma_i}{\sum_{a=1}^{j} 1/\sigma_a} w_0, & i \leq j, \\ 0, & i > j. \end{cases}$$

Notice that the allocations do not depend on c. Only the degree of diversification j does.

Remarks

- $x_i \sigma_i$ is constant for all the assets the manager invests in.
- The robust optimization **selects the number of assets** j the manager will invest in.
- When the manager invests in all assets, the allocation is similar to Markovitz's allocation but the σ_i have a different meaning.
- When assets are uncorrelated, the diversification index j increases with Γ, until η becomes zero and we invest in the stock with the highest worst-case return only.

Diversification (Cont'd)



Correlated Assets - Formulation

- The behavior of stock prices, is replaced by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sqrt{T} Z_i,$$

  where the random vector Z is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{Q}$.

- We define:

$$\mathbf{Y} = \mathbf{Q}^{-1/2}\mathbf{Z},$$

  where $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{Q}^{1/2}$ is the square-root of the covariance matrix $\mathbf{Q}$, i.e., the unique symmetric positive definite matrix $\mathbf{S}$ such that $\mathbf{S}^2 = \mathbf{Q}$.

Formulation (Cont'd)

The robust optimization model becomes:

$$\begin{array}{lll} \max\limits_{x} & \min\limits_{\tilde{y}} & \displaystyle\sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T + \sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_j\right)\right] \\ & \text{s.t.} & \displaystyle\sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma, \\ & & |\tilde{y}_j| \leq 1, \ \forall j, \\ \text{s.t.} & & \displaystyle\sum_{i=1}^{n} x_i = w_0, \\ & & x_i \geq 0, \ \forall i. \end{array}$$

Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem with correlated assets is: $w_0 \exp(F(\Gamma))$*, where* F *is the function defined by:*

$$
\begin{aligned}
F(\Gamma) = \max_{\eta,\chi,\xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sqrt{T}\, c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \chi_j \right) \geq 0, \ \forall i, \\
& \eta + \xi_i + \sqrt{T}\, c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \chi_j \right) \geq 0, \ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i.
\end{aligned}
$$

(ii) The optimal amount of money invested at time 0 *in stock* i *is* $\chi_i\, w_0$*, for all* i*.*

Numerical Experiments

Goal: to compare the proposed Log-robust approach with the robust optimization approach that has been traditionally implemented in portfolio management.

$$
\begin{aligned}
\max_{x,\,p,\,q,\,r} \quad & \sum_{i=1}^{n} x_i \, \exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right) T\right] E\left[\exp\left(\sum_{j=1}^{n} Q_{ij}^{1/2} Z_j\right)\right] - \Gamma\, p - \sum_{i=1}^{n} q_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& p + q_i \geq c\, r_i, \ \forall i, \\
& -r_i \leq \sum_{k=1}^{n} M_{ki}^{1/2} x_k \leq r_i, \ \forall i, \\
& p,\ q_i,\ r_i,\ x_i \geq 0, \ \forall i,
\end{aligned}
$$

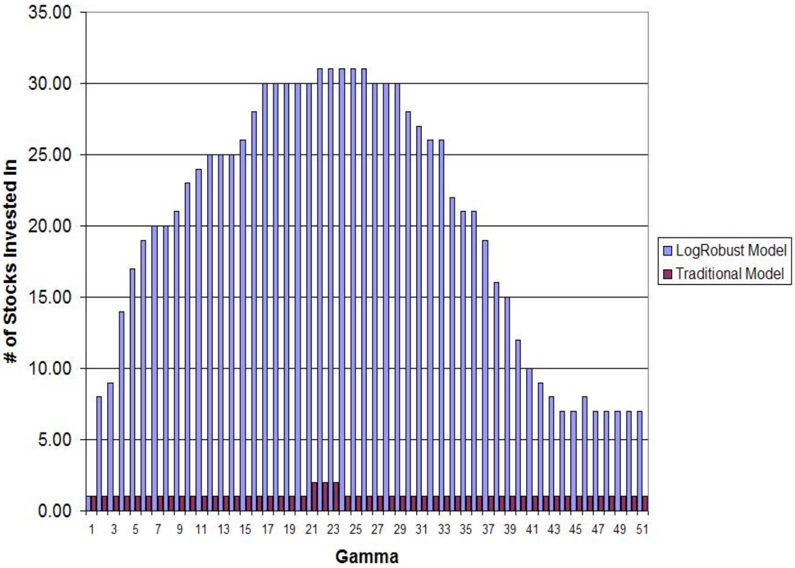
with $M^{1/2}$ the square root of the covariance matrix of $\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sqrt{T}\left(\sum_{j=1}^{n} Q_{ij}^{1/2} Z_j\right)\right]$

Numerical Experiments (Cont'd)

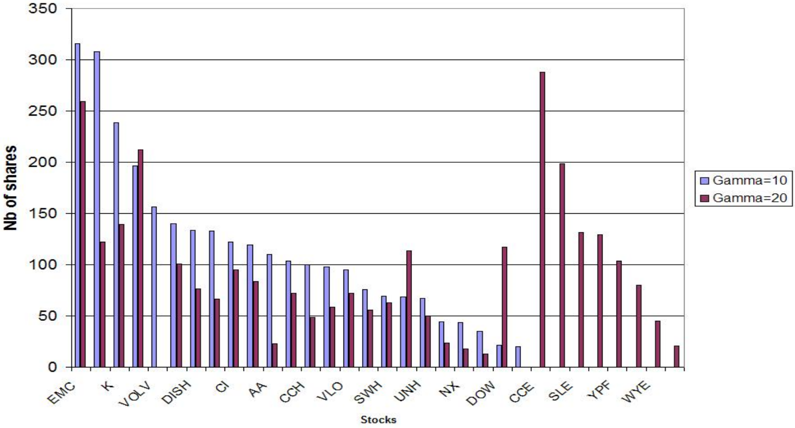
We will see that:

- The Log-robust approach yields far greater diversification in the optimal asset allocation.
- It outperforms the traditional robust approach, when performance is measured by percentile values of final portfolio wealth, if at least one of the following two conditions is satisfied:
  - The budget of uncertainty parameter is relatively small, or
  - The percentile considered is low enough.
- This means that the Log-robust approach shifts the **left** tail of the wealth distribution to the right, compared to the traditional robust approach; how much of the whole distribution ends up being shifted depends on the choice of the budget of uncertainty.

Number of stocks in optimal portfolio vs Γ



Number of shares in optimal Log-robust portfolio for $\Gamma = 10$ and $\Gamma = 20$

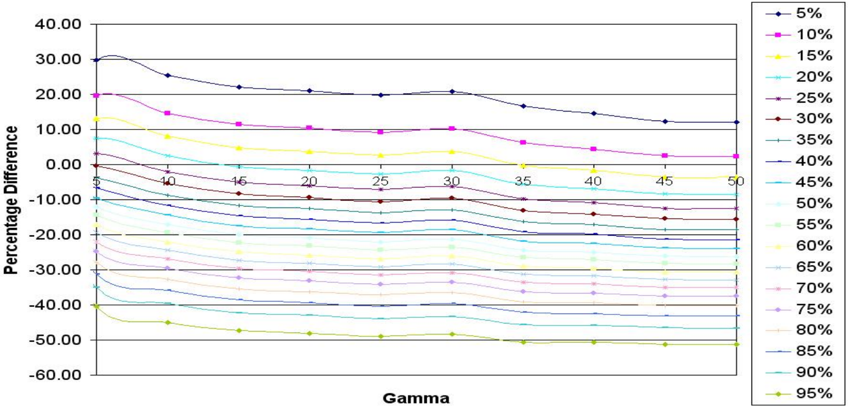


Numerical Experiments (Cont'd)

Γ	Traditional	Log-Robust	Relative Gain
5	70958.81	107828.94	51.96%
10	70958.81	104829.93	47.73%
15	70958.81	102502.79	44.45%
20	70958.81	101707.00	43.33%
25	70958.81	100905.96	42.40%
30	70958.81	101763.58	43.41%
35	70958.81	98445.23	38.74%
40	70958.81	96120.18	35.46%
45	70958.81	94253.62	32.83%
50	70958.81	94032.09	32.52%

Table: 99% VaR as a function of Γ for Gaussian distribution.

Relative gain of the Log-robust model compared to the Traditional robust model - Gaussian Distribution



Numerical Experiments (Cont'd)

Γ	Traditional	Log-Robust	Relative Gain
5	68415.97	108234.32	58.20%
10	68415.97	105146.66	53.69%
15	68415.97	102961.66	50.49%
20	68415.97	102124.75	49.27%
25	68415.97	101294.347	48.06%
30	68415.97	102206.73	49.39%
35	68415.97	98508.69	43.98%
40	68415.97	95940.01	40.23%
45	68415.97	93841.05	37.16%
50	68415.97	93562.59	36.76%

Table: 99% VaR as a function of Γ for Logistic distribution.

Conclusions

- We have presented an approach to uncertainty in stock prices returns that does not require the knowledge of the underlying distributions.
- It builds upon observed dynamics of stock prices while addressing limitations of the Log-Normal model.
- It leads to tractable linear formulations.
- We have characterized the structure of the optimal solution without correlation and explained diversification.

Conclusions

- The model is more aligned with the finance literature than the traditional robust model that does not address the true uncertainty drivers.
- The traditional robust optimization approach does not achieve diversification for real-life financial data like our model.
- Better performance for the ambiguity-averse manager maximizing his 99% VaR (or 95% or 90% VaR).

- Short-selling is the practice of borrowing a security and selling it, in the hope that the asset price will decrease.
- Short-selling provides the decision maker with additional profit opportunities. Therefore it is an important step in making the log-robust portfolio management model appealing to practitioners.

Notation

n :	the number of stocks,
T :	the length of the time horizon,
p :	leverage parameter,
$S_i(0)$:	the initial (known) value of stock i,
$S_i(T)$:	the (random) value of stock i at time T,
w_0 :	the initial wealth of the investor,
μ_i :	the drift of the process for stock i,
σ_i :	the infinitesimal standard deviation of the process for stock i,
$\tilde{x}_i$:	the number of shares invested in stock i,
x_i :	the amount of money invested in stock i.

p limits the amount of money that can be short-sold (borrowed) as a percentage of the manager's initial wealth.

Formulation

The log-robust portfolio management model with short sales can be formulated as:

$$\begin{aligned}
\max_{x} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2}) T + \sigma_i \sqrt{T}\, c\, \tilde{z}_i \right] \\
& \quad \text{s.t.} \ \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\
& \qquad\quad |\tilde{z}_i| \leq 1 \ \forall i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& \sum_{i \,|\, x_i < 0} -x_i \leq p\, w_0.
\end{aligned}$$

Tractable Reformulation

- Additional notation:

k_i:	return of stock i without uncertainty,
z_i^+:	scaled deviation for assets that are not short sold,
z_i^-:	scaled deviation for assets that are short sold,
Γ^+:	budget of uncertainty for assets not short sold,
Γ^-:	budget of uncertainty for assets short sold.

  Specifically, $k_i = \exp\left((\mu_i - \frac{\sigma_i^2}{2})T\right)$ for all i.

- We distinguish between assets that are short-sold ($x_i < 0$) and not short-sold ($x_i \geq 0$), allocating a budget of uncertainty (to be optimized) Γ^- and Γ^+ to each group.

$$\max_{x} \min_{\Gamma^+,\Gamma^-}$$

$$\left(\begin{array}{ll} \min_{\tilde{z}^+} \sum_{i|x_i\geq 0}^{n} x_i k_i \exp(\sigma_i \sqrt{T} c \tilde{z}_i^+) + & \min_{\tilde{z}^-} \sum_{i|x_i<0}^{n} x_i k_i \exp(\sigma_i \sqrt{T} c \tilde{z}_i^-) \\ \text{s.t} \quad \sum_{i|x_i\geq 0}^{n} |\tilde{z}_i^+| \leq \Gamma^+, & \text{s.t} \quad \sum_{i|x_i<0}^{n} |\tilde{z}_i^-| \leq \Gamma^-, \\ \quad |\tilde{z}_i^+| \leq 1 \, \forall\, i \text{ s.t. } x_i \geq 0. & \quad |\tilde{z}_i^-| \leq 1 \, \forall\, i \text{ s.t. } x_i < 0. \end{array} \right)$$

$$\text{s.t} \quad \Gamma^+ + \Gamma^- = \Gamma,$$

$$\Gamma^+, \Gamma^- \geq 0 \text{ integer.}$$

$$. \quad \sum_{i=1}^{n} x_i = w_0, \qquad \sum_{i|x_i<0}^{n} -x_i \leq p w_0.$$

- At optimality, $0 \leq \tilde{z}_i^- \leq 1$ for all stocks that are short-sold (the worst case is to have returns no lower than their nominal value), and the minimization problem in $\tilde{z}_i^-$ is equivalent to the linear programming problem:

$$\begin{aligned} \min_{z^-} \quad & \sum_{i|x_i<0}^{n} x_i\, k_i (1 - z_i^-) + x_i\, k_i\, \exp(\sigma_i \sqrt{T}\, c) z_i^- \\ \text{s.t.} \quad & \sum_{i|x_i<0}^{n} z_i^- \leq \Gamma^-, \\ & 0 \leq z_i^- \leq 1, \ \forall i \text{ s.t. } x_i < 0. \end{aligned}$$

Optimal Strategy

Theorem (Optimal Strategy)

(i) At optimality, either the manager short-sells the maximum amount allowed, or he does not short-sell at all.
(ii) The optimal wealth is the maximum between the optimal wealth in the no-short-sales model and the convex problem:

$$\max_{\theta \geq 0} \quad w_0 \cdot \left(\theta \left[1 + \ln \left(\frac{(1+p)}{\theta} \right) \right] + F_p(\theta, \Gamma) \right),$$

where F_p is defined by:

Theorem (Optimal Strategy (Cont'd))

$$
\begin{aligned}
F_p(\theta, \Gamma) = \max_{\eta, \xi, \widetilde{\chi}} \quad & \sum_{i|x_i \geq 0} \widetilde{\chi}_i \ln k_i - \sum_{i|x_i<0} \widetilde{\chi}_i \, k_i - \eta \, \Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sigma_i \sqrt{T} c \, \widetilde{\chi}_i \geq 0, && \forall i | x_i \geq 0, \\
& \eta + \xi_i - k_i \left[\exp(\sigma_i \sqrt{T} c) - 1 \right] \widetilde{\chi}_i \geq 0, && \forall i | x_i < 0, \\
& \sum_{i|x_i \geq 0} \widetilde{\chi}_i = \theta, \\
& \sum_{i|x_i < 0} \widetilde{\chi}_i = p, \\
& \eta \geq 0, \, \xi_i \geq 0, \, \widetilde{\chi}_i \geq 0, && \forall i.
\end{aligned}
$$

(iii) The optimal fraction of money χ_i allocated to asset i is $(1+p)\frac{\widetilde{\chi}_i}{\theta}$ if the stock is invested in and $-\widetilde{\chi}_i$ if the stock is short-sold.

Corollary (Optimal Allocation)

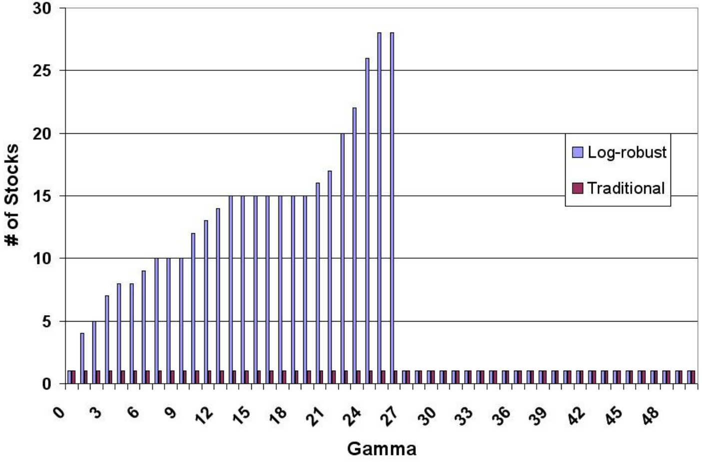
If it is optimal to short-sell, there exist indices j and l, $j < l$ such that the decision-maker:

- *invests in stocks 1 to j,*
- *neither invests in nor short-sells stocks $j+1$ to $l-1$,*
- *short-sells stocks l to n.*

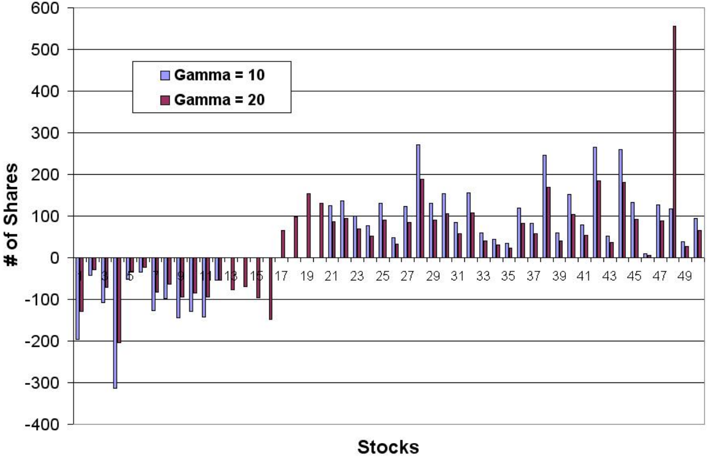
The traditional robust model with short sales is given by:

$$
\begin{aligned}
\max_{x,s,q,r} \quad & \sum_{i=1}^{n} (x_i^+ - x_i^-) \exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right) T\right] E\left[\exp\left(\sum_{j=1}^{n} Q_{ij}^{1/2} Z_j\right)\right] \\
& -\Gamma\, s - \sum_{i=1}^{n} q_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} (x_i^+ - x_i^-) = w_0, \\
& s + q_i \geq c\, r_i, \ \forall i, \\
& -r_i \leq \sum_{k=1}^{n} M_{ki}^{1/2} (x_k^+ - x_k^-) \leq r_i, \ \forall i, \\
& \sum_{i=1}^{n} x_i^- \leq p\, w_0 \\
& s,\ q_i,\ r_i,\ x_i^+,\ x_i^- \geq 0, \ \forall i,
\end{aligned}
$$

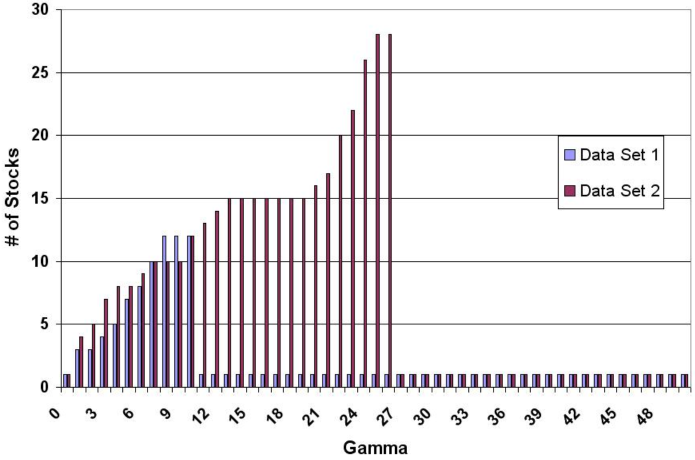
Numerical Experiments - Uncorrelated Assets



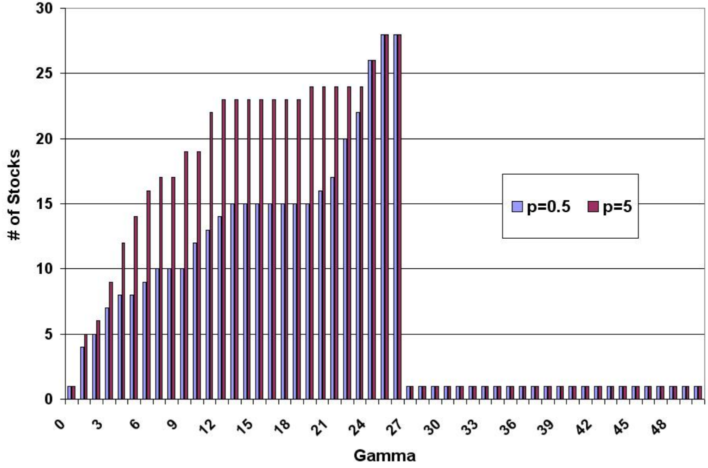
Number of Shares per Stock in the Log-robust model



Number of Stocks Short Sold for Two Data Sets



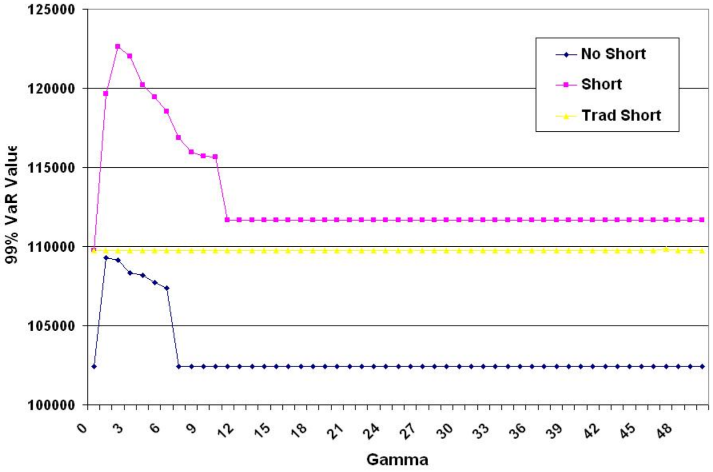
Number of Stocks Short Sold for $p = 0.5$ and $p = 5$



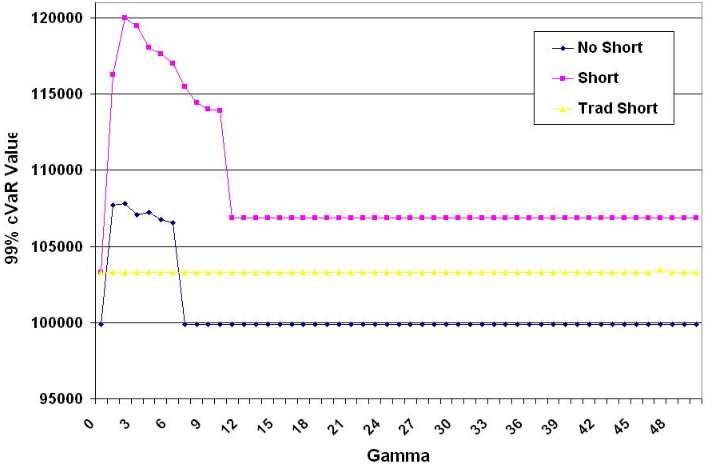
Impact of Γ on Stocks Allocation and Diversification

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
MMM	4210	1367	809	633	476	425	360	307	286	274	268	4210	4210	4210	4210	4210
SYT	0	973	576	451	339	303	256	219	204	195	191	0	0	0	0	0
AA	0	684	405	317	238	213	180	154	143	137	134	0	0	0	0	0
APD	0	669	395	309	233	208	176	150	140	134	131	0	0	0	0	0
FCX	0	0	416	326	245	219	185	158	147	141	138	0	0	0	0	0
AU	0	0	427	334	251	224	190	162	151	145	142	0	0	0	0	0
APA	0	0	417	326	245	219	185	158	147	141	138	0	0	0	0	0
E	0	0	0	297	223	200	169	144	134	129	126	0	0	0	0	0
ESV	0	0	0	346	260	233	197	168	156	150	147	0	0	0	0	0
HP	0	0	0	187	238	213	180	154	143	137	135	0	0	0	0	0
OKS	0	0	0	0	156	140	118	101	94	90	88	0	0	0	0	0
TPP	0	0	0	0	234	209	177	151	141	135	132	0	0	0	0	0
EPE	0	0	0	0	229	204	173	147	138	132	129	0	0	0	0	0
HMC	0	0	0	0	280	251	212	181	169	161	158	0	0	0	0	0
MT	0	0	0	0	279	410	347	296	276	264	259	0	0	0	0	0
NCX	0	0	0	0	0	215	182	155	145	138	136	0	0	0	0	0
CR	0	0	0	0	0	0	357	304	284	272	266	0	0	0	0	0
LII	0	0	0	0	0	0	129	110	102	98	96	0	0	0	0	0
DAI	0	0	0	0	0	0	290	355	331	317	311	0	0	0	0	0
TRW	0	0	0	0	0	0	0	149	138	133	130	0	0	0	0	0
BUD	0	0	0	0	0	0	0	317	295	283	277	0	0	0	0	0
FMX	0	0	0	0	0	0	0	588	594	569	557	0	0	0	0	0
BTI	0	0	0	0	0	0	0	0	126	121	118	0	0	0	0	0
DEO	0	0	0	0	0	0	0	0	45	43	42	0	0	0	0	0
RAI	0	0	0	0	0	0	0	0	0	114	112	0	0	0	0	0
CLX	0	0	0	0	0	0	0	0	0	17	52	0	0	0	0	0
HSY	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CBY	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HNZ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CPB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RAH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
AMG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
PFG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HOLX	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TRIB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TAC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
STE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SHS	0	0	0	0	0	0	0	0	-18	-55	-53	0	0	0	0	0
NYT	0	0	0	0	0	0	0	0	-54	-51	-50	0	0	0	0	0
NDSN	0	0	0	0	0	0	0	-103	-95	-91	-89	0	0	0	0	0
ITC	0	0	0	0	0	0	0	-58	-54	-51	-50	0	0	0	0	0
DTE	0	0	0	0	0	0	-119	-100	-93	-88	-86	0	0	0	0	0
DRS	0	0	0	0	0	-18	-185	-156	-145	-137	-134	0	0	0	0	0
ATU	0	0	0	0	0	-178	-150	-126	-117	-111	-109	0	0	0	0	0
WEC	0	0	0	0	-446	-397	-334	-282	-261	-248	-242	0	0	0	0	0
ATG	0	0	0	-218	-161	-143	-120	-102	-94	-89	-87	0	0	0	0	0
GCO	0	-227	-603	-489	-360	-321	-270	-228	-211	-200	-220	-1095	-1095	-1095	-1095	-1095
APOG	0	-744	-421	-324	-239	-212	-179	-151	-140	-132	-130	0	0	0	0	0
ATTR	-2093	-906	-513	-394	-290	-258	-218	-184	-170	-161	-158	0	0	0	0	0

99% VaR - Gaussian Distribution



99% cVaR - Gaussian Distribution



Portfolio Management with Short Sales
Correlated Assets

The log-robust optimization model with short sales and correlation is:

$$
\begin{aligned}
\max_{x} \quad \min_{\tilde{y}} \quad & \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T + \sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_j\right)\right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma, \\
& |\tilde{y}_j| \leq 1, \ \forall j, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& \sum_{i | x_i < 0}^{n} -x_i \leq p w_0.
\end{aligned}
$$

Heuristics

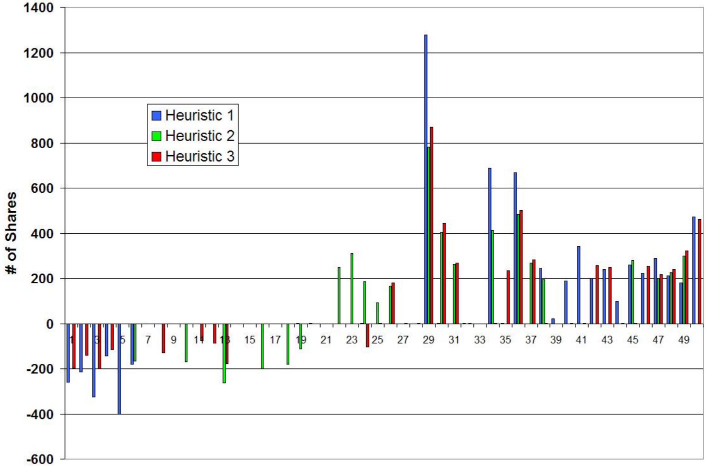
The heuristics aim at allowing us to use the results of the independent-assets case.

1. Heuristic 1: No correlation for assets short-sold.
2. Heuristic 2: Approximating the off-diagonal elements by their average and use budget of uncertainty.
3. Heuristic 3: Approximating the off-diagonal elements by a conservative estimate of their worst-case value.

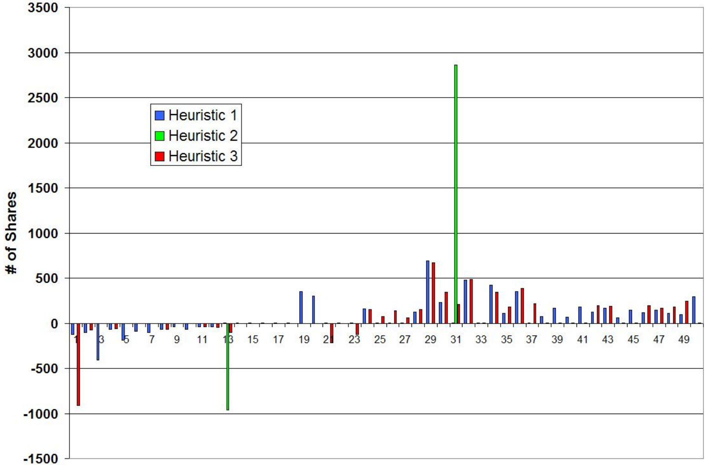
Impact of Γ on stock allocation and diversification for correlated stocks.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
MMM	4210	1403	1018	609	528	474	372	367	309	266	295	4210	4210	4210	4210
SYT	0	893	567	245	192	181	154	148	135	102	93	0	0	0	0
AA	0	717	476	298	240	211	154	136	106	115	108	0	0	0	0
APD	0	672	510	284	291	289	241	222	197	164	145	0	0	0	0
FCX	0	0	451	328	231	222	207	227	168	126	116	0	0	0	0
AU	0	0	599	382	283	259	219	198	151	156	144	0	0	0	0
APA	0	0	0	150	128	97	103	96	85	76	63	0	0	0	0
E	0	0	0	0	256	241	231	241	203	167	169	0	0	0	0
ESV	0	0	0	278	226	201	157	154	146	140	125	0	0	0	0
HP	0	0	0	353	354	341	243	219	210	182	184	0	0	0	0
OKS	0	0	0	0	156	188	131	123	106	76	69	0	0	0	0
TPP	0	0	0	0	0	21	222	225	213	180	163	0	0	0	0
EPE	0	0	0	0	241	244	185	155	126	92	73	0	0	0	0
HMC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
MT	0	0	0	812	717	668	524	443	414	350	354	0	0	0	0
NCX	0	0	0	0	0	0	0	22	162	142	113	0	0	0	0
CR	0	0	0	815	743	688	555	496	479	420	421	0	0	0	0
LII	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DAI	0	0	0	0	0	0	0	0	293	486	478	0	0	0	0
TRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
BUD	0	0	0	0	0	0	417	350	308	232	227	0	0	0	0
FMX	0	0	0	0	0	1278	982	938	840	760	690	0	0	0	0
BTI	0	0	0	0	0	0	0	0	0	0	124	0	0	0	0
DEO	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RAI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CLX	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HSY	0	0	0	0	0	0	219	205	191	162	163	0	0	0	0
CBY	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HNZ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CPB	0	0	0	0	0	0	118	442	404	315	302	0	0	0	0
RAH	0	0	0	0	0	0	0	0	0	359	353	0	0	0	0
AMG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
PFG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HOLX	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TRIB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TAC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
STE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SHS	0	0	0	0	0	0	0	0	-55	-45	-42	0	0	0	0
NYT	0	0	0	0	0	0	0	0	-51	-42	-39	0	0	0	0
NDSN	0	0	0	0	0	0	0	-103	-91	-75	-70	0	0	0	0
ITC	0	0	0	0	0	0	-64	-58	-51	-42	-39	0	0	0	0
DTE	0	0	0	0	0	0	-112	-100	-88	-73	-68	0	0	0	0
DRS	0	0	0	0	0	0	-174	-156	-137	-114	-106	0	0	0	0
ATU	0	0	0	0	0	-180	-141	-126	-111	-92	-86	0	0	0	0
WEC	0	0	0	-239	-446	-401	-313	-282	-248	-206	-191	0	0	0	0
ATG	0	0	0	-187	-161	-144	-113	-102	-89	-74	-69	0	0	0	0
GCO	0	-227	-487	-420	-360	-324	-253	-228	-200	-353	-406	-1095	-1095	-1095	-1095
APOG	0	-744	-521	-278	-239	-214	-168	-151	-132	-110	-102	0	0	0	0
ATTR	-2093	-906	-634	-338	-290	-261	-204	-184	-161	-134	-124	0	0	0	0

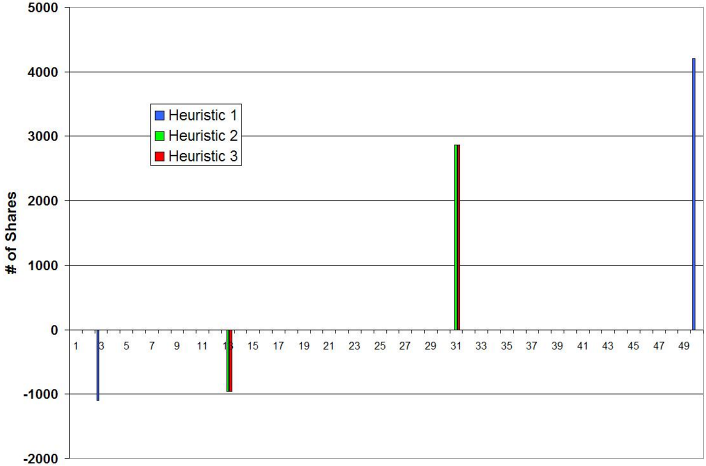
Allocation for the three heuristics, $\Gamma = 5$



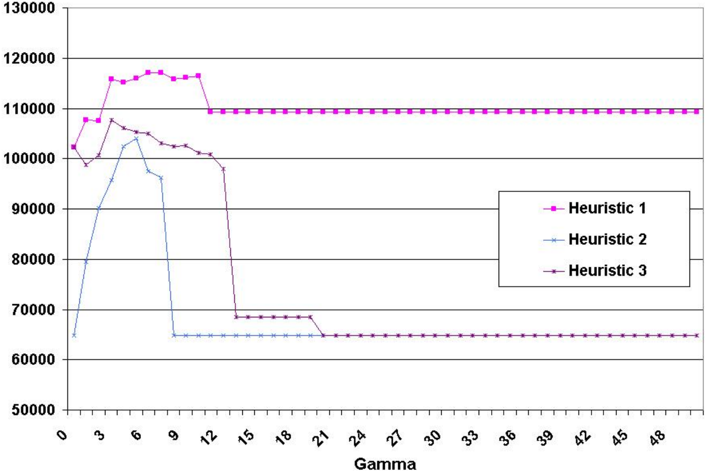
Allocation for the three heuristics, $\Gamma = 10$



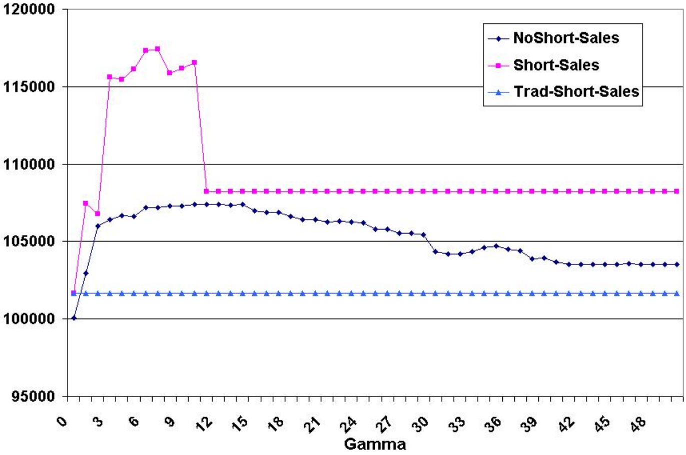
Allocation for the three heuristics, $\Gamma = 20$



Comparison of the three heuristics with Normal distribution using cVaR



99% cVaR for Gaussian distribution



Conclusions

- We have derived tractable reformulations to the portfolio management problem with short sales.
- We have proved that it is optimal for the manager to either short-sell as much as he can, or not short-sell at all, and provided optimal allocations in this case.
- We have also seen that diversification arises naturally from the log-robust optimization approach.