Indifference valuation for guaranteed annuity options using the explicit solution for a class of stochastic optimization problems

Sebastiano Silla
(with M. R. Grasselli)

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Implicit options in life insurance contracts

- Insurance policies often include implicit options.
- Hedging such options is difficult. Reserving for such option was improper in the past.
- Fair valuation for implicit option is *now* required by financial reporting.

Guaranteed Annuity Options (GAOs)

- GAOs belong to the class of long term guarantees.
- GAOs are rights available to holders of certain policies, very common in U.S. tax-sheltered plans and U.K. retirement savings.
- Under a GAO, the insurer guarantees to convert the policyholder's accumulated funds into a life-long annuity at a guaranteed rate, which is specified when the policy is purchased.
- The underlings?

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Literature review

- Both *financial* and *actuarial* approaches try to handle with GAOs. An overview is offered by Gatzert & Schmeiser (2006) and Bacinello (2006). In the contribution of Milevsky & Promislow (2001), both stochastic interest and hazard rates are considered.
- GAOs are analyzed by Boyle & Hardy (2003), Hardy (2003), Ballotta & Haberman (2003, 2006), Pelsser (2003).
- Stochastic and Dynamic mortality is also studied by Pitacco (2004), Olivieri & Pitacco (2003), Dahl (2004), Dahl and Möller (2006), Möller (2002, 2003ab) Cairns et al. (2005, 2006).

The indifference utility approach

We consider an indifference approach for valuing the GAO, from the insurer's point of view.

- Mortality risk and non-traded assets.
- This approach is actually a continuous time portfolio selection problem.

Agenda

1. Assumptions on the financial and the insurance market
2. The indifference model from the insurer's point of view
3. A class of stochastic optimal control problems
4. Future research.

Policy design

Assume the option to convert the policyholder's accumulated funds A, into a life long annuity, at a given rate h.

- The time $t_0 \geqslant 0$ when the policy is purchased.
- The guaranteed rate h. The accumulated funds A. The premium is payed at a nominal instantaneous rate P per year.
- The time of conversion or when the policy expires $T > t_0$.
- The agent is aged x at time 0.

Actuarial present value of a life annuity

- In a continuous compounding setting, the actuarial present value of a one-euro per year life annuity:

$$\bar{a}_{x+t} := \int_t^{+\infty} e^{-r(s-t)}\, {}_{s-t}p^{O}_{x+t}\, \mathrm{d}s$$

- The conversion rate $h = 1/\bar{a}_{x+t}$

The guaranteed conversion rate

- Once the guaranteed rate for time T is specified, a subjective implicit discount rate r_h is defined:

$$\frac{1}{h} = \overline{a}^{(h)}_{x+T} := \int_T^{+\infty} e^{-r_h(s-T)} \, {}_{s-T}p^{O}_{x+T} \, \mathrm{d}s$$

- At time T the annuitant can be assured by a cash flow stream at a nominal continuously compounded rate $H := A \cdot h$ per year.

The subjective implicit discount rate r_h.

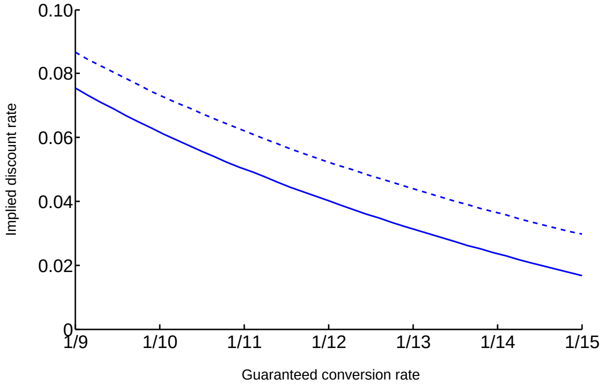

Figure: Discount rates over different levels of the conversion rate, implied for a female, using survival tables available in 1970 (solid) and 2004 (dashed).

Investment strategies

- Assume the policyholder holds her wealth in a portfolio partly invested in a risky stock and partly in a risk free asset, for $s \geqslant t_0$.

$$\begin{cases} dS_s = \mu S_s \, ds + \sigma S_s \, dB_s \\ S_{t_0} > 0 \end{cases} ; \qquad \begin{cases} d\zeta_s = r\zeta_s \, ds \\ \zeta_{t_0} > 0 \end{cases}$$

- $\{B_s\}$ is a Brownian's motion over some probability space $(\Omega, \mathscr{F}, P)$, and $0 < \mu < \sigma$, and $r > 0$.
- The policyholder's asset allocation: w_{t_0}, π_s and $W_s - \pi_s$.
- π_s is progressively measurable w.r.t. $\{\mathscr{F}_s\}_{s \geqslant t_0}$, where $\mathscr{F}_s$ is the augmentation of $\sigma(B_t : t_0 \leqslant t \leqslant s)$, and $\int_{t_0}^{s} \pi_t^2 \, dt < \infty$.

Comparing the policyholder's strategies at time T

We consider the following strategies:

1. The option is not embedded in the policy
2. The option is embedded in the policy. In this case:
   - 2.a The individual can exercise the option at time T.
   - 2.b The individual does not exercise the option at time T.

Strategy 2.a: The option is exercised

If the option is exercised at time T, the policyholder seeks to solve the following problem:

$$V(w_T, T) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)} {}_{s-T}p^S_{x+T} \cdot u(c_s)\, \mathrm{d}s \,\middle|\, W_T = w_T \right]$$

subject to the wealth dynamics:

$$\begin{cases} \mathrm{d}W_s = r(W_s - \pi_s) + \pi_s \mathrm{d}S_s + (H - c_s)\,\mathrm{d}s \\ \qquad\;\; = [rW_s + (\mu - r)\pi_s + H - c_s]\,\mathrm{d}s + \sigma \pi_s \mathrm{d}B_s \\ W_T = w_T \end{cases}$$

Strategy 2.b: The option is not exercised

The policyholder withdraws the accumulated funds A and she will seek to solve a standard Merton's problem given by:

$$U(w_T + A, T) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)} {}_{s-T}p^S_{x+T} \times \times \; u(c_s)\,\mathrm{d}s \,\middle|\, W_T = w_T + A\right]$$

subject to the following wealth dynamics

$$\begin{cases} \mathrm{d}W_s = r(W_s - \pi_s) + \pi_s \mathrm{d}S_s - c_s \mathrm{d}s \\ \qquad = [rW_s + (\mu - r)\pi_s - c_s]\,\mathrm{d}s + \sigma\pi_s \mathrm{d}B_s \\ W_T = w_T + A \end{cases}$$

Strategy 1: No option embedded in the policy

We assume that value function U also represents the expected reward if the option was not embedded in the policy:

$$U(w_T + A, T) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_T^{+\infty} e^{-r(s-T)}\,{}_{s-T}p^S_{x+T} \times \times\; u(c_s)\,\mathrm{d}s \,\middle|\, W_T = w_T + A\right]$$

subject to the following wealth dynamics

$$\begin{cases} \mathrm{d}W_s = r(W_s - \pi_s) + \pi_s \,\mathrm{d}S_s - c_s \,\mathrm{d}s \\ \qquad\;\; = [rW_s + (\mu - r)\pi_s - c_s]\,\mathrm{d}s + \sigma \pi_s \,\mathrm{d}B_s \\ W_T = w_T + A \end{cases}$$

Technical assumptions

We assume that $\{c_s\}$ and $\{\pi_s\}$ are measurable w.r.t. $\{\mathscr{F}_s\}_{s \geqslant T}$, where $\mathscr{F}_s$ is the augmentation of $\sigma(B_t : T \leqslant t \leqslant s)$. Also, for $t \geqslant T$, we assume

$$\int_T^t \pi_s^2 \, \mathrm{d}t < \infty$$

$$c_t \geqslant 0 \quad \text{and} \quad \int_T^t c_s \, \mathrm{d}s < \infty$$

and that the results in previous equations are unique. See Karatzas & Shreve (1998)

Indifference valuation at time T

Assuming the policy is embedded in the policy, at time T the policyholder compares the two expected rewards arising from strategies (2.a) and (2.b), described above. We assume she will choose to exercise the guaranteed annuity option as long as

$$\boxed{U(w_T + A, T) \leqslant V(w_T, T)}$$

Comparing the policyholder's strategies at time t_0

Assuming the agent's income is given by the gains she can realize trading in the stock market (labor income is considered later), between time t_0 and time T, her wealth obeys to the following dynamics:

$$\begin{aligned} dW_s &= r(W_s - \pi_s) + \pi_s \, dS_s - (P + c_s)\, ds \\ &= [rW_s + (\mu - r)\pi_s - P - c_s]\, ds + \sigma \pi_s \, dB_s \end{aligned}$$

under the initial condition $W_{t_0} = w_0 > 0$, where P is such that

$$\int_{t_0}^{T} P e^{r(T-s)}\, ds = A$$

The policyholder decides not to embed the GAO

If at time t_0 the option is not embedded in the policy, the annuitant's expected reward will be given by:

$$\mathcal{U}(w_0, t_0) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_{t_0}^{T} e^{-r(s-t_0)}\, {}_{s-t_0}p^S_{x+t_0} \cdot u(c_s)\,\mathrm{d}s + \right.$$
$$\left. + \; e^{-r(T-t_0)}\, {}_{T-t_0}p^S_{x+t_0} \cdot U(W_T + A, T) \,\middle|\, W_{t_0} = w_0 \right]$$

The policyholder decides to embed the G.A.O.

Assumption. The annuitant is asked to pay a lump sum L_0 at time t_0, if she decides to embed the G.A.O. in the policy.

If at time t_0 the option is embedded in the policy, the annuitant's expected reward will be given by:

$$\mathcal{V}(w_0 - L_0, t_0) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\Bigg[\int_{t_0}^{T} e^{-r(s-t_0)}\,{}_{s-t_0}p^S_{x+t_0} \cdot u(c_s)\,\mathrm{ds} +$$

$$+ e^{-r(T-t_0)}\,{}_{T-t_0}p^S_{x+t_0} \times$$

$$\times \max\{U(W_T + A, T),\ V(W_T, T)\} \,\Bigg|\, W_{t_0} = w_0 - L_0\Bigg]$$

Technical assumptions

We assume that $\{c_s\}$ and $\{\pi_s\}$ are measurable w.r.t. $\{\mathscr{F}_s\}_{t_0 \leqslant s < T}$, where $\mathscr{F}_s$ is the augmentation of $\sigma(B_t : t_0 \leqslant t < s)$. Also:

$$\int_{t_0}^{T} \pi_s^2 \, ds < \infty; \quad \text{and} \quad \int_{t_0}^{T} c_s \, ds < \infty$$

where, for $t_0 \leqslant t < T$, we restrict $c_t \geqslant 0$, and that the results in previous equations are unique.

Indifference valuation at time t_0

We postulate that the agent will decide to embed a guaranteed annuity option in her policy, as long as the following inequality holds:

$$\boxed{\mathscr{U}(w_0, t_0) \leqslant \mathscr{V}(w_0 - L_0, t_0)}$$

The policyholder's expected reward at the initial time t_0

The stochastic problems considered so far can be embedded in the following class of stochastic problems:

$$\mathcal{G}(w_0, t_0) := \sup_{\{c_s, \pi_s\}} \mathbb{E}\left[\int_{t_0}^{T} e^{-\int_{t_0}^{s} r+\lambda^S_{x+\eta}\,\mathrm{d}\eta} v(c_s)\,\mathrm{d}s + \right.$$
$$\left. + \; e^{-r(T-t_0)} \, {}_{T-t_0}p^S_{x+t_0} \cdot \frac{(W_T-\Delta)^{1-\gamma}}{1-\gamma} \,\middle|\, Y_{t_0} = y_0 \right]$$

subject to the following dynamics:

$$\begin{cases} \mathrm{d}Y_s = [rY_s + (\mu - r)\pi_s - c_s - \Delta_1]\,\mathrm{d}s + \sigma \pi_s \,\mathrm{d}B_s \\ Y_{t_0} = y_0 \end{cases}$$

Assumptions and remarks

Assumptions:

- We assume a CRRA utility function $v(c) = c^{1-\gamma}/(1-\gamma)$, for $\gamma > 0$, $\gamma \neq 1$.
- We assume $\Delta < 0$ and $\Delta_1 > 0$;
- We assume $r > (1-\gamma)\delta$, where $\delta := r + 1/(2\gamma) \cdot (\mu - r)^2/\sigma^2$

Remarks. The problem is difficult because:

- The horizon is finite: $T < +\infty$;
- The following function acts as a bequest motive and is not null

$$g(W_T, t_0; T) = e^{-r(T-t_0)} {}_{T-t_0}p^S_{x+t_0} \cdot \frac{(W_T - \Delta)^{1-\gamma}}{1-\gamma}$$

- The drift of previous wealth dynamics specification.

Solution

Construct a value function that measure the remaining utility since time $t \geqslant t_0$, given a positive wealth y:

$$\widetilde{\mathscr{G}}(y, t) := \sup_{\{c_s, \pi_s\}} \mathbb{E} \Bigg[\int_t^T e^{-\int_t^T r + \lambda_{x+\eta}^S \, \mathrm{d}\eta} \cdot v(c_s) \, \mathrm{d}s + \\ + \; e^{-r(T-t)} \, {}_{T-t}p_{x+t}^S \cdot \frac{(W_T - \Delta)^{1-\gamma}}{1-\gamma} \, \Bigg| \, W_t = y \Bigg]$$

Solution

Henceforth, value function $\widetilde{\mathscr{G}}(y,t)$ need to satisfy the following HJB equation:

$$0 = \sup_{c \geqslant 0, \pi} \left\{ \widetilde{\mathscr{G}}_t(y,t) - (r + \lambda_{x+t}) \widetilde{\mathscr{G}}(y,t) + v(c) + \left(\mathscr{A}^{c,\pi} \widetilde{\mathscr{G}} \right)(y,t) \right\}$$

where the differential operator associated to policyholder's wealth dynamics is given by:

$$(\mathscr{A}^{c,\pi})(y,t) := \left[ry + (\mu - r)\pi - c - \Delta_1 \right] \frac{\partial}{\partial y} + \frac{1}{2}\sigma^2 \pi^2 \frac{\partial^2}{\partial y^2}$$

Solution

The HJB equation leads to the following partial differential equation

$$\tilde{\mathcal{G}}_t - (r + \lambda_{x+t})\tilde{\mathcal{G}} + (ry - \Delta_1)\tilde{\mathcal{G}}_y + \\ + \frac{\gamma}{1-\gamma}\tilde{\mathcal{G}}_y^{(\gamma-1)/\gamma} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\tilde{\mathcal{G}}_y^2}{\tilde{\mathcal{G}}_{yy}} = 0$$

under the terminal condition

$$\tilde{\mathcal{G}}(y, T) = g(y, T; T) = \frac{(W_T - \Delta)^{1-\gamma}}{1-\gamma}$$

Solution

Motivated by the contributions proposed by Koo (1998) and Kingston & Thorp (2005), we consider the following change of variable:

$$\widehat{\xi}(t) := \frac{\Delta_1}{r}\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)}$$

$$\xi(t) := y - \widehat{\xi}(t)$$

that lead to write

$$y = \xi + \widehat{\xi} = \xi + \frac{\Delta_1}{r}\left(1 - e^{r(t-T)}\right) + \Delta\, e^{r(t-T)}$$

so then

$$g(y, T; T) = \frac{1}{1-\gamma}\xi^{1-\gamma}(T)$$

Solution

Under this change of variable we guess a solution of the following form:

$$\widetilde{\mathcal{G}}(y,t) = \frac{1}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma}(t)$$

for which, taking derivatives:

$$\widetilde{\mathcal{G}}_t = \xi^{-\gamma}\left(\Delta_1 e^{r(t-T)} - r\Delta e^{r(t-T)}\right)\alpha^{\gamma} + \frac{\gamma}{1-\gamma}\xi^{1-\gamma}\alpha^{\gamma-1}\alpha'$$

$$\widetilde{\mathcal{G}}_y = \xi^{-\gamma}\alpha^{\gamma}$$

$$\widetilde{\mathcal{G}}_{yy} = -\gamma\xi^{-\gamma-1}\alpha^{\gamma}$$

Solution

Plugging into the PDE, it finally lead to the following ordinary differential equation:

$$\alpha'(t) + \left[\frac{(1-\gamma)\delta - (r + \lambda^S_{x+t})}{\gamma}\right]\alpha(t) = -1$$

Under the boundary condition the solution of the previous ordinary differential equation is

$$\psi(t) = e^{-b(T-t)} \cdot \left({}_{T-t}p^S_{x+t}\right)^{1/\gamma} + \int_t^T e^{-b(s-t)} \cdot \left({}_{s-t}p^S_{x+t}\right)^{1/\gamma} \mathrm{d}s \qquad (1)$$

therefore, value function $\mathcal{G}$ is given by the following expression:

$$\mathcal{G}(w_0, t_0) = \frac{1}{1-\gamma}\left[y - \widehat{\xi}(t_0)\right]^{1-\gamma} \cdot \psi^\gamma(t_0)$$

Solution

For well posed solutions, we need:

- $y - \widehat{\xi}(t) > 0$, that is $\widehat{\xi}(t) \leqslant 0$, for every $t_0 \leqslant t < T$. It is straightforward to see that this condition is assured by the assumptions on Δ and Δ_1, i.e.: $\Delta < 0$ and $\Delta_1 > 0$.
- By preliminary assumptions, i.e. $r > (1-\gamma)\delta$, we also have:

$$|\psi(t)| \leqslant \left| e^{-b(T-t)} \cdot \left({}_{T-t}p^S_{x+t} \right)^{1/\gamma} \right| + \left| \int_t^T e^{-b(s-t)} \cdot \left({}_{s-t}p^S_{x+t} \right)^{1/\gamma} \mathrm{d}s \right| < +\infty$$

coherently with Karatzas et al. (1986).

Explicit form for value functions U and V

We assume an instantaneous utility function of the form:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1$$

We are able to find a closed form for the value function U and the value function V:

$$U(w_T + A, T) = \frac{1}{1-\gamma}(w_T + A)^{1-\gamma} \cdot \varphi^\gamma(T)$$

$$V(w_T, T) = \frac{1}{1-\gamma}\left(w_T + \frac{H}{r}\right)^{1-\gamma} \cdot \varphi^\gamma(T)$$

where $\varphi(T) = \int_T^{+\infty} e^{-b(s-t)} \cdot \left({}_{s-T}p^S_{x+T}\right)^{1/\gamma} \mathrm{d}s$, $b := -[(1-\gamma)\delta - r]/\gamma$

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Inequality relations between U and V

For every $\gamma > 0$, $\gamma \neq 1$, we have

$$\boxed{\max\left\{U(w_T + A, T);\; V(w_T, T)\right\} = \begin{cases} U(w_T + A, T), & \text{if } r \geqslant h \\ V(w_T, T), & \text{if } r < h \end{cases}}$$

Explicit form for value functions $\mathscr{U}$ and $\mathscr{V}$

Using the characterization above we arrive at:

$$\mathscr{V}(w_0, t_0) = \begin{cases} \mathscr{U}(w_0, t_0) & \text{if } r \geqslant h \\ \frac{1}{1-\gamma}\left(w_0 - \widehat{\xi}_{\mathscr{V}}(t_0)\right)^{1-\gamma} \varphi^{\gamma}(t_0) & \text{if } r < h \end{cases}$$

and we also find

$$\mathscr{U}(w_0, t_0) = \frac{1}{1-\gamma}\left(w_0 - \widehat{\xi}_{\mathscr{U}}(t_0)\right)^{1-\gamma} \varphi^{\gamma}(t_0)$$

where $\widehat{\xi}_{\mathscr{V}}$ and $\widehat{\xi}_{\mathscr{U}}$ are the transformations associated to functions $\mathscr{U}$ and $\mathscr{V}$.

The shape of value function $\mathscr{U}$ and $\mathscr{V}$

Dependence of $\mathscr{U}$ and $\mathscr{V}$ on the initial wealth w_0

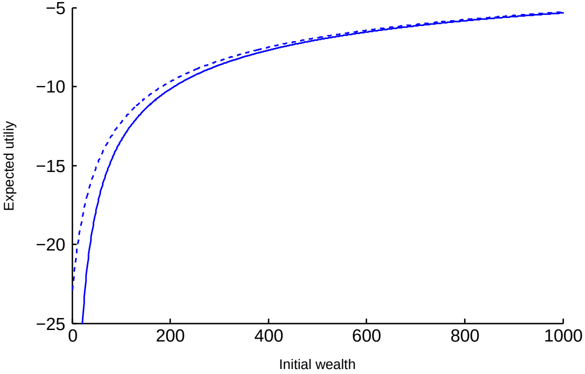

Figure: Value functions $\mathscr{U}$ (solid) and $\mathscr{V}$ (dashed), $\gamma = 1.4$, $r = 0.07$, $\mu = 0.08$, $\sigma = 0.12$, $A := \$350,000$, $t_0 = 0$ and $T = 35$, $h := 1/9$.

The indifference valuation at time t_0

For a given initial wealth w_0, let

$$L_0^* := \sup\left\{L_0 : \mathscr{U}(w_0, t_0) \leqslant \mathscr{V}(w_0 - L_0, t_0),\ w_0 - L_0 > 0\right\}$$

is the indifference price for the guaranteed annuity option, if the following equality holds

$$\mathscr{U}(w_0, t_0) = \mathscr{V}(w_0 - L_0^*, t_0)$$

Otherwise L_0^* is the maximum sum that the agent is willing to pay in order to embed the guaranteed annuity option in her policy. If the indifference price exists, it is given by

$$\boxed{L_0^* = \left(\frac{H}{r} - A\right) e^{-r(T-t0)}}$$

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The indifference valuation at time T

For a given wealth at time T of conversion w_T, let

$$L_T^* := \sup\left\{L_T : U(w_T + A, T) \leqslant V(w_T - L_T, T),\ w_T - L_T > 0\right\}$$

is the indifference price for the guaranteed annuity option, if the following equality holds

$$U(w_T + A, T) = V(w_T - L_T^*, T)$$

If at T, an indifference price exists, it can be written as follows:

$$\boxed{L_T^* = \frac{H}{r} - A}$$

If both indifference prices L_0^* and L_T^* exist, we have:

$$\boxed{L_0^* = e^{-r(T-t_0)} \cdot L_T^*}$$

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Numerical experiment. Implementation

Assumptions:

- We consider a policy written for the period 1970-2005.
- The annuitant is a female aged 40, from Ontario, Canada.
- $t_0 := 0$ and $T := 35$ (where we may think that T coincides with the time of retirement).
- Accumulated funds: $A := \$350,000$.
- Guaranteed annuity option (G.A.O.): $h := 1/9$. Thus we obtain: $H \approx 38.89$ and $r_h \approx 0.0754$.
- The parameters μ and σ, describing the financial market, are consistent with the following assumptions on r.

Numerical experiment

Table: Premium and indifference valuation associated to the policy, depending on the current interest rate.

r	P	L_0^*	p_{12}	l_{12}	Total
0.035	\$6,594	\$266,342	\$550	\$419	\$969
0.050	\$5,026	\$ 95,450	\$420	\$115	\$535
0.085	\$2,519	\$ 8,395	\$211	\$ 5	\$216

Remark. L_0^* represents a lump sum referring to a period of 35 years. P is the instantaneous rate necessary to accumulate funds A. Therefore, P and L_0^* do not represent homogeneous quantities.

Numerical experiment

In order to give an intuitive economic meaning to P and L_0^*, we use an (independent) auxiliary discretization: We convert P and L_0^* with respect to a monthly annuity from t_0 to T. We define l_{12} and p_{12} according to:

$$L_0^* = l_{12} \cdot a_{\overline{T \times 12}|i_{12}} \qquad A = p_{12} \cdot s_{\overline{T \times 12}|i_{12}}$$

where $i_{12} := e^{r/12} - 1$ is the monthly interest rate equivalent to the instantaneous rate r and where

$$a_{\overline{n}|i} := \frac{1 - (1+i)^{-n}}{i} \qquad s_{\overline{n}|i} := (1+i)^n \cdot a_{\overline{n}|i}$$

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Remark. By construction p_{12} and l_{12} are homogeneous quantities.

Remark. It is interesting to see the case $r = 0.085$, coherent with prevalent rates observed in 1970s.

The indifference price L_0^*

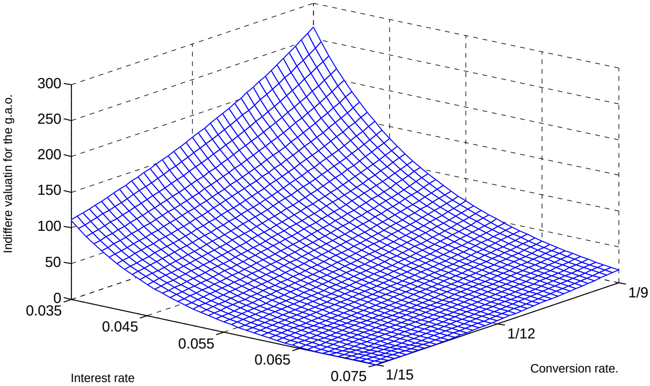

Figure: Indifference price L_0^* depending on the guaranteed conversion rate h and the market interest rate r.

The impact of the force of mortality

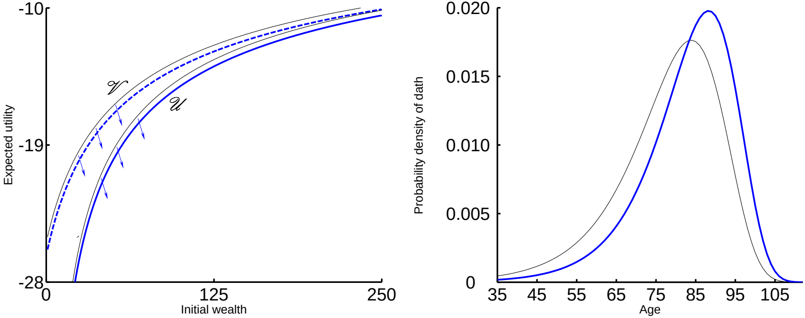

Figure: Impact of the force of mortality on value functions $\mathscr{U}$ and $\mathscr{V}$

Extension: Labor income

The results can be proved assuming constant labor income, payed at a nominal instantaneous rate $0 < \Lambda < P$. For instance, the wealth dynamics, between time t_0 and time T is

$$\begin{aligned} \mathrm{d}W_s &= r(W_s - \pi_s) + \pi_s \,\mathrm{d}S_s + \Lambda - (P + c_s)\,\mathrm{d}s \\ &= [rW_s + (\mu - r)\pi_s + \Lambda - (P + c_s)]\,\mathrm{d}s + \sigma \pi_s \,\mathrm{d}B_s \end{aligned}$$

under the initial condition $W_{t_0} = w_0 > 0$.

Since, this new variable is not associated with any randomness, as expected we find that it does not influence the indifference valuation of the implicit guaranteed annuity option.

Why an indifference model?

Why an indifference model with *deterministic* interest rate, volatility, and force of mortality?

Future research

The model can be generalized in several ways:

- The model characterize the exercise of the option assuming deterministic interest rates.
- Changes in mortality are considered by comparing different longevity scenarios.
- From the policyholder's point of view, an unrestricted market can be considered, where she can annuitize anything anytime.
- Policyholder's strategies can contemplate stochastic labor income.

Thank You!