Factor Alignment Problems in Optimized Portfolio Construction

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Mean Variance Optimization Notation

Maximize Expected Return

$$\alpha^t h$$

$$h^t Q h \le R^2$$

Risk Model

Risk Target

How are Q and alpha constructed?

- Factor model of returns

$$r = Xf + \varepsilon$$

- Expected Returns Model

$$E[r] = XE[f]$$

- Factor Risk Model

$$\begin{aligned} Var[r] &= XE[ff^T]X^T + E[\varepsilon\varepsilon^T] \\ &= X\Omega X^T + \Delta \end{aligned}$$

Risk Models and Alpha Interaction

- Are optimal MVO portfolios “biased” with respect to certain risk models?
- How does a risk model used in MVO affect the optimal portfolio?
- Why do risk estimates provided by risk models that were used to construct an MVO portfolio tend to underestimate risk?
- Can an existing risk model be modified to be unbiased when used in optimization?
- And more importantly, can “mean-variance efficiency” be made more efficient?

Long Only Strategy

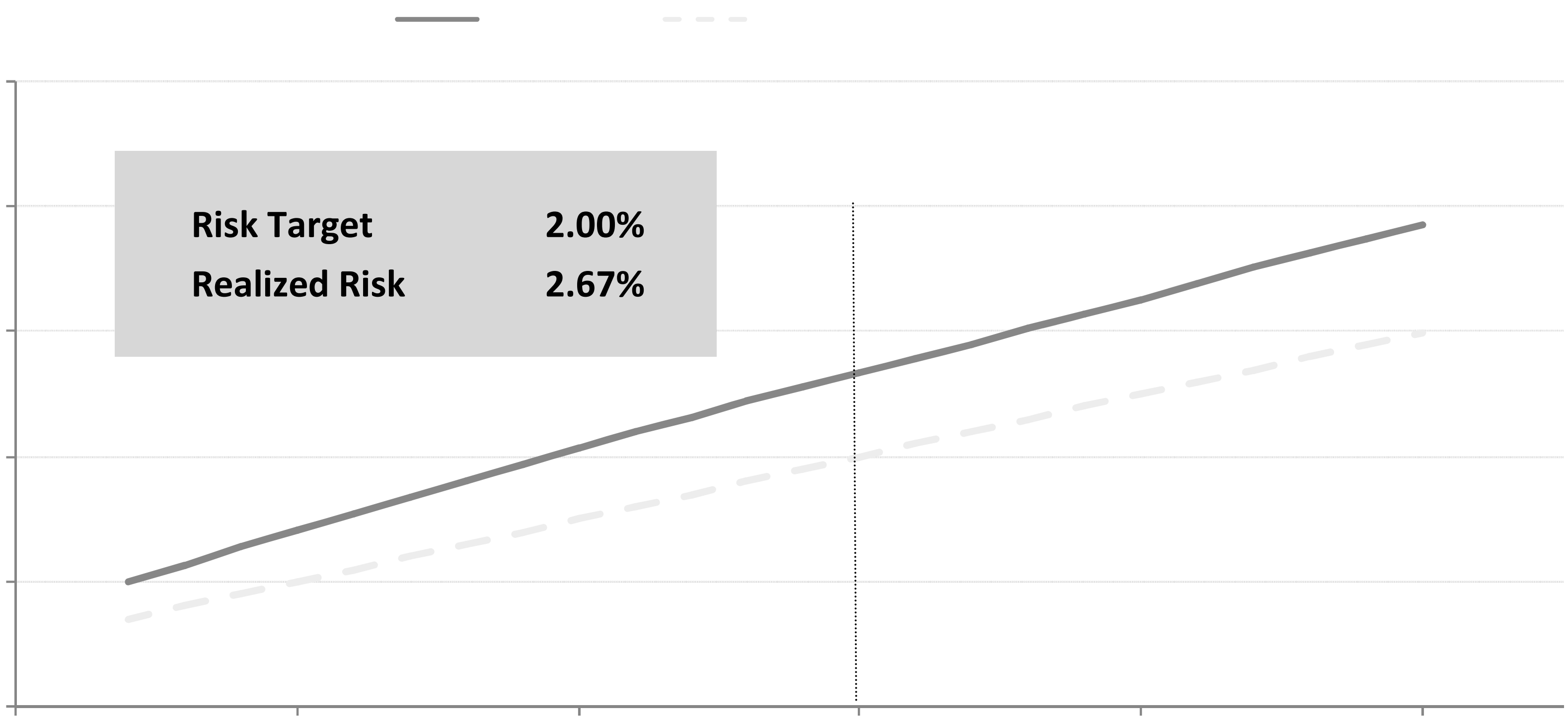
Maximize Expected Return

s.t

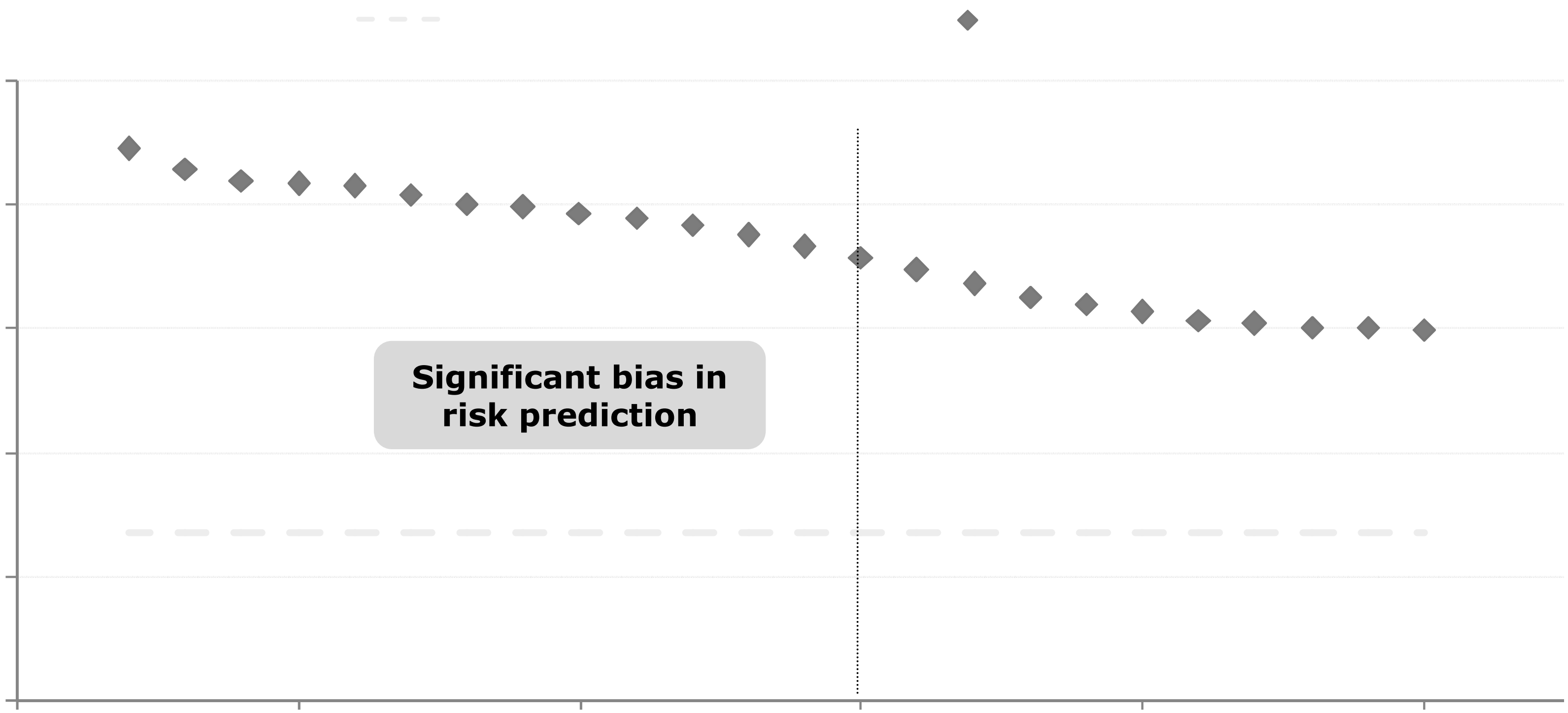
- **Fully invested long only portfolio**
- **GICS Industry exposure constraints**
- **Active asset bounds constraint**
- **Active Risk Constraint (2%)**

Risk Model = US2AxiomaMH
Benchmark = Russell 1000 Growth
Monthly backtest, 1999- 2007 time period
Alphas are not completely aligned with risk factors

Predicted vs Realized Active Risk

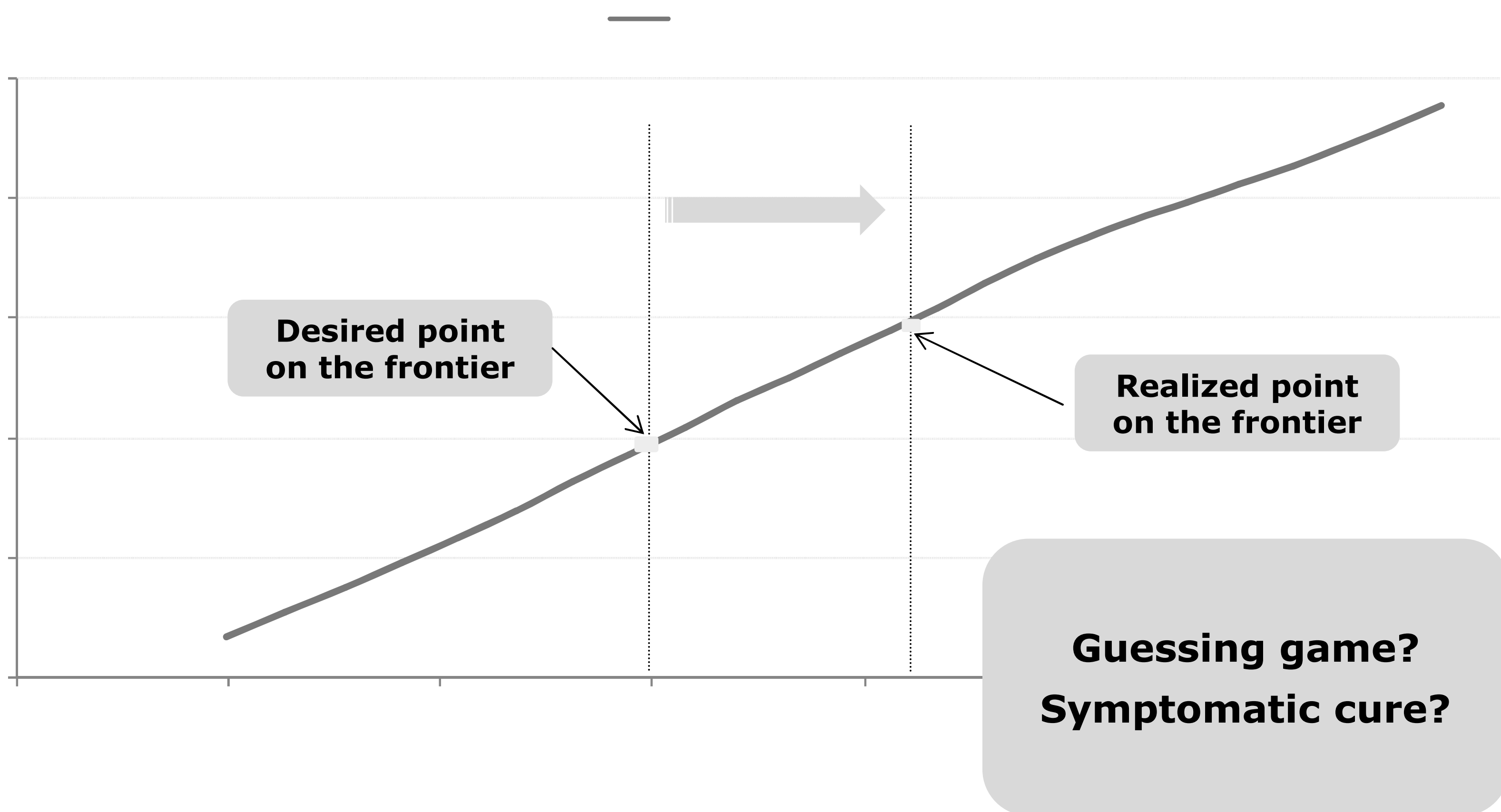


Bias Statistic

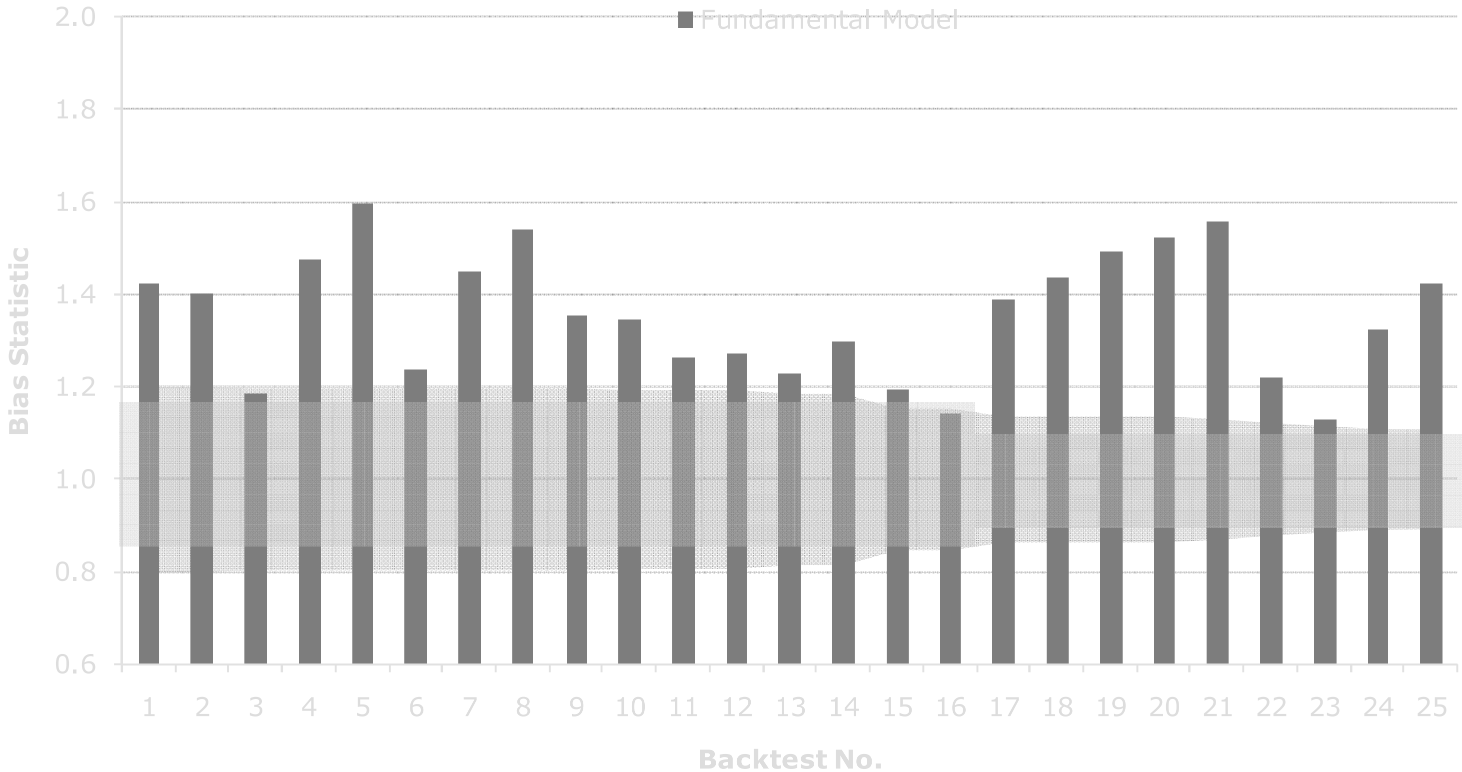


- For our purposes, the bias statistic is (realized risk/predicted risk)

Risk Return Frontier



Pervasive Problem



Fact Sheet

1. Risk under-estimation of optimized portfolios

How do we explain this phenomenon?

Unconstrained MVO

$$Q = X\Omega X^T + \Delta$$

The portion of alpha explained by the risk factors is referred to as the spanned component.

The residual obtained by regressing the alphas against the factors in the risk model is referred to as the orthogonal component of alpha.

Unconstrained MVO

The optimizer sees no systematic risk in the orthogonal component of alpha and is hence likely to load up on it

Previous Research: The Unconstrained Case

- Consider the Unconstrained Mean-Variance Optimization Problem

- Single Factor Models (*Lee & Stefek '08)*

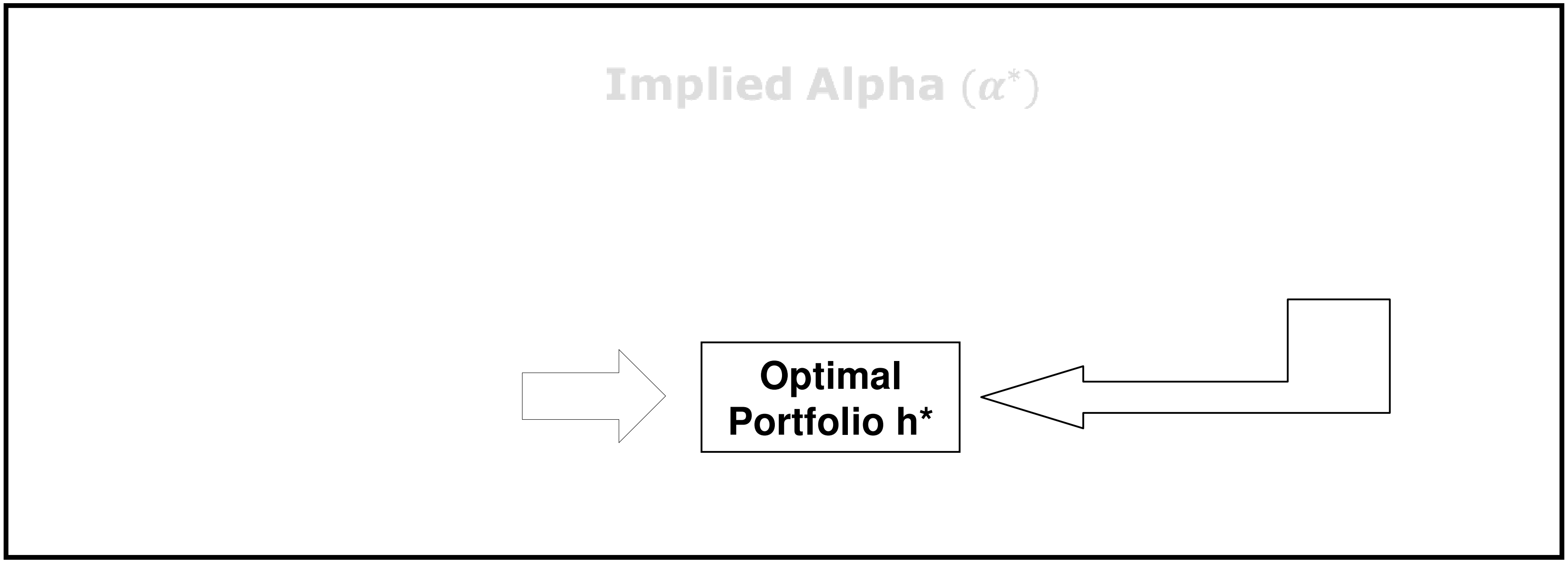
$$\frac{\sigma_s^2}{n\sigma_f^2 + \sigma_s^2}$$

< 1

- Clearly, is overweighted in the optimal solution relative to

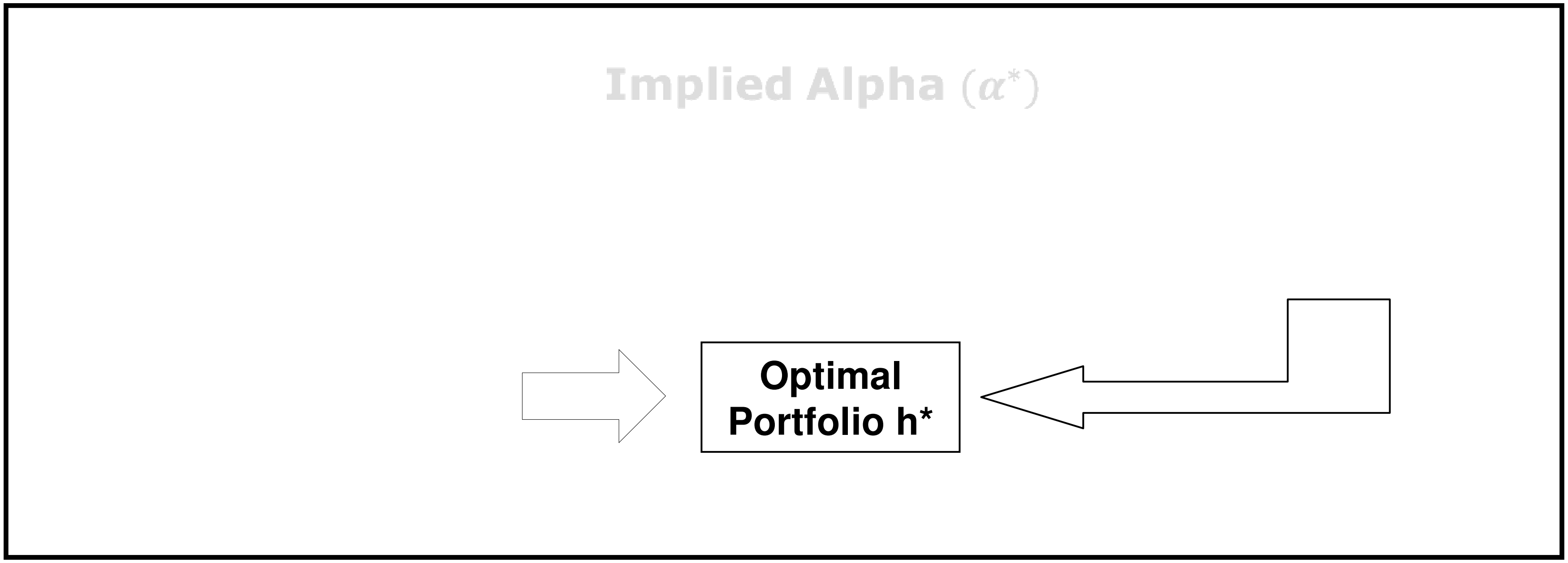
- Multi-Factor Models (*Saxena & Stubbs '10)*

Constrained MVO



- Implied alpha acts as the *de facto* alpha in the case of constrained MVO problems
- Optimizer sees no systematic risk in the orthogonal component of implied alpha and is hence likely to load up on it.
- Implied alpha is a *dynamic* entity determined by the interaction of alpha, risk factors and constraints via KKT conditions

Constrained MVO

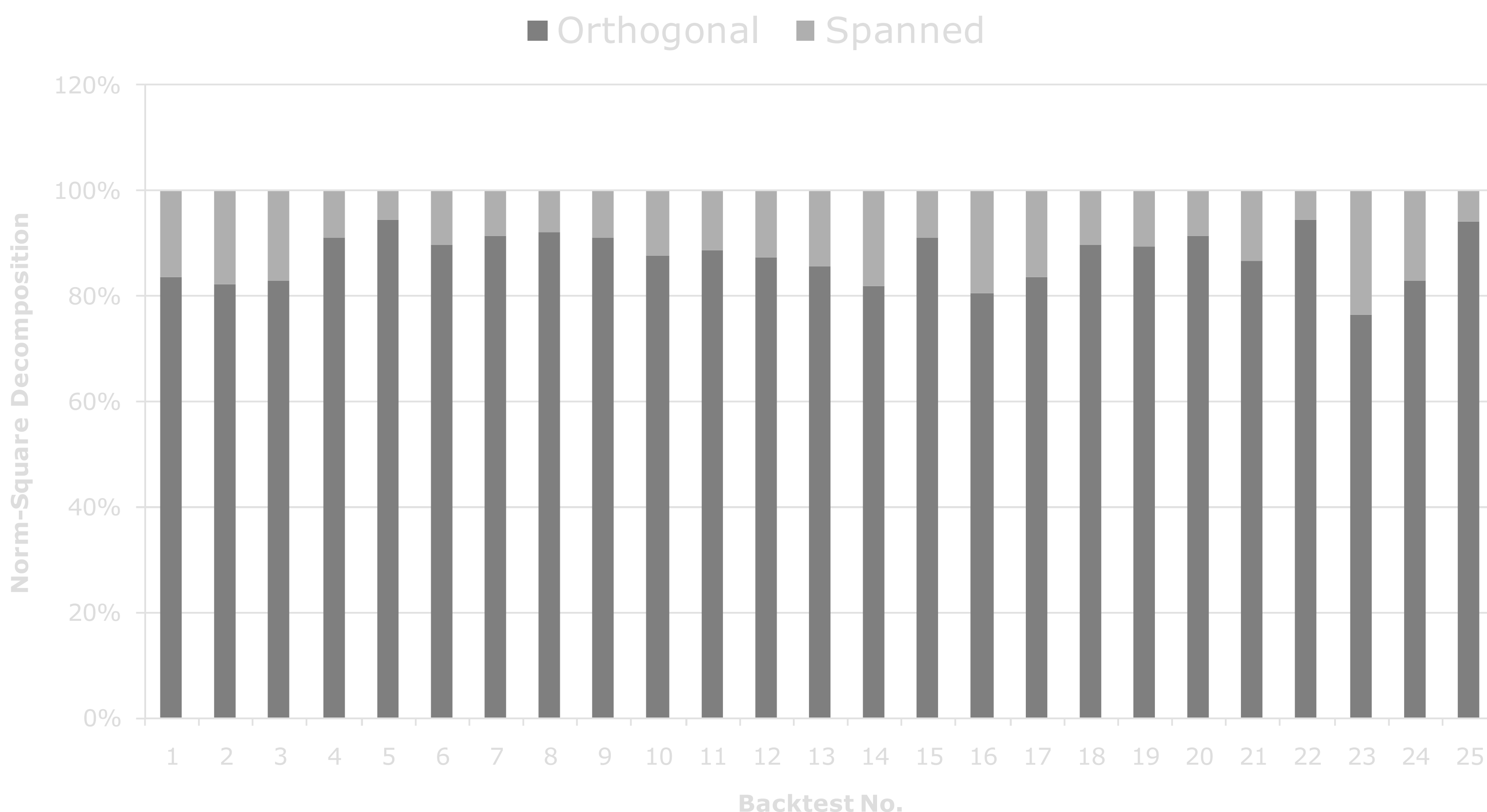


$$AAF = \frac{1}{|h_\perp|} h_\perp = \frac{1}{|\alpha^*_\perp|} \alpha^*_\perp$$

An important result that allows us to penalize exposure of the portfolio to the orthogonal component of implied alpha

AAF = Alpha Alignment Factor

Extent of overloading?

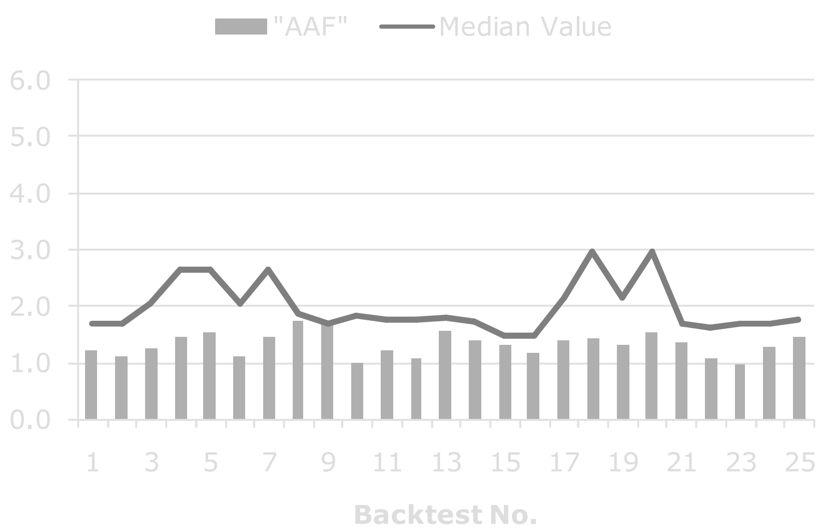


Fact Sheet

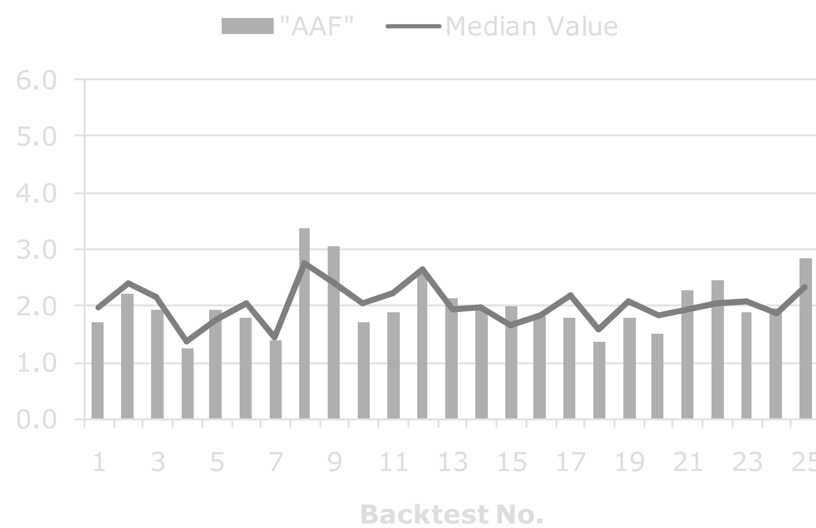
1. **Risk under-estimation of optimized portfolios**
2. **The optimizer assumes that the orthogonal component of the portfolio has no systematic risk and hence overloads on it – by a factor of 8!!**

Is it really true that the orthogonal component of the portfolio has no systematic risk?

How does AAF compared with regular risk factors: **RMS t-statistics**

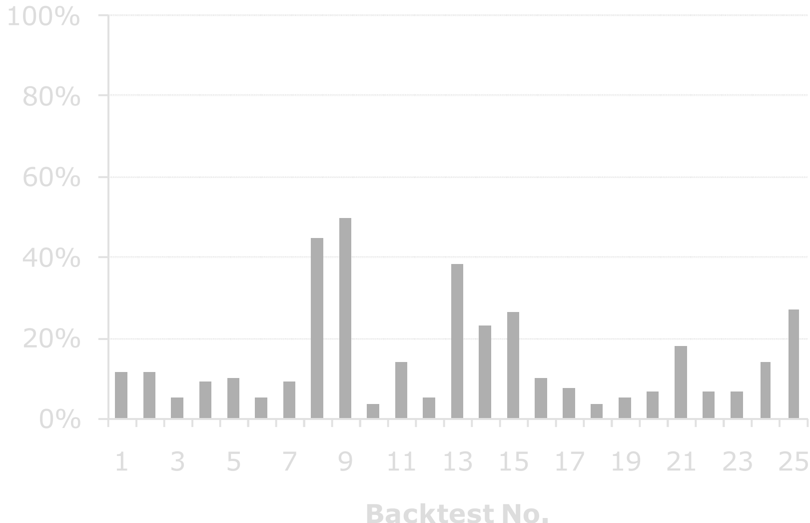


Unbiased Setup

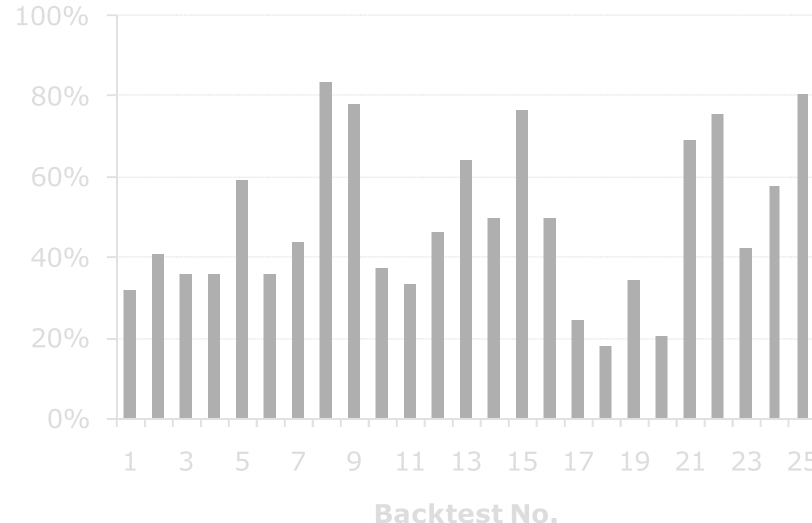


Portfolio Biased Setup

How does AAF compared with regular risk factors: **AAF Percentile Ranking**

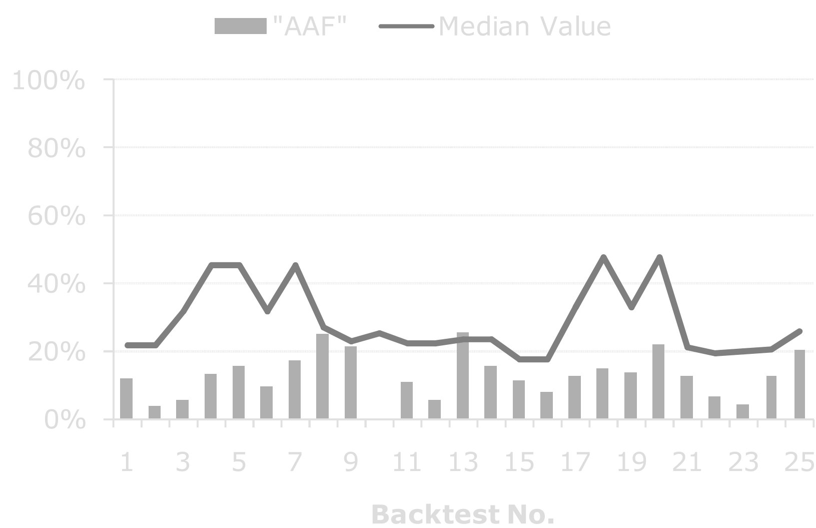


Unbiased Setup

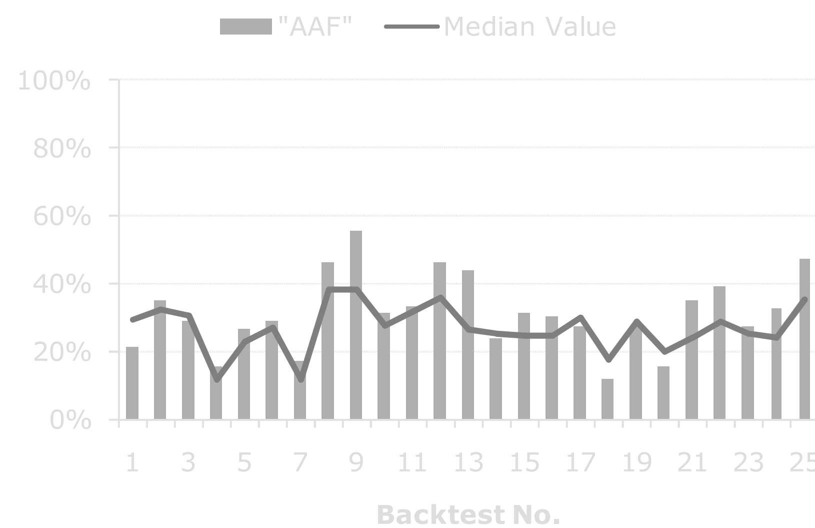


Portfolio Biased Setup

How does AAF compared with regular risk factors: **Statistical Significance of AAF**

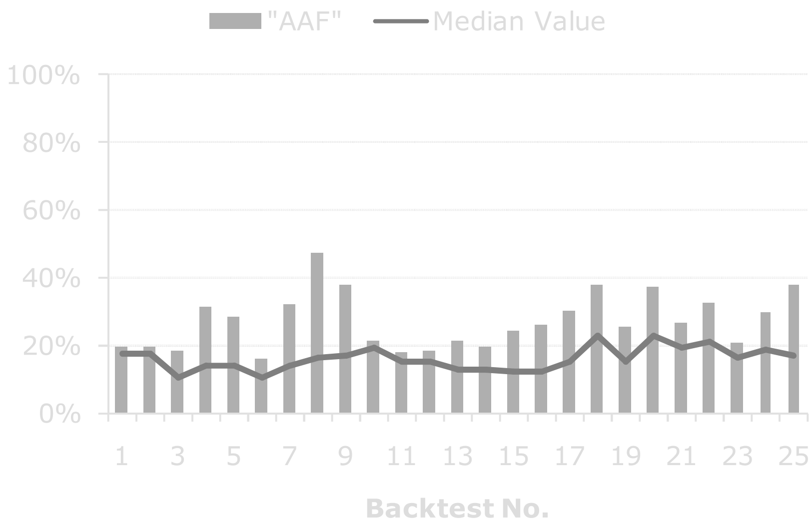


Unbiased Setup

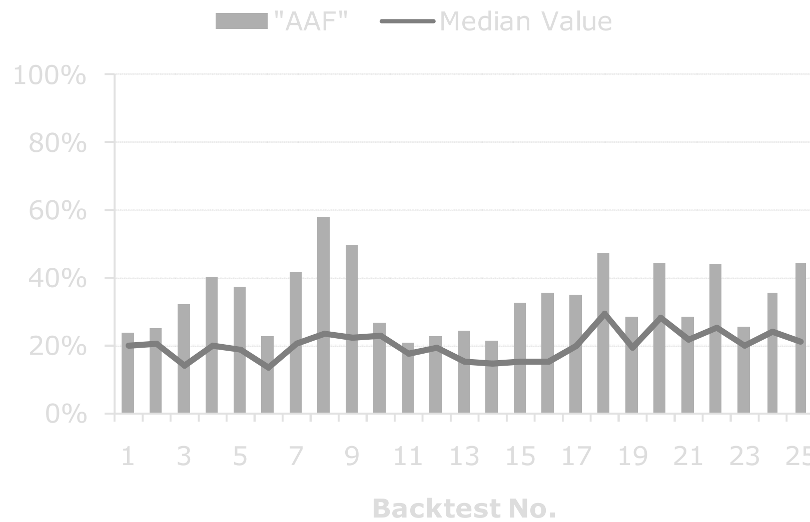


Portfolio Biased Setup

How does AAF compared with regular risk factors: **Realized Volatility of AAF**



Unbiased Setup



Portfolio Biased Setup

Fact Sheet

1. Risk under-estimation of optimized portfolios
2. The optimizer assumes that the orthogonal component of the portfolio has no systematic risk and hence overloads on it.
3. The orthogonal component of the portfolio has latent systematic risk.

    $h_{\perp}$ is not an excellent generic risk factor

    $h_{\perp}$ can be a useful factor in explaining the volatility of the optimal portfolio

What is the source of this latent systematic risk?

Sources of Factor Misalignment

- Alpha Factors
  Proprietary definition of style factors
  e.g. EBITDA/EV vs E/P
- Constraints
  Limiting exposure to custom factors
  e.g. Exposures to liquidity factor
- Proprietary definition of technical factors
  Using different parameters to define common factors
  e.g. Momentum Factors

Is Misalignment Bad?

- When we work in “theory”, we assume that the orthogonal component of alpha is all noise
  - In this case, we do not want exposure to orthogonal alpha (Hence, Misalignment is Bad)

- However, in practice, the orthogonal component of alpha may have positive IC, so, a positive exposure to the orthogonal component of alpha will increase returns (Hence, Misalignment is good)

- But, what if the orthogonal component of alpha also contains systematic risk? (Then, we should be managing the tradeoff given by the misalignment)

Augmented **Risk Models**

- Assume that the true covariance matrix Q_T is given by

- Z denotes systematic risk factors which are missing from the user risk model Q.

- Key Idea: Construct an augmented risk model by adding one additional risk factor to the original factor model that can capture the effect of the missing factors.

- Augmented Covariance Matrix

- Goal: Evaluate the effect of augmenting a risk model on the optimal portfolio

Augmented Risk Models

- denote the optimal portfolio associated with the original covariance matrix Q

- denote the optimal portfolio associated with the augmented covariance matrix

- *U(Q)* denotes the true utility function of *h(Q)*

- $U(Q_y)$ denotes the true utility function of $h(Q_y)$

Pushing Frontiers

Theorem

Using an Augmented Risk Model

- **Pushes the ex-post frontier upwards**
- **Eliminates the bias in risk prediction**
- **Restores Markowitz MVO efficiency**

Augmented Risk Models

Theorem

- If $y^T\alpha_\perp = 0$ then the augmented risk model has no impact on the holdings
- Thus if we want to materially affect the optimal portfolio, we need to choose an augmenting factor y which is correlated with $\alpha_\perp$.
- Using an augmented risk model is tantamount to tilting the portfolio in a direction away from y.

Augmented Risk Models

Theorem

- If then

By calibrating the *v* parameter we can control the extent of overloading on the orthogonal component of alpha.

Manage Exposure to Orthogonal Implied Alpha

- We really need the implied alpha, not alpha, which means, this process would require us to solve optimization problems to identify the implied alpha

- We would need to re-estimate the whole risk model with the new factor

- We would need to iterate this process, until the orthogonal component of the implied alpha is zero

- Not practical, may not converge -- not recommended

- Alternatively, penalize exposure to the orthogonal implied alpha

Penalizing Exposure to the Orthogonal Implied Alpha: Iterative Refinement

Optimal Portfolio

Determine Implied Alpha ( )

Penalize exposure to

Solve Rebalancing

Does this process reach an equilibrium?
Not necessarily…

Finding the Alpha Alignment Factor

- We solve an optimization problem to find an additional factor that increases risk the most, given a fixed portfolio (*w)*

$$\underset{f}{\text{Max}} \quad w^t \underbrace{\begin{bmatrix} B & f \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & v^2 \end{bmatrix} \begin{bmatrix} B & f \end{bmatrix}^T}_{Q'} w$$

$$\text{st.} \quad \left\| f \right\| = 1$$

$$B^t f = 0$$

v^2	a fixed constant (new factor volatility)
$B^T f = 0$	new factor is “orthogonal” to existing factors
$\|\| f \|\| = 1$	normalize new factor exposures

Integrating the AAF and MVO

Maximize Expected Return

Traditional MVO

$$
\begin{aligned}
\text{Max} \quad & \boxed{\alpha^t w} \\
\text{st.} \quad & \sum_{i \in U} w_i = 1 \\
& \boxed{w^t Q w \le R^2} \\
& w \ge 0
\end{aligned}
$$

MVO with Alpha Factor

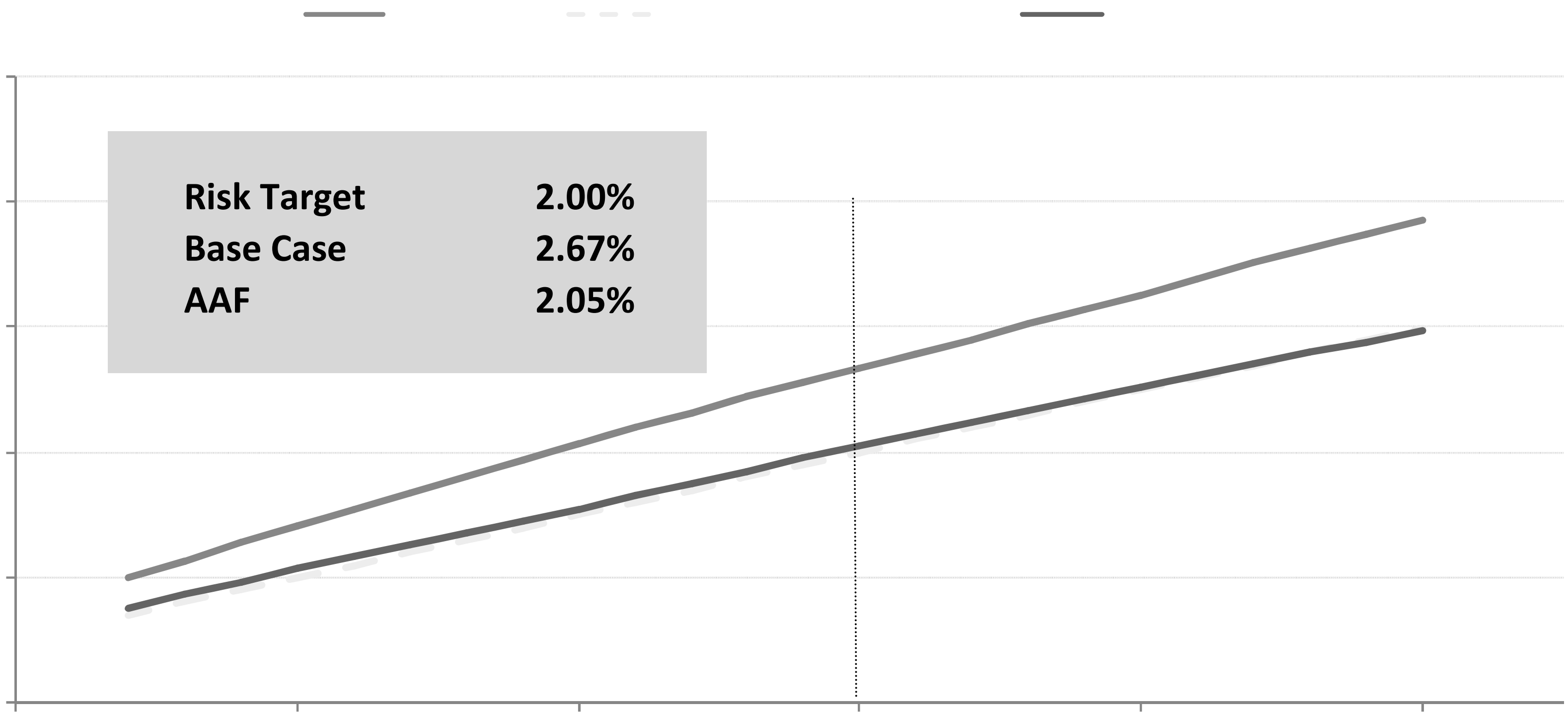
$$
\begin{aligned}
\text{Max} \quad & \boxed{\alpha^t w} \\
\text{st.} \quad & \sum_{i \in U} w_i = 1 \\
& \boxed{(\max_{f, \|f\|=1, B^T f = 0} w^t Q' w) \le R^2} \\
& w \ge 0
\end{aligned}
$$

Risk Target

Robust Optimization provides AAF

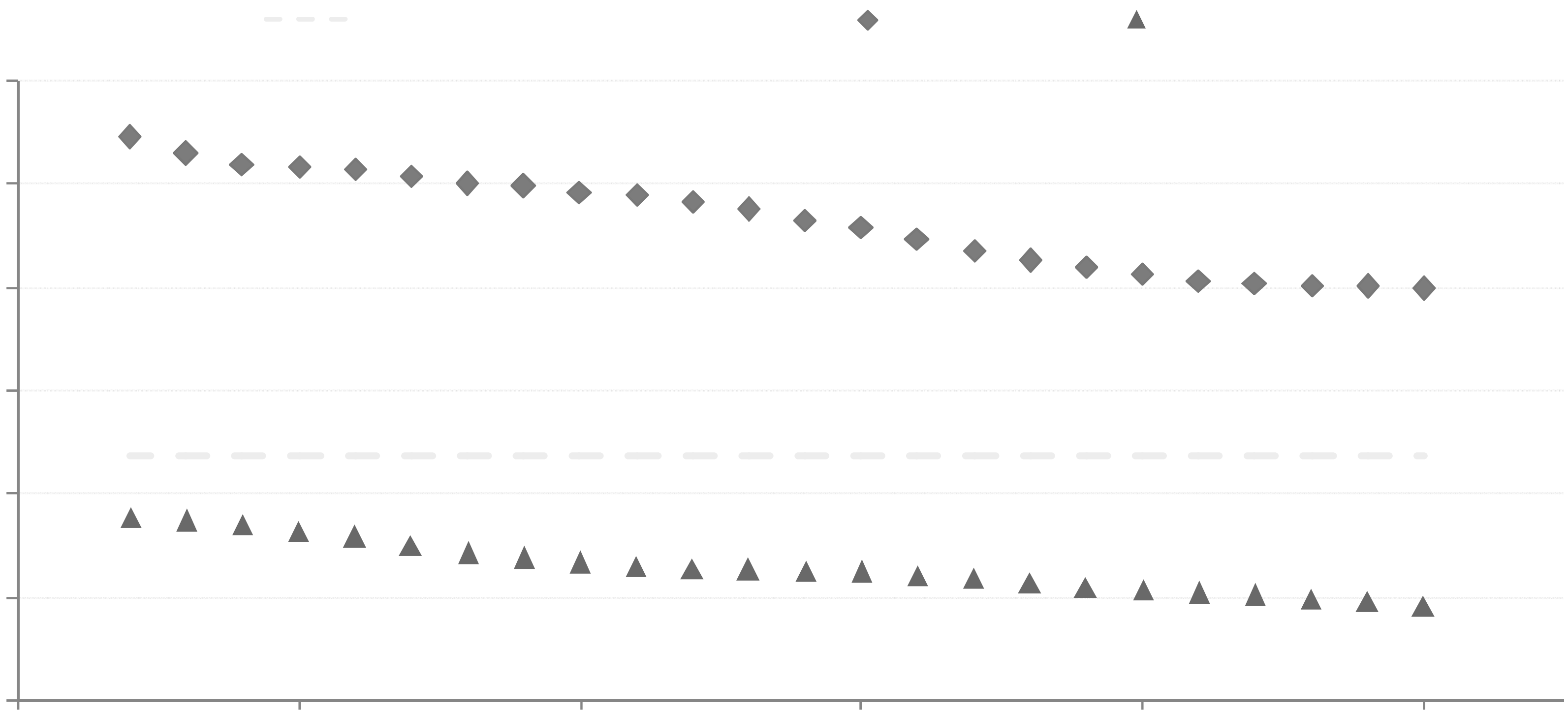
- Robust portion has closed form solution: $f = \frac{w_\perp}{\|w_\perp\|}$
- "Alpha Alignment Factor" (f) is the orthogonal part of implied alpha
- Adding the Alpha Factor to the optimization problem as we proposed before is, in fact, penalizing the exposure to the orthogonal implied alpha
- Still have to manage the tradeoff; fortunately, it is only one parameter v, the "volatility" of the AAF

Long-only Strategy Revisited



Unbiased Risk Prediction

Long-only Strategy Revisited



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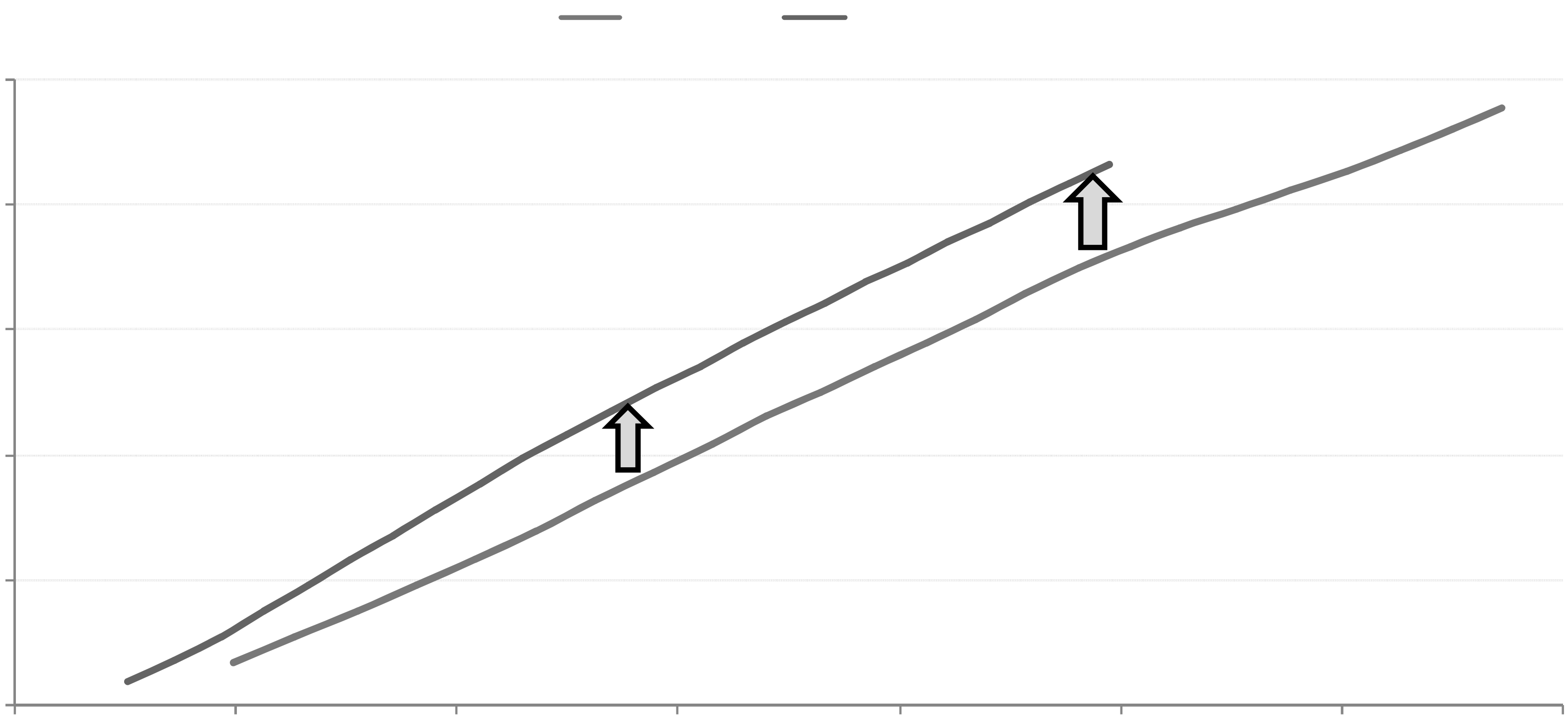
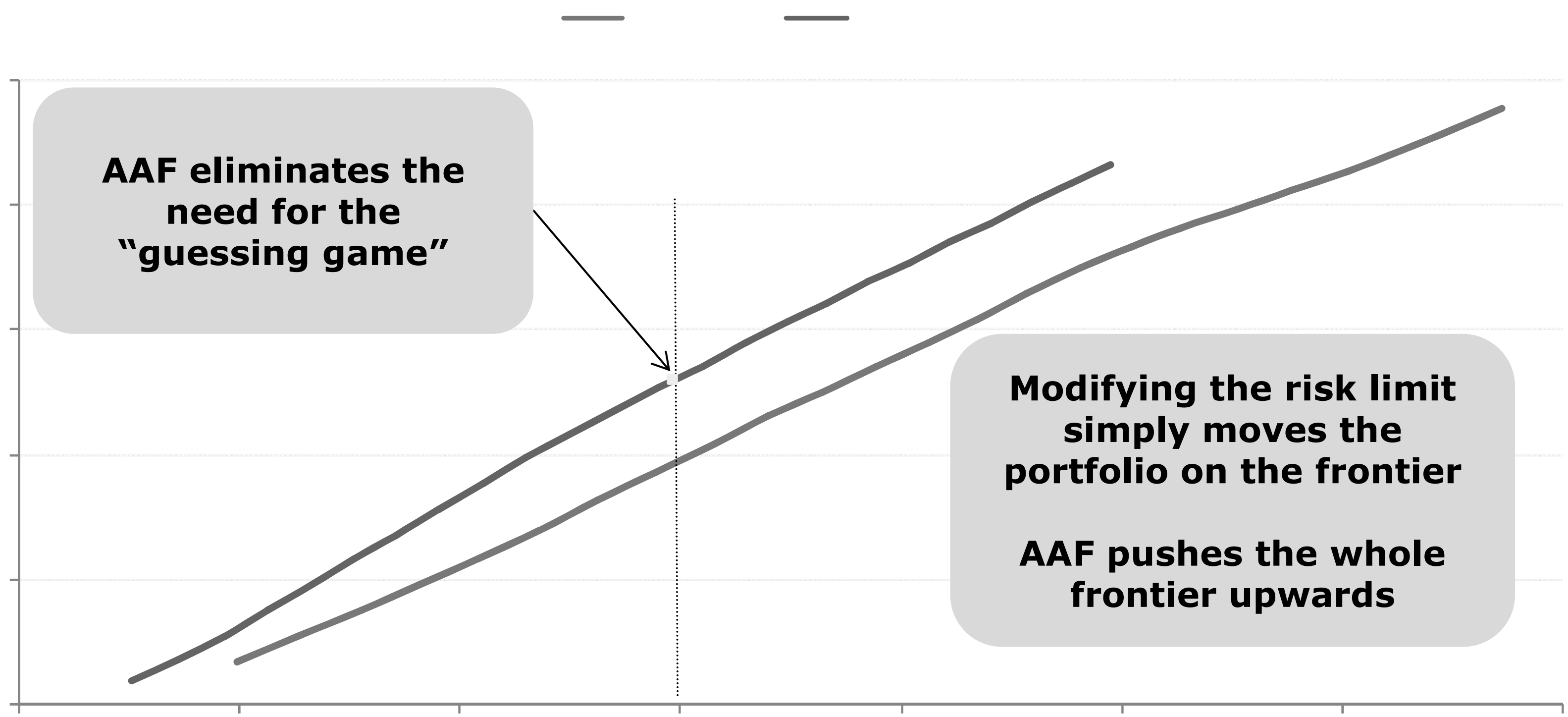


Pushing Frontiers, literally!

Long-only Strategy Revisited



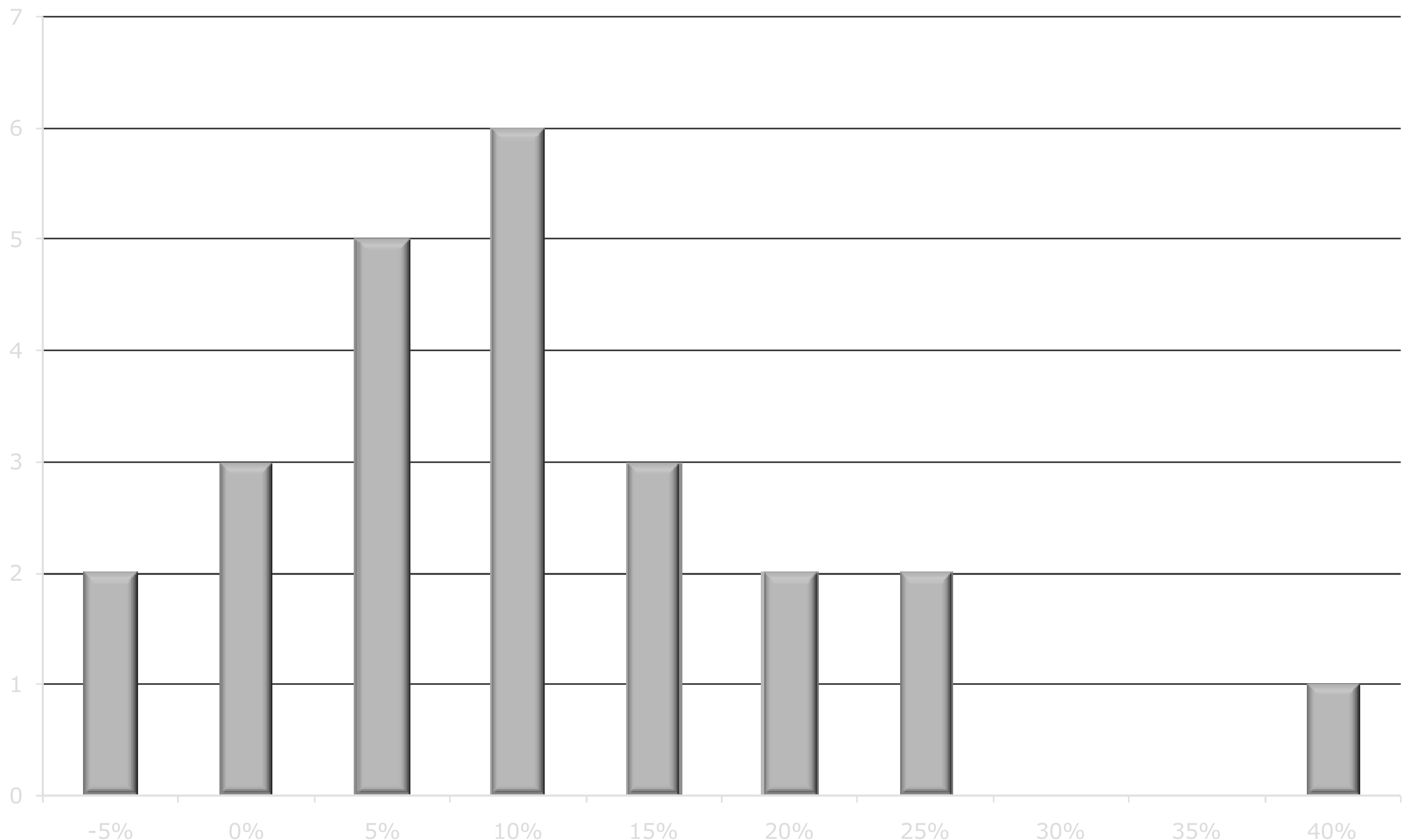
Ex-post performance improvement?

Q. How can an approach designed exclusively to remove bias in risk prediction also improve ex-post performance?

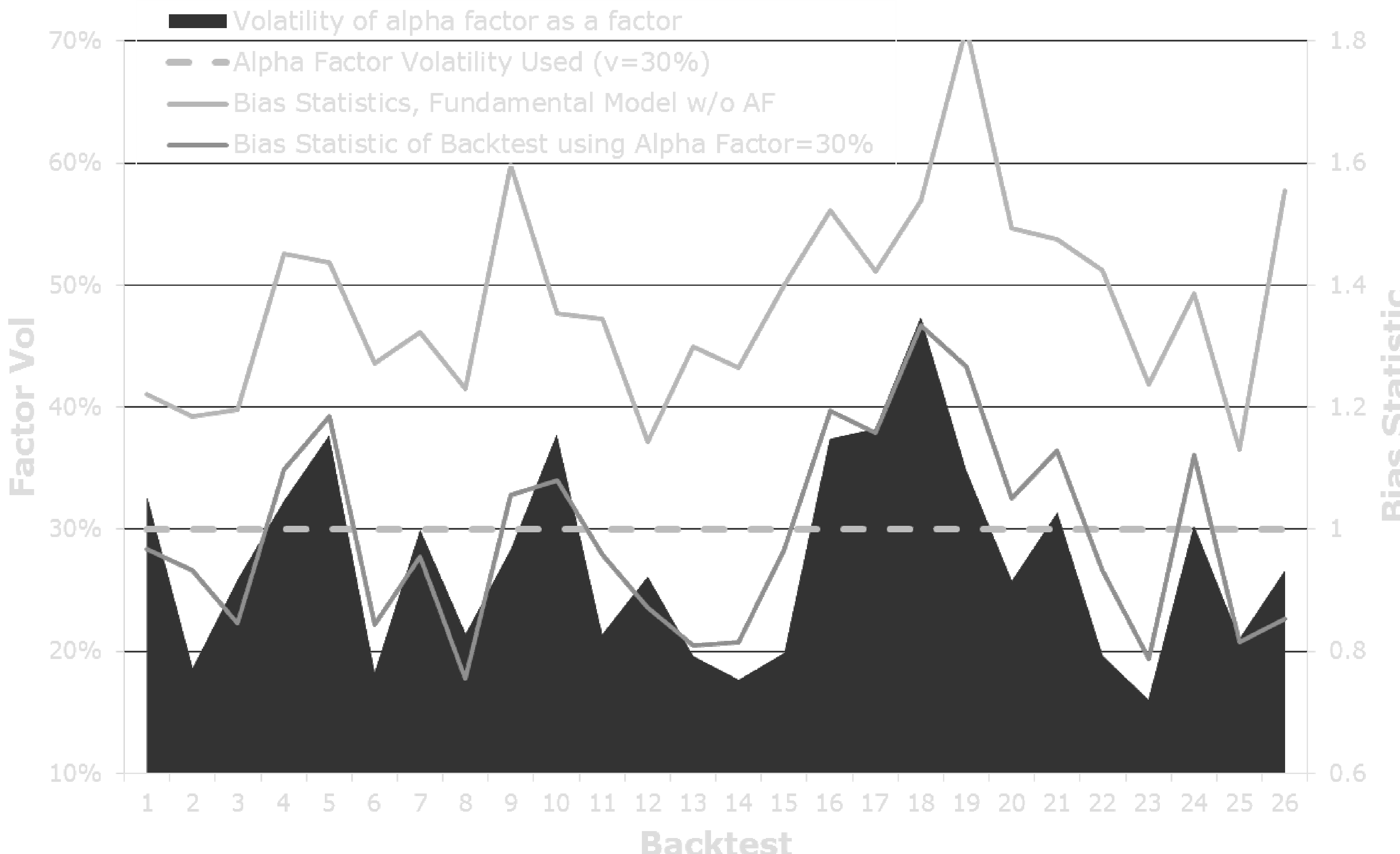
A.

- **Application of risk models in quantitative portfolio construction is not just limited to risk forecasting**
- **Risk models materially affect the composition of optimal holdings**
- **Risk models indirectly affect Sharpe/Information ratio, transfer coefficient, risk/budget allocation across various securities, etc.**
- **Bias in risk models naturally leads to inefficient budget allocation across various securities**
- **Using AAF not only removes unintended systematic bets but also improves the "efficiency" of the budget allocation process itself**

Improvements in IR (distribution)



Relationship between Bias Statistic and Misestimation of Volatility of Alpha Factor



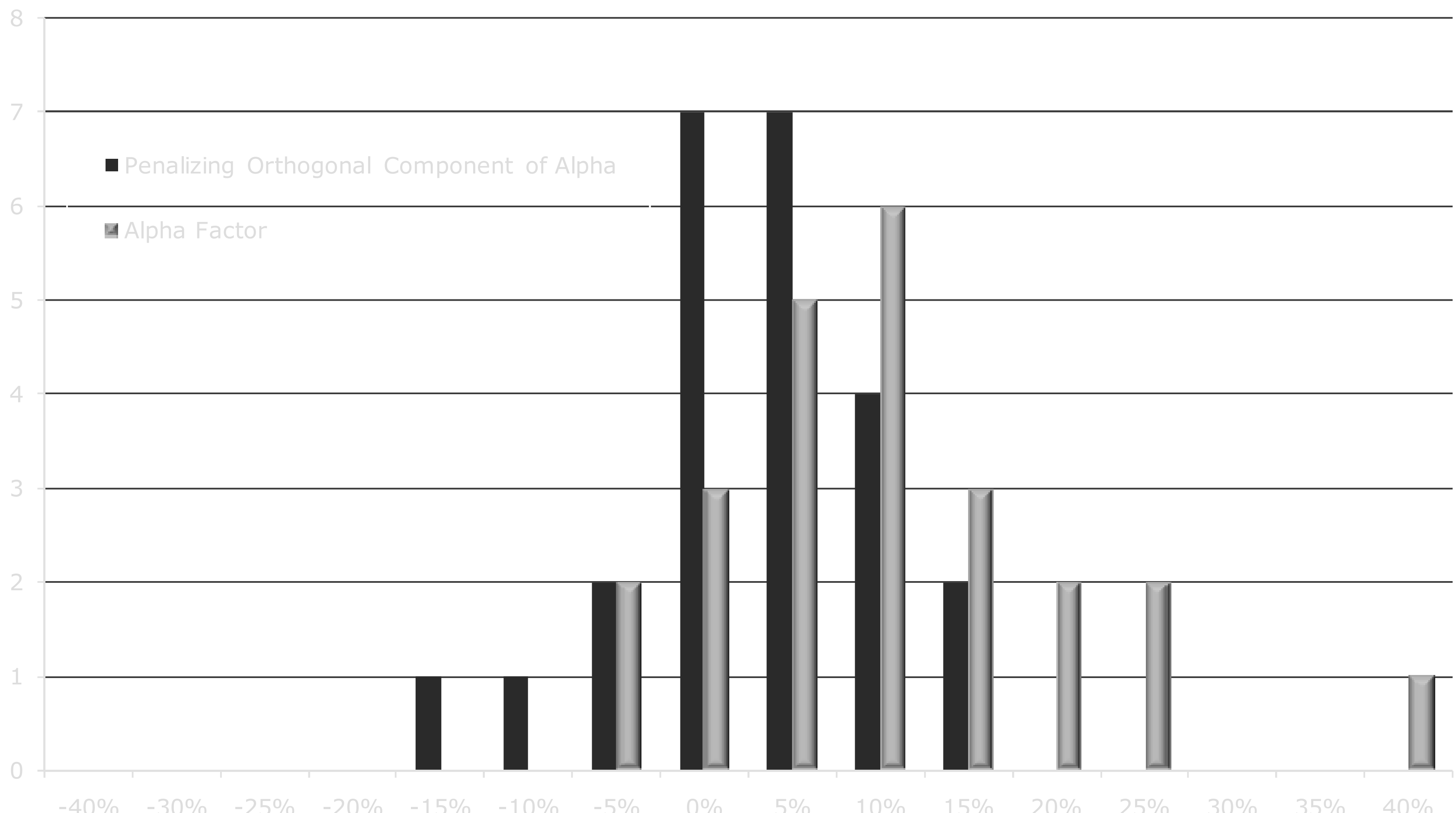
Fact Sheet

1. Risk under-estimation of optimized portfolios
2. The optimizer assumes that the orthogonal component of the portfolio has no systematic risk and hence overloads on it
3. The orthogonal component of the portfolio has latent systematic risk
4. Misalignment between alpha factors, risk factors and constraints
5. *Pushing Frontiers, literally!*

Do we really need AAF?

Can we just penalize exposure to $\alpha_{\perp}$?

Improvements in IR (distribution)



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4. Misalignment between alpha factors, risk factors and constraints
5. *Pushing Frontiers, literally!*
6. Penalizing the exposure to $\alpha_{\perp}$ is only a partial solution to the alignment problem, and inferior to using the AAF methodology.

Theoretical Foundation: Our Contribution

- **Augmented Risk Models**
  - Risk model obtained by adding a single auxiliary factor *y* that is uncorrelated with all the user risk factors.
  - Our research is based on detailed analysis of augmented risk models and their capability in capturing the effect of missing systematic risk factors.

- **Main Theoretical Results**
  - There is improvement in the ex-post utility function *if and only if* the auxiliary factor *y* is correlated with the orthogonal component of implied alpha.
  - Factor alignment problems arise *if and only if* the orthogonal component of implied alpha has systematic risk.
  - The effect of all the missing factors can be captured by just one auxiliary factor. The **Alpha Alignment Factor** is such a factor.
  - Using AAF pushes the ex-post risk-return frontier upwards. In other words, it allows the PM to reach a given return target at a lower value of realized risk than obtained without using **Alpha Alignment Factor**.

Research Papers (www.axiomainc.com)

- Saxena, A. and Stubbs, R.A., Alpha alignment factor: A solution to the underestimation of risk for optimized active portfolios. Technical report, Axioma, Inc. Research Report #015, February 2010. Submitted to *Journal of Risk*.

- Saxena, A. and Stubbs, R.A., Pushing the Frontier (literally) with the Alpha Alignment Factor. Technical report, Axioma, Inc. Research Report #022, September 2010.

- Ceria, S., Saxena, A. and Stubbs, R.A., Factor Alignment Problems and Quantitative Investing. To appear in *Journal of Portfolio Management, 2012.*

- Saxena, A. and Stubbs, R.A., An empirical case study of factor alignment problems using the United States Expected Returns (USER) model. Technical report, Axioma, Inc. Research Report #036, October 2011. Submitted to *Journal of Investing.*

- Saxena, A. and Stubbs, R.A., Axioma's Alpha alignment factor demystified. Axioma Advisor, September 2010.

Next Steps...

- Custom risk models