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3 Portfolio Management with Short Sales
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Motivation – The LogNormal Model


- If there is no correlation, random stock price of asset $i$ at time $T$, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T} Z_i.$$

where $Z_i$ obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0, 1)$, and:

- $T$: the length of the time horizon,
- $S_i(0)$: the initial (known) value of stock $i$,
- $\mu_i$: the drift of the process for stock $i$,
- $\sigma_i$: the infinitesimal standard deviation of the process for stock $i$,

- Widely used in industry, especially for option pricing.
Other distributions have been investigated by:

- Fama (1965),
- Blattberg and Gonedes (1974),
- Kon (1984),
- Jansen and deVries (1991),
- Richardson and Smith (1993),

In real life, the distribution of stock prices have fat tails (Jansen and deVries (1991), Cont (2001))
Jansen and deVries (1991) states:

“Numerous articles have investigated the distribution of share prices, and find that the returns are **fat-tailed**. Nevertheless, there is still controversy about the amount of probability mass in the tails, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven **hard to resolve**.”

The Gaussian distribution in the Log-Normal model leads the manager to take more risk than he is willing to accept.
Numerous studies suggest that the continuously compounded rates of return are indeed the true drivers of uncertainty.

There does not seem to be one good distribution for these rates of return.

Managers want to protect their portfolio from adverse events.

This makes robust optimization particularly well-suited for the problem at hand.
Robust Optimization:

- Models random variables as uncertain parameters belonging to known intervals.

- Optimizes the worst-case objective.

- All (independent) random variables are not going to reach their worst case simultaneously! They tend to cancel each other out. (Law of large numbers.)

- Key to the performance of the approach is to take the worst case over a “reasonable uncertainty set.”

- Tractability of max-min approach depends on the ability to rewrite the problem as one big maximization problem using strong duality.

- Setting of choice: objective linear in the uncertainty.
Robust Optimization (Cont’d)

- Theory of Robust Optimization:
  - Ben-Tal and Nemirovski (1999),
  - Bertsimas and Sim (2004).

- Applications to Finance:
  - Fabozzi et. al. (2007).
  - Erdogan et. al. (2004).
All the researchers who have applied robust optimization to portfolio management before us have modeled the returns $S_i(T)$ as the uncertain parameters.

This matters because of the nonlinearity (exponential term) in the asset price equation.

To the best of our knowledge, we are the first ones to apply robust optimization to the true drivers of uncertainty.
Contributions

- We incorporate randomness on the continuously compounded rates of return using range forecasts and a budget of uncertainty.

- We maximize the worst-case portfolio value at the end of the time horizon in a one-period setting.

- For the model without short-sales, we derive a tractable robust formulation, specifically, a linear programming problem, with only a moderate increase in the number of constraints and decision variables.

- For the model with short-sales and independent assets, we devise an exact algorithm that involves solving a series of LP problems and of convex problems of one variable.

- For the model with short-sales and correlated assets, we study some heuristics.

- We gain insights into the worst-case scaled deviations and the structures of the optimal strategies.
We use the following notation:

- $n$: the number of stocks,
- $T$: the length of the time horizon,
- $S_i(0)$: the initial (known) value of stock $i$,
- $S_i(T)$: the (random) value of stock $i$ at time $T$,
- $w_0$: the initial wealth of the investor,
- $\mu_i$: the drift of the process for stock $i$,
- $\sigma_i$: the infinitesimal standard deviation of the process for stock $i$,
- $x_i$: the amount of money invested in stock $i$. 
Assumptions:

- Short sales are not allowed.
- All stock prices are independent.

In the traditional Log-Normal model, the random stock price $i$ at time $T$, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} Z_i.$$ 

$Z_i$ obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0, 1)$. 
We model $Z_i$ as uncertain parameters with nominal value of zero and known support $[-c, c]$ for all $i$.

$$Z_i = c \tilde{z}_i,$$

$\tilde{z}_i \in [-1, 1]$ represents the *scaled deviation* of $Z_i$ from its mean, which is zero.

**Budget of uncertainty constraint:**

$$\sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma,$$
The robust portfolio management problem can be formulated as a maximization of the worst-case portfolio wealth:

$$\max_x \min_{\tilde{z}} \sum_{i=1}^{n} x_i \exp \left[ \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} \tilde{c} \tilde{z}_i \right]$$

s.t. \( \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma \),

\(|\tilde{z}_i| \leq 1 \ \forall i\),

s.t. \( \sum_{i=1}^{n} x_i = w_0 \).

\( x_i \geq 0 \ \forall i \).

The problem is **linear** in the asset allocation and nonlinear but **convex** in the scaled deviations.
Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem is: $w_0 \exp(F(\Gamma))$, where $F$ is the function defined by:

$$F(\Gamma) = \max_{\eta, \chi, \xi} \sum_{i=1}^{n} \chi_i \ln k_i - \eta \Gamma - \sum_{i=1}^{n} \xi_i$$

s.t. $\eta + \xi_i - \sigma_i \sqrt{T} c_\chi i \geq 0$, $\forall i$,

$$\sum_{i=1}^{n} \chi_i = 1,$$

$\eta \geq 0$, $\chi_i$, $\xi_i \geq 0$, $\forall i$.

(ii) The optimal amount of money invested at time 0 in stock $i$ is $\chi_i w_0$, for all $i$. 

Dr. Aurélie Thiele (Lehigh University)
Theorem

Assume assets are ordered in decreasing order of the stock returns without uncertainty $k_i = \exp((\mu_i - \sigma_i^2/2)T)$ (i.e., $k_1 > \cdots > k_n$).

There exists an index $j$ such that the optimal asset allocation is given by:

$$x_i = \begin{cases} 
\frac{1/\sigma_i}{\sum_{a=1}^{j} 1/\sigma_a} w_0, & i \leq j, \\
0, & i > j.
\end{cases}$$

Notice that the allocations do not depend on $c$. Only the degree of diversification $j$ does.
• \( \chi_i \sigma_i \) is constant for all the assets the manager invests in.

• The robust optimization selects the number of assets \( j \) the manager will invest in.

• When the manager invests in all assets, the allocation is similar to Markovitz’s allocation but the \( \sigma_i \) have a different meaning.

• When assets are uncorrelated, the diversification index \( j \) increases with \( \Gamma \), until \( \eta \) becomes zero and we invest in the stock with the highest worst-case return only.
The behavior of stock prices is replaced by:

$$\ln \left( \frac{S_i(T)}{S_i(0)} \right) = \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sqrt{T} Z_i,$$

where the random vector $Z$ is normally distributed with mean $0$ and covariance matrix $Q$.

We define:

$$Y = Q^{-1/2}Z,$$

where $Y \sim \mathcal{N}(0, I)$ and $Q^{1/2}$ is the square-root of the covariance matrix $Q$, i.e., the unique symmetric positive definite matrix $S$ such that $S^2 = Q$. 
The robust optimization model becomes:

\[
\max_x \min_{\tilde{y}} \sum_{i=1}^{n} x_i \exp \left[ (\mu_i - \sigma_i^2/2) T + \sqrt{T} c \left( \sum_{j=1}^{n} Q_{ij}^{1/2} \tilde{y}_j \right) \right]
\]

s.t. \[
\sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma, \]
\[
|\tilde{y}_j| \leq 1, \forall j,
\]

s.t. \[
\sum_{i=1}^{n} x_i = w_0,
\]
\[
x_i \geq 0, \forall i.
\]
Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem with correlated assets is: $w_0 \exp(F(\Gamma))$, where $F$ is the function defined by:

$$F(\Gamma) = \max_{\eta, \chi, \xi} \sum_{i=1}^{n} \chi_i \ln k_i - \eta \Gamma - \sum_{i=1}^{n} \xi_i$$

s.t. \[
\begin{align*}
\eta + \xi_i - \sqrt{T} c \left( \sum_{j=1}^{n} Q_{ij}^{1/2} \chi_j \right) & \geq 0, \quad \forall i, \\
\eta + \xi_i + \sqrt{T} c \left( \sum_{j=1}^{n} Q_{ij}^{1/2} \chi_j \right) & \geq 0, \quad \forall i,
\end{align*}
\]

$$\sum_{i=1}^{n} \chi_i = 1,$$\[ \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i. \]

(ii) The optimal amount of money invested at time 0 in stock $i$ is $\chi_i \ w_0$, for all $i$. 

Numerical Experiments

**Goal:** to compare the proposed Log-robust approach with the robust optimization approach that has been traditionally implemented in portfolio management.

\[
\begin{align*}
\max_{x, p, q, r} & \quad \sum_{i=1}^{n} x_i \exp \left[ \left( \mu_i - \frac{\sigma_i^2}{2} \right) T \right] E \left[ \exp \left( \sum_{j=1}^{n} Q_{ij}^{1/2} Z_j \right) \right] - \Gamma p - \sum_{i=1}^{n} q_i \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_i = w_0, \\
& \quad p + q_i \geq c r_i, \quad \forall i, \\
& \quad -r_i \leq \sum_{k=1}^{n} M_{ki}^{1/2} x_k \leq r_i, \quad \forall i, \\
& \quad p, q_i, r_i, x_i \geq 0, \quad \forall i,
\end{align*}
\]

with \( M^{1/2} \) the square root of the covariance matrix of

\[
\exp \left[ \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sqrt{T} \left( \sum_{j=1}^{n} Q_{ij}^{1/2} Z_j \right) \right]
\]
We will see that:

- The Log-robust approach yields far greater diversification in the optimal asset allocation.

- It outperforms the traditional robust approach, when performance is measured by percentile values of final portfolio wealth, if at least one of the following two conditions is satisfied:
  - The budget of uncertainty parameter is relatively small, or
  - The percentile considered is low enough.

- This means that the Log-robust approach shifts the left tail of the wealth distribution to the right, compared to the traditional robust approach; how much of the whole distribution ends up being shifted depends on the choice of the budget of uncertainty.
Number of stocks in optimal portfolio vs $\Gamma$

![Bar chart showing the number of stocks invested in the optimal portfolio against Gamma. The chart compares LogRobust Model with Traditional Model.](image)
Number of shares in optimal Log-robust portfolio for \( \Gamma = 10 \) and \( \Gamma = 20 \)
<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>Traditional</th>
<th>Log-Robust</th>
<th>Relative Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>70958.81</td>
<td>107828.94</td>
<td>51.96%</td>
</tr>
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<td>70958.81</td>
<td>104829.93</td>
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<td>102502.79</td>
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<td>20</td>
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</tr>
<tr>
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<td>94253.62</td>
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</tr>
<tr>
<td>50</td>
<td>70958.81</td>
<td>94032.09</td>
<td>32.52%</td>
</tr>
</tbody>
</table>

**Table:** 99% VaR as a function of $\Gamma$ for Gaussian distribution.
Relative gain of the Log-robust model compared to the Traditional robust model - Gaussian Distribution
<table>
<thead>
<tr>
<th>Γ</th>
<th>Traditional</th>
<th>Log-Robust</th>
<th>Relative Gain</th>
</tr>
</thead>
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<td>68415.97</td>
<td>108234.32</td>
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</tr>
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<td>102961.66</td>
<td>50.49%</td>
</tr>
<tr>
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<td>68415.97</td>
<td>102124.75</td>
<td>49.27%</td>
</tr>
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<td>48.06%</td>
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<td>43.98%</td>
</tr>
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<td>40.23%</td>
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<td>45</td>
<td>68415.97</td>
<td>93841.05</td>
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</tr>
<tr>
<td>50</td>
<td>68415.97</td>
<td>93562.59</td>
<td>36.76%</td>
</tr>
</tbody>
</table>

**Table:** 99% VaR as a function of Γ for Logistic distribution.
Conclusions

- We have presented an approach to uncertainty in stock prices returns that does not require the knowledge of the underlying distributions.

- It builds upon observed dynamics of stock prices while addressing limitations of the Log-Normal model.

- It leads to tractable linear formulations.

- We have characterized the structure of the optimal solution without correlation and explained diversification.
The model is more aligned with the finance literature than the traditional robust model that does not address the true uncertainty drivers.

The traditional robust optimization approach does not achieve diversification for real-life financial data like our model.

Better performance for the ambiguity-averse manager maximizing his 99% VaR (or 95% or 90% VaR).
Short-selling is the practice of borrowing a security and selling it, in the hope that the asset price will decrease.

Short-selling provides the decision maker with additional profit opportunities. Therefore it is an important step in making the log-robust portfolio management model appealing to practitioners.
Notation

\[ n : \text{ the number of stocks,} \]
\[ T : \text{ the length of the time horizon,} \]
\[ p : \text{ leverage parameter,} \]
\[ S_i(0) : \text{ the initial (known) value of stock } i, \]
\[ S_i(T) : \text{ the (random) value of stock } i \text{ at time } T, \]
\[ w_0 : \text{ the initial wealth of the investor,} \]
\[ \mu_i : \text{ the drift of the process for stock } i, \]
\[ \sigma_i : \text{ the infinitesimal standard deviation of the} \]
\[ \text{ process for stock } i, \]
\[ \tilde{x}_i : \text{ the number of shares invested in stock } i, \]
\[ x_i : \text{ the amount of money invested in stock } i. \]

\( p \) limits the amount of money that can be short-sold (borrowed) as a percentage of the manager’s initial wealth.
The log-robust portfolio management model with short sales can be formulated as:

$$\begin{align*}
\max_x \ & \min_{\tilde{z}} \sum_{i=1}^n x_i \exp \left[ (\mu_i - \frac{\sigma_i^2}{2}) T + \sigma_i \sqrt{T} \ c \tilde{z}_i \right] \\
\text{s.t.} \ & \sum_{i=1}^n |\tilde{z}_i| \leq \Gamma, \\
\ & |\tilde{z}_i| \leq 1 \ \forall i, \\
\text{s.t.} \ & \sum_{i=1}^n x_i = w_0, \\
\ & \sum_{i \mid x_i < 0} -x_i \leq p w_0.
\end{align*}$$
Tractable Reformulation

- Additional notation:
  - $k_i$: return of stock $i$ without uncertainty,
  - $z_i^+$: scaled deviation for assets that are not short sold,
  - $z_i^-$: scaled deviation for assets that are short sold,
  - $\Gamma^+$: budget of uncertainty for assets not short sold,
  - $\Gamma^-$: budget of uncertainty for assets short sold.

  Specifically, $k_i = \exp \left( (\mu_i - \frac{\sigma_i^2}{2}) T \right)$ for all $i$.

- We distinguish between assets that are short-sold ($x_i < 0$) and not short-sold ($x_i \geq 0$), allocating a budget of uncertainty (to be optimized) $\Gamma^-$ and $\Gamma^+$ to each group.
Tractable Reformulation (Cont’d)

\[ \begin{array}{c}
\max \min_{\Gamma^+, \Gamma^-} \\
\Gamma^+ \leq z^+_i \leq \Gamma^- \\
\text{s.t.} \sum_{i | x_i \geq 0} |z^+_i| \leq \Gamma^+, \\
|z^+_i| \leq 1 \forall i \text{ s.t. } x_i \geq 0.
\end{array} \]

\[ \begin{array}{c}
\min_{\tilde{z}^+} \sum_{i | x_i \geq 0} x_i k_i \exp(\sigma_i \sqrt{T} \tilde{c} \tilde{z}^+_i) + \min_{\tilde{z}^-} \sum_{i | x_i < 0} x_i k_i \exp(\sigma_i \sqrt{T} \tilde{c} \tilde{z}^-_i) \\
\text{s.t.} \sum_{i | x_i \geq 0} |\tilde{z}^+_i| \leq \Gamma^+, \\
|\tilde{z}^+_i| \leq 1 \forall i \text{ s.t. } x_i \geq 0.
\end{array} \]

\[ \begin{array}{c}
\text{s.t. } \Gamma^+ + \Gamma^- = \Gamma, \\
\Gamma^+, \Gamma^- \geq 0 \text{ integer.}
\end{array} \]

\[ \begin{array}{c}
\sum_{i=1}^{n} x_i = w_0, \\
\sum_{i | x_i < 0} -x_i \leq pw_0.
\end{array} \]
At optimality, $0 \leq \tilde{z}_i^- \leq 1$ for all stocks that are short-sold (the worst case is to have returns no lower than their nominal value), and the minimization problem in $\tilde{z}_i^-$ is equivalent to the linear programming problem:

$$\min_{z^-} \sum_{i \mid x_i < 0} x_i \cdot k_i (1 - z_i^-) + x_i \cdot k_i \exp(\sigma_i \sqrt{T} \cdot c) z_i^-$$

subject to:

$$\sum_{i \mid x_i < 0} z_i^- \leq \Gamma^-,$$

$$0 \leq z_i^- \leq 1, \ \forall i \text{ s.t. } x_i < 0.$$
(i) At optimality, either the manager short-sells the maximum amount allowed, or he does not short-sell at all.

(ii) The optimal wealth is the maximum between the optimal wealth in the no-short-sales model and the convex problem:

$$\max_{\theta \geq 0} \ w_0 \cdot \left( \theta \left[ 1 + \ln \left( \frac{1 + p}{\theta} \right) \right] + F_p(\theta, \Gamma) \right),$$

where $F_p$ is defined by:
Theorem (Optimal Strategy (Cont’d))

\[ F_p(\theta, \Gamma) = \max_{\eta, \xi, \tilde{\chi}} \sum_{i | x_i \geq 0} \tilde{\chi}_i \ln k_i - \sum_{i | x_i < 0} \tilde{\chi}_i k_i - \eta \Gamma - \sum_{i=1}^{n} \xi_i \]

s.t.

\[ \eta + \xi_i - \sigma_i \sqrt{T} c \tilde{\chi}_i \geq 0, \quad \forall i | x_i \geq 0, \]

\[ \eta + \xi_i - k_i \left[ \exp(\sigma_i \sqrt{T} c) - 1 \right] \tilde{\chi}_i \geq 0, \quad \forall i | x_i < 0, \]

\[ \sum_{i | x_i \geq 0} \tilde{\chi}_i = \theta, \]

\[ \sum_{i | x_i < 0} \tilde{\chi}_i = p, \]

\[ \eta \geq 0, \xi_i \geq 0, \tilde{\chi}_i \geq 0, \quad \forall i. \]

(iii) The optimal fraction of money \( \chi_i \) allocated to asset \( i \) is \( (1 + p) \frac{\tilde{\chi}_i}{\theta} \) if the stock is invested in and \( -\tilde{\chi}_i \) if the stock is short-sold.
Corollary (Optimal Allocation)

If it is optimal to short-sell, there exist indices $j$ and $l$, $j < l$ such that the decision-maker:

- invests in stocks 1 to $j$,
- neither invests in nor short-sells stocks $j + 1$ to $l - 1$,
- short-sells stocks $l$ to $n$. 
The traditional robust model with short sales is given by:

$$\begin{align*}
\max_{x, s, q, r} & \quad \sum_{i=1}^{n} (x_i^+ - x_i^-) \exp \left[ \left( \mu_i - \frac{\sigma_i^2}{2} \right) T \right] E \left[ \exp \left( \sum_{j=1}^{n} Q_{ij}^{1/2} Z_j \right) \right] \\
- & \Gamma s - \sum_{i=1}^{n} q_i \\
\text{s.t.} & \quad \sum_{i=1}^{n} (x_i^+ - x_i^-) = w_0, \\
& \quad s + q_i \geq c r_i, \quad \forall i, \\
& \quad -r_i \leq \sum_{k=1}^{n} M_{ki}^{1/2} (x_k^+ - x_k^-) \leq r_i, \quad \forall i, \\
& \quad \sum_{i=1}^{n} x_i^- \leq p w_0 \\
& \quad s, q_i, r_i, x_i^+, x_i^- \geq 0, \quad \forall i,
\end{align*}$$
Numerical Experiments - Uncorrelated Assets

![Bar Chart Image]

- X-axis: Gamma
- Y-axis: # of Stocks
- Legend:
  - Log-robust
  - Traditional
Number of Shares per Stock in the Log-robust model

- Gamma = 10
- Gamma = 20

# of Shares

0 100 200 300 400 500 600

-400 -300 -200 -100 0 100 200 300 400

Stocks
Number of Stocks Short Sold for Two Data Sets

![Graph showing the number of stocks short sold for two data sets. The x-axis represents Gamma, and the y-axis represents the number of stocks. Two bars are shown, one for Data Set 1 and another for Data Set 2.](image-url)
Number of Stocks Short Sold for $p = 0.5$ and $p = 5$
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99% VaR - Gaussian Distribution

The graph illustrates the 99% Value at Risk (VaR) for different portfolios and gamma values. The x-axis represents the gamma values, while the y-axis shows the 99% VaR value. Three types of portfolios are compared:

- **No Short**
- **Short**
- **Trad Short**

The graph shows how the VaR changes as gamma varies for each of these portfolios.
99% cVaR - Gaussian Distribution

![Graph of 99% cVaR for Gaussian Distribution](image)

- **No Short**
- **Short**
- **Trad Short**

99% cVaR Value vs Gamma
The log-robust optimization model with short sales and correlation is:

$$\max_x \min_{\tilde{y}} \sum_{i=1}^{n} x_i \exp \left[ \left( \mu_i - \frac{\sigma_i^2}{2} \right) T + \sqrt{T} c \left( \sum_{j=1}^{n} Q_{ij}^{1/2} \tilde{y}_j \right) \right]$$

s.t. $$\sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma,$$

$$|\tilde{y}_j| \leq 1, \forall j,$$

s.t. $$\sum_{i=1}^{n} x_i = w_0,$$

$$\sum_{i \mid x_i < 0} -x_i \leq pw_0.$$
The heuristics aim at allowing us to use the results of the independent-assets case.

1. **Heuristic 1**: No correlation for assets short-sold.
2. **Heuristic 2**: Approximating the off-diagonal elements by their average and use budget of uncertainty.
3. **Heuristic 3**: Approximating the off-diagonal elements by a conservative estimate of their worst-case value.
Impact of $\Gamma$ on stock allocation and diversification for correlated stocks.

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Allocation for the three heuristics, $\Gamma = 5$
Allocation for the three heuristics, $\Gamma = 10$
Allocation for the three heuristics, $\Gamma = 20$
Comparison of the three heuristics with Normal distribution using cVaR
99% cVaR for Gaussian distribution
We have derived tractable reformulations to the portfolio management problem with short sales.

We have proved that it is optimal for the manager to either short-sell as much as he can, or not short-sell at all, and provided optimal allocations in this case.

We have also seen that diversification arises naturally from the log-robust optimization approach.