Multi-Objective and Robust Optimization in Finance and Risk Management

Oleksandr Romanko

Quantitative Research, Algorithmics Inc.
Dept. of Computing and Software, McMaster University

Joint work with **Helmut Mausser**

Industrial-Academic Workshop on Optimization in Finance and Risk Management, October 3, 2011

Outline

- **Multi-objective optimization** – simultaneously optimizing two or more conflicting objectives subject to certain constraints
  - introduction
  - solution techniques
  - computing efficient frontiers
- **Portfolio selection** – selecting optimal portfolios based on multiple criteria
  - computing 2D efficient frontiers
  - computing 3D efficient surfaces
- **Robust multi-objective portfolio selection** – robustness as one of the objectives
  - box uncertainty sets
  - ellipsoidal uncertainty sets
- **Risk management** – market-credit risk optimization
  - Value-at-Risk – Conditional Value-at-Risk efficient frontiers
- **Conclusions and future work**

Algorithmics

& Multi-Objective Optimization

Algorithmics

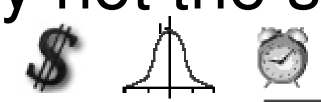
Multi-Objective Optimization

- **Multi-objective optimization**: simultaneously optimizing two or more conflicting objectives subject to certain constraints

$$\begin{array}{rl} \text{minimize} & \{f_1(x), f_2(x), \ldots, f_k(x)\} \\ \text{subject to} & x \in \Omega \end{array}$$

- Examples:
  - Finance: Minimize **risk** & Maximize **return**
  - Business: Minimize **cost** & Minimize **environmental impact**

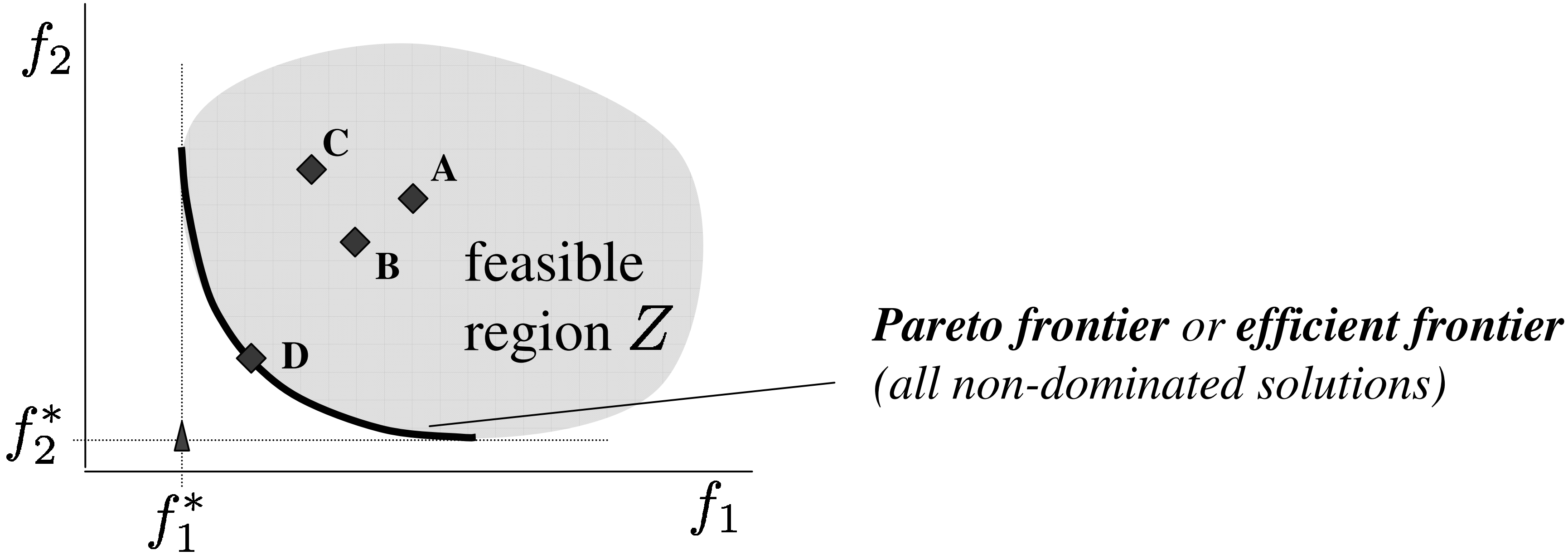
- Units of the objectives are typically not the same:
  dollars, probability, units of time, …



Algorithmics

Bi-Objective Example

- $\min\ f_1 = \text{risk},\ \min\ f_2 = \text{loss}:$



- Pareto efficiency: solutions with characteristics like **D**, are called tradeoff, Pareto optimal or non-dominated
- **Multi-objective optimization goal**: find solution(s) on the efficient frontier according to the decision maker preferences

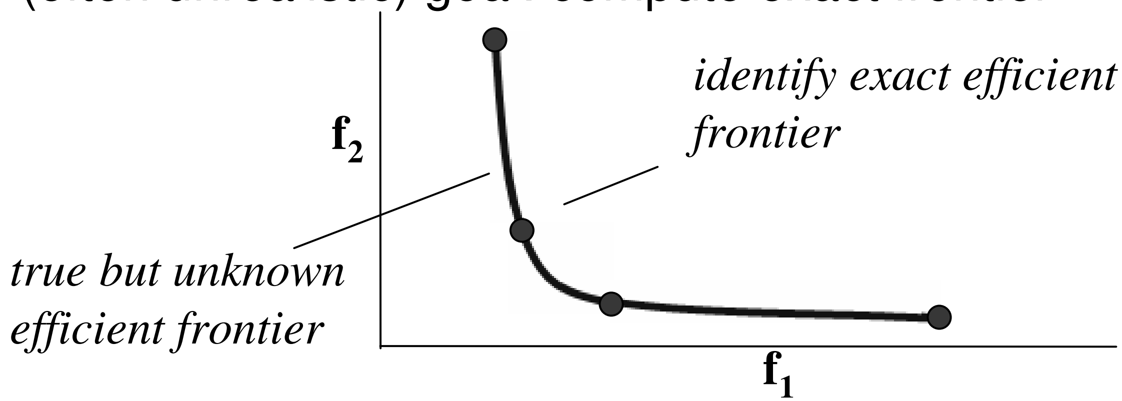
Algorithmics

Computing Efficient Frontiers - Possibilities

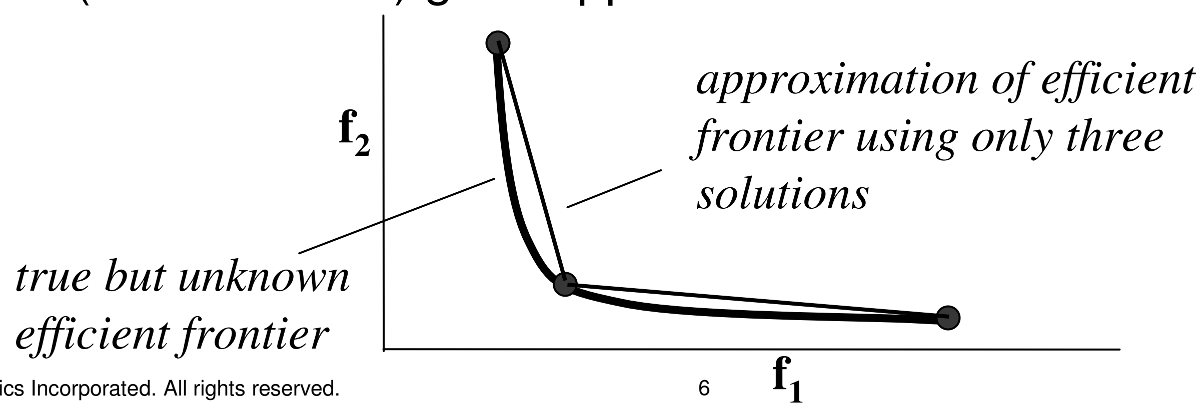
Multi-objective analysis involves **computing the efficient frontier**, evaluating it (if possible, out-of-sample) and selecting the final solution based on the decision maker preferences

Computing efficient frontiers:

- Ideal (often unrealistic) goal: compute exact frontier



- Typical (more realistic) goal: approximate the frontier



Algorithmics

Solving Multi-Objective Optimization Problems

- **<u>Convert multi-objective optimization problem to a series of single-objective optimization problems</u>**

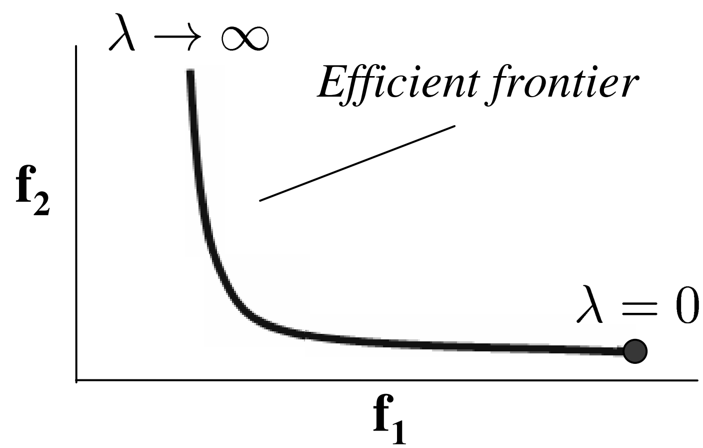
- Methods:
  - Weighting Method
  - ε-Constraint (Hierarchical) Method

Weighting Method

- Assign weights to each objective
- Optimize the weighted sum of the objectives
- Multi-objective optimization with weighting method:

$$\begin{array}{rl} \min & w_1 \cdot f_1(x) + w_2 \cdot f_2(x) \\ \text{s.t} & x \in \Omega \end{array}$$

f_i is linear or convex quadratic,
$\Omega \subseteq \mathbb{R}^n$ (convex),
$w_i \in \mathbb{R}$ is the weight of the i-th objective,
$w_i \geq 0$, $i = 1, 2$ and $w_1 + w_2 = 1$



- Easier formulation: $\lambda = \dfrac{w_1}{w_2}$

$$\begin{array}{rl} \min & \lambda f_1(x) + f_2(x) \\ \text{s.t} & x \in \Omega \end{array}$$

ε-Constrained Method

- Optimize one objective
- Convert other objectives into constraints

- Multi-objective optimization with ε-constrained method:

First step

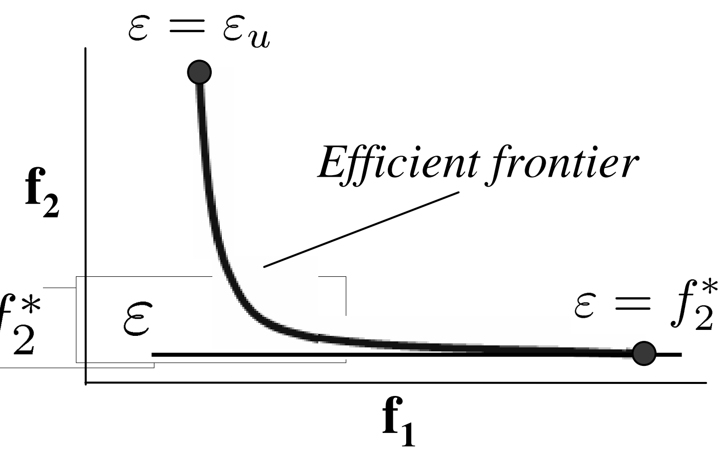
$$\begin{aligned} \min \quad & f_2(x) \\ \text{s.t} \quad & x \in \Omega \end{aligned}$$

$$\Downarrow$$

$$f_2^*$$

Second step

$$\begin{aligned} \min \quad & f_1(x) \\ \text{s.t} \quad & x \in \Omega \\ & f_2(x) \leq (1+\epsilon) f_2^* \end{aligned}$$



Multi-Objective Optimization Examples

- Multi-objective optimization in finance and risk management:

$$\min \quad w_1 \cdot (-\text{performance measure}) + w_2 \cdot (\text{risk measure})$$

$$\min \quad w_1 \cdot (\text{risk measure 1}) + w_2 \cdot (\text{risk measure 2})$$

$$\min \quad \text{risk measure}$$

$$\text{s.t.} \quad \text{performance measure} \geq \varepsilon$$

$$+ w_2 \cdot (\text{perform. measure 2})$$

- Firm's performance measures: profits, sales, stock price, growth, liquidity, market share, …
- Bank's performance measures: return, profit, liquidity, tracking error, …
- Risk measures:
  - variance
  - Value-at-Risk (VaR)
  - Conditional Value-at-Risk (CVaR)
- Robustness as model's performance measure?

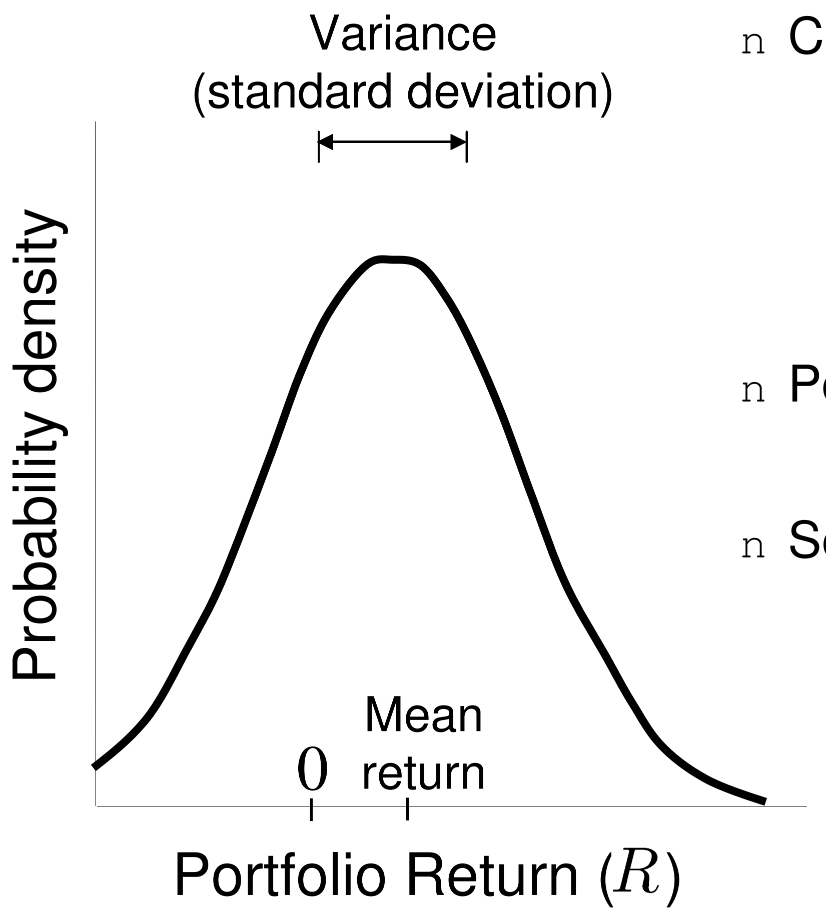
Algorithmics

& Portfolio Selection

Algorithmics

Market Risk Measures and Portfolio Selection

Portfolio return distribution (F_R) is assumed to be Gaussian (Normal)



- Consider n assets with random returns:
  - $x_i,\ i = 1, ..., n$ proportion invested in asset i
  - μ_i, σ_i exp. return and standard dev. of the return of asset i
  - $Q_{ij} = \rho_{ij}\sigma_i\sigma_j$ variance-covariance matrix
- Portfolio exp. return and variance:

  $$\mathbb{E}[x] = \mu^T x \qquad \mathrm{Var}[x] = x^T Q x$$

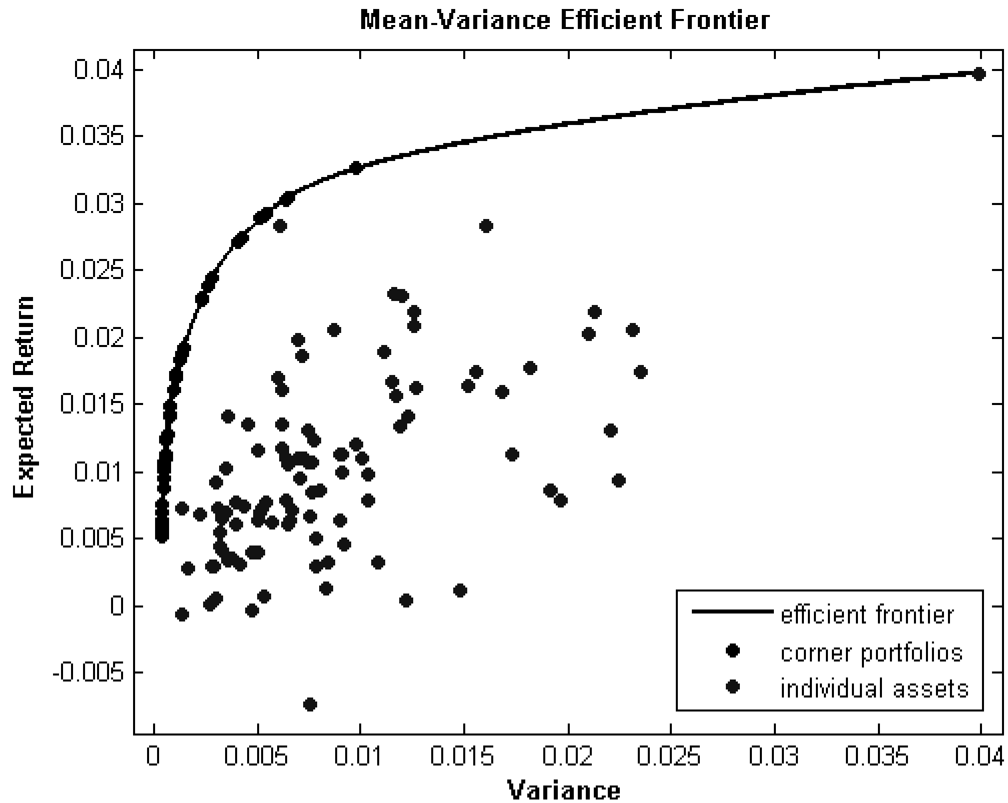
- Set of admissible portfolios:

  $$\mathcal{F} = \{x : \textstyle\sum_i x_i = 1,\ x \geq 0\}$$

Portfolio Selection

- **Mean-variance (Markowitz, 1952) portfolio optimization problem with market risk** – two objectives:

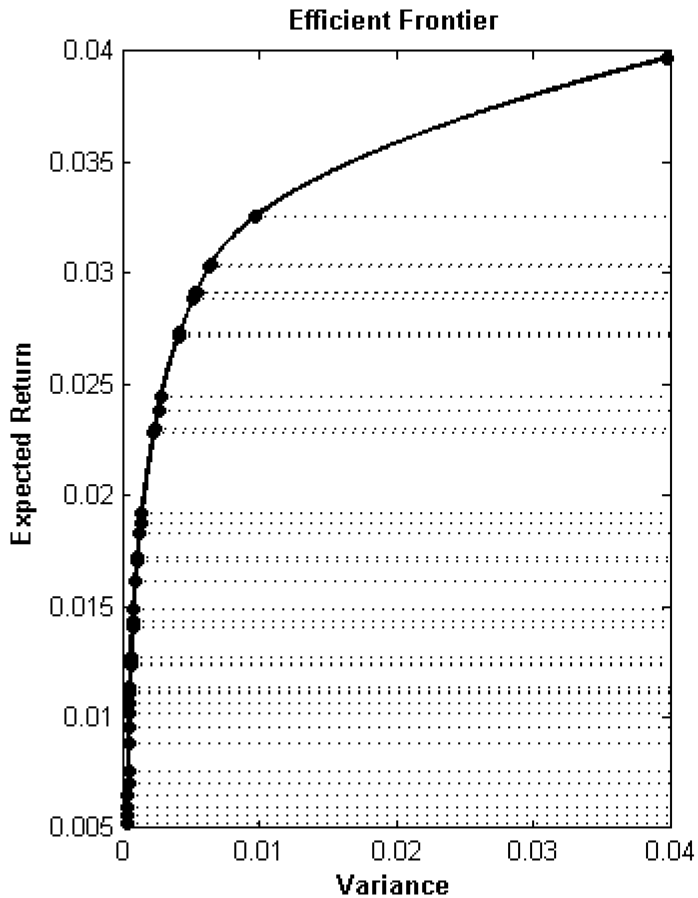
$$\begin{aligned} \min_x \quad & x^T Q x \\ \text{s.t} \quad & \mu^T x \geq \varepsilon \\ & \textstyle\sum_i x_i = 1 \\ & x \geq 0 \end{aligned}$$

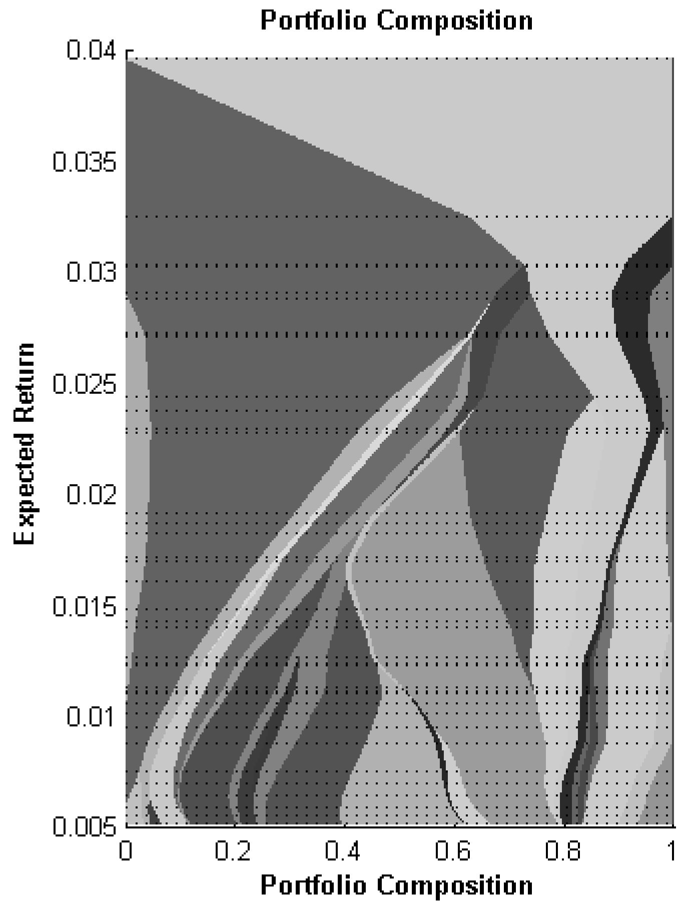


Extensions of mean-variance model: introduce transaction costs

Portfolio Selection

- **Mean-variance portfolio optimization problem with market risk** – efficient frontier and portfolio composition:



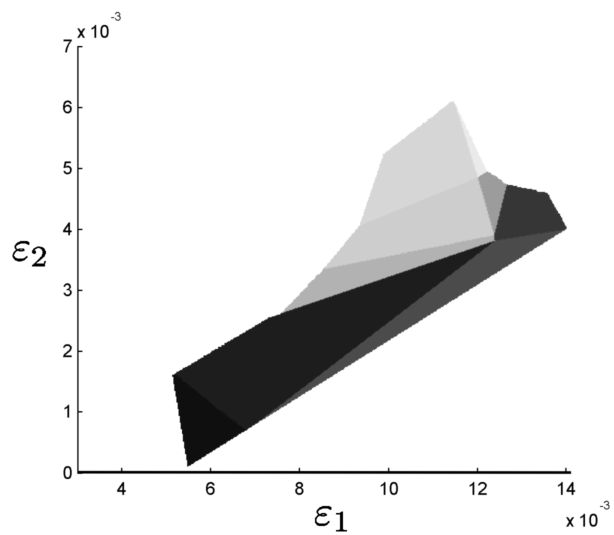


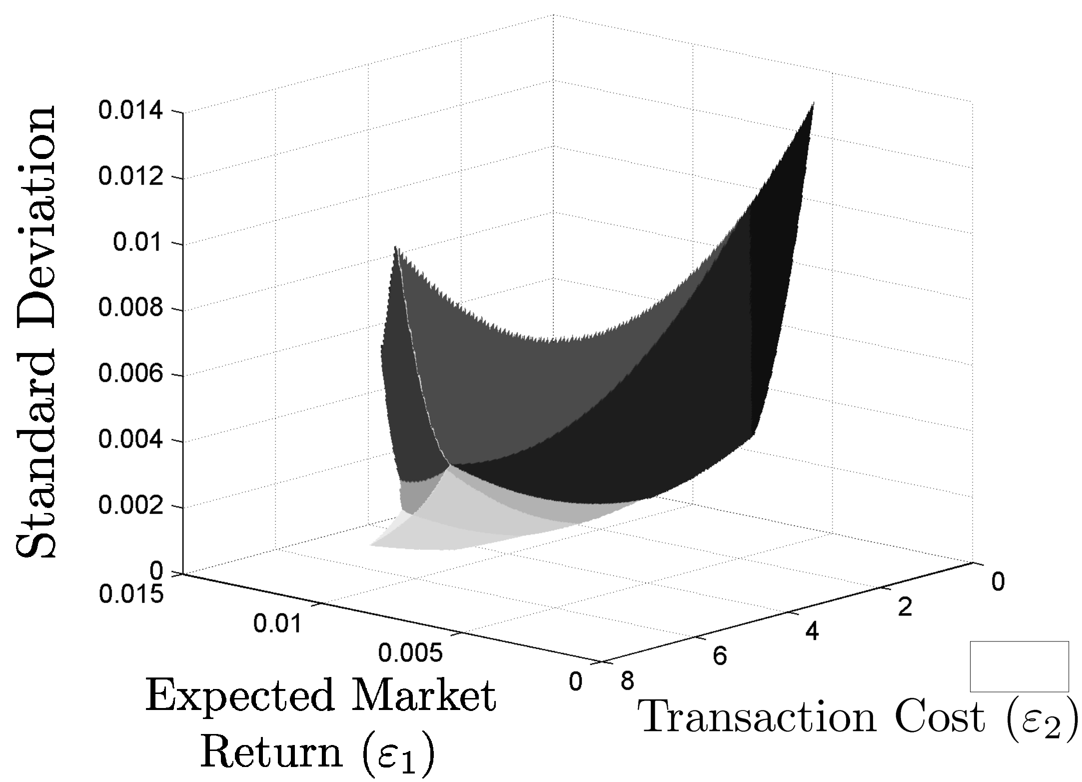
Algorithmics

Portfolio Selection

- **Mean-variance portfolio optimization problem with market risk and transaction cost** – three objectives:

$$\begin{aligned} \min_x \quad & x^T Q x \\ \text{s.t} \quad & \mu^T x \geq \varepsilon_1 \\ & c^T x \leq \varepsilon_2 \\ & \textstyle\sum_i x_i = 1 \\ & x \geq 0 \end{aligned}$$





Algorithmics

& Robust Portfolio Selection

Algorithmics

Robust Optimization 1

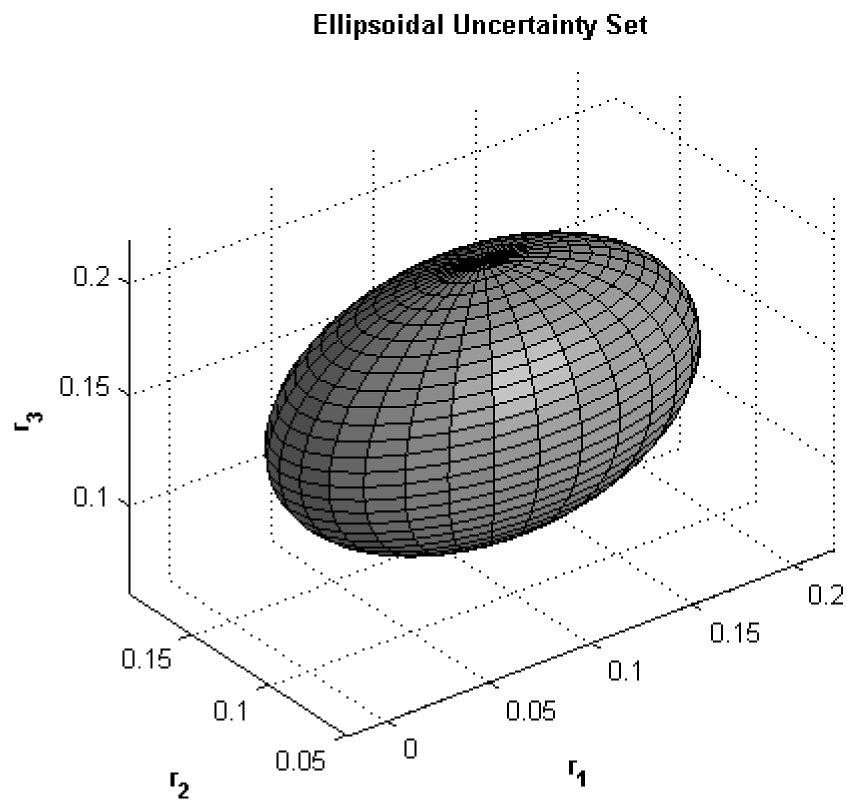
- The vector of true expected returns r lies in the ellipsoidal uncertainty set:

$$r \in \mathcal{U}(\mu)_\delta = \{r \,:\, (r-\mu)^T \Theta^{-1} (r-\mu) \le \delta^2\}$$

- Robust portfolio optimization:

$$\begin{array}{ll} \min & -r^T x + \lambda x^T Q x \\ \text{s.t.} & \sum_i x_i = 1 \\ & \quad\quad x \ge 0 \\ & \quad \forall\, r \in \mathcal{U}(\mu)_\delta \end{array}$$

- Ellipsoidal uncertainty set:



Robust Optimization 2

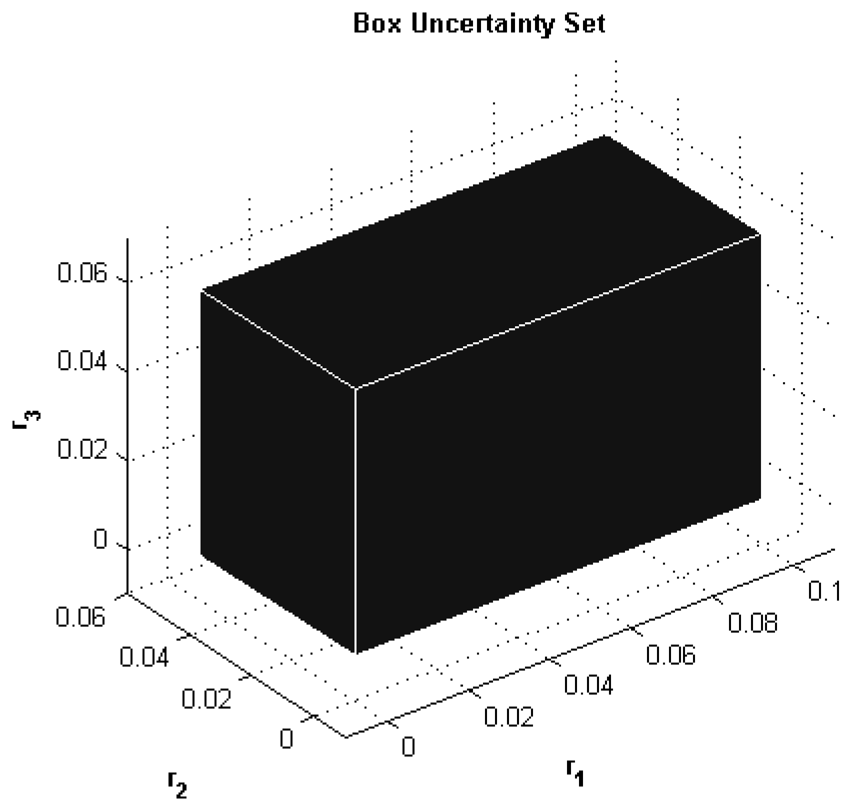
- The vector of true expected returns r lies in the box uncertainty set:

$$r \in \mathcal{U}(\mu)_\delta = \{r \,:\, |r_i - \mu_i| \leq \delta_i,\ i = 1, \ldots, n\}$$

- Robust portfolio optimization:

$$\begin{array}{ll} \min & -r^T x + \lambda x^T Q x \\ \text{s.t.} & \sum_i x_i = 1 \\ & \quad x \geq 0 \\ & \forall\, r \in \mathcal{U}(\mu)_\delta \end{array}$$

- Box uncertainty set:



Robust Optimization 1 and 2

- The vector of true expected returns r lies in the box uncertainty set or ellipsoidal uncertainty set :

$$r \in \mathcal{U}(\mu)_\delta = \{r \, : \, |r_i - \mu_i| \leq \delta_i, \; i = 1, \ldots, n\}$$

$$r \in \mathcal{U}(\mu)_\delta = \{r \, : \, (r - \mu)^T \Theta^{-1} (r - \mu) \leq \delta^2\}$$

- Objectives:
  - minimize variance of portfolio return $x^T Q x$
  - maximize portfolio expected return $\mu^T x$
  - minimize portfolio return estimation error
    - box uncertainty set: $\delta^T |x| = \|Dx\|_1, \; D = \mathrm{diag}(\delta_j)$
    - ellipsoidal uncertainty set: $\|\Theta^{1/2} x\| = \sqrt{x^T \Theta x}$

- Constraints:

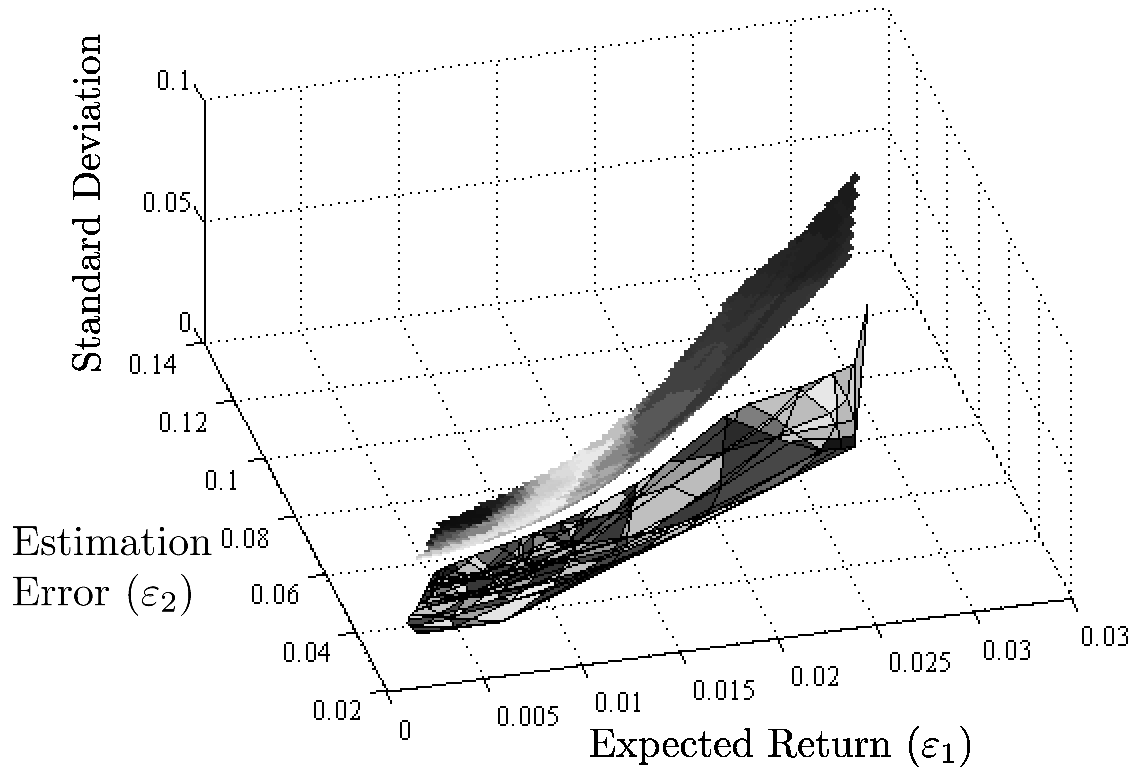
$$\begin{aligned} \sum_i x_i &= 1 \\ x &\geq 0 \end{aligned}$$

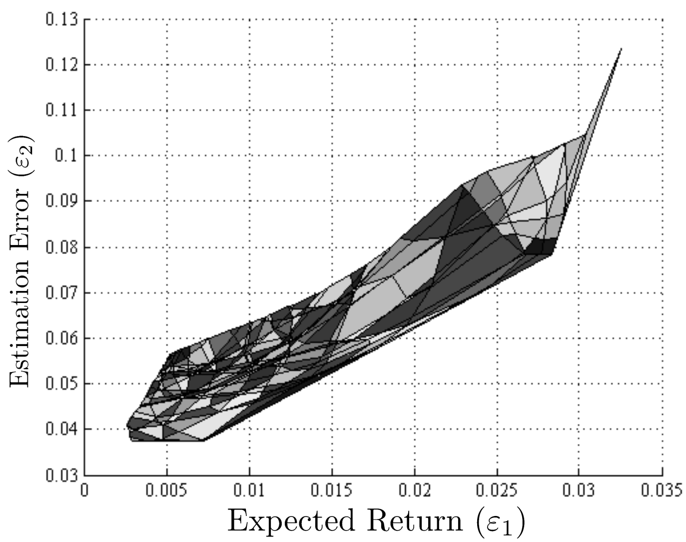
Algorithmics

Robust Optimization 2

Multi-objective robust optimization:

$$
\begin{aligned}
\min \quad & x^T Q x \\
\text{s.t.} \quad & \mu^T x \geq \varepsilon_1 \\
& \delta^T |x| \leq \varepsilon_2 \\
& \textstyle\sum_{i=1}^{100} x_i = 1 \\
& x \geq 0
\end{aligned}
$$



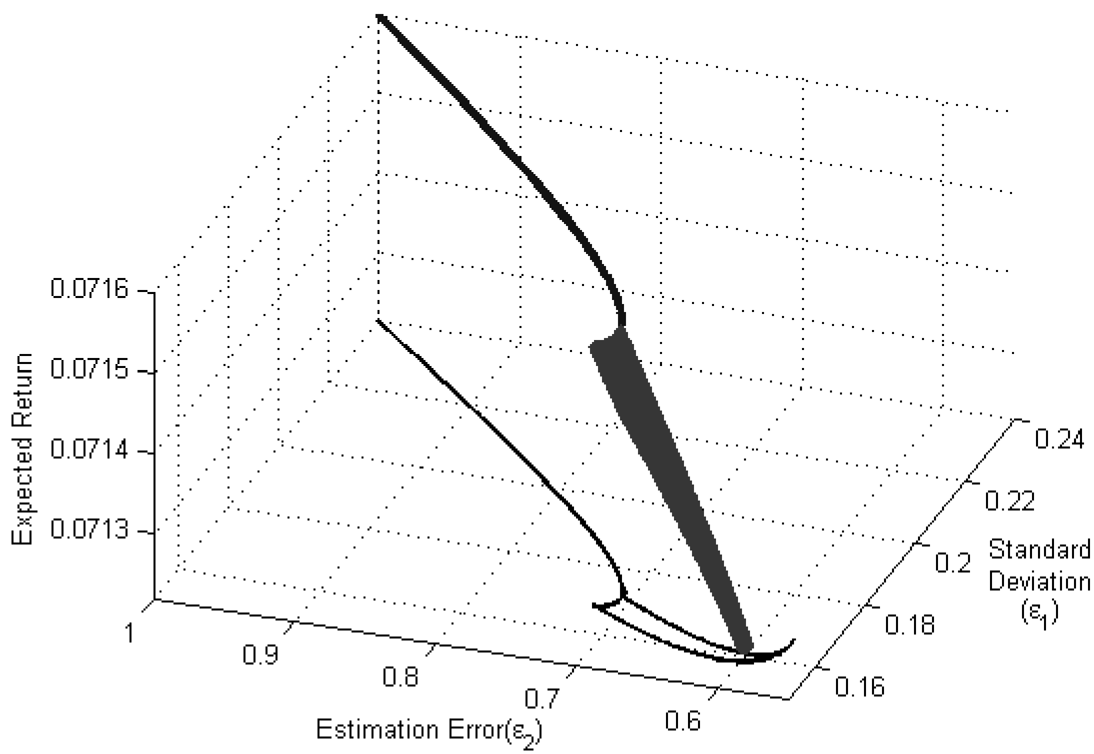


Robust Optimization 1

Multi-objective robust optimization:

$$\begin{array}{ll} \max & \mu^T x \\ \text{s.t.} & x^T Q x \leq \varepsilon_1 \\ & \|\Theta^{1/2} x\| \leq \varepsilon_2 \\ & \sum_{i=1}^{n} x_i = 1 \\ & x \geq 0 \end{array}$$

Robust portfolio optimization problem solution – **efficient surface**



& Risk Management

Algorithmics

Conditional Value-at-Risk Optimization

- Conditional Value-at-Risk optimization problem:

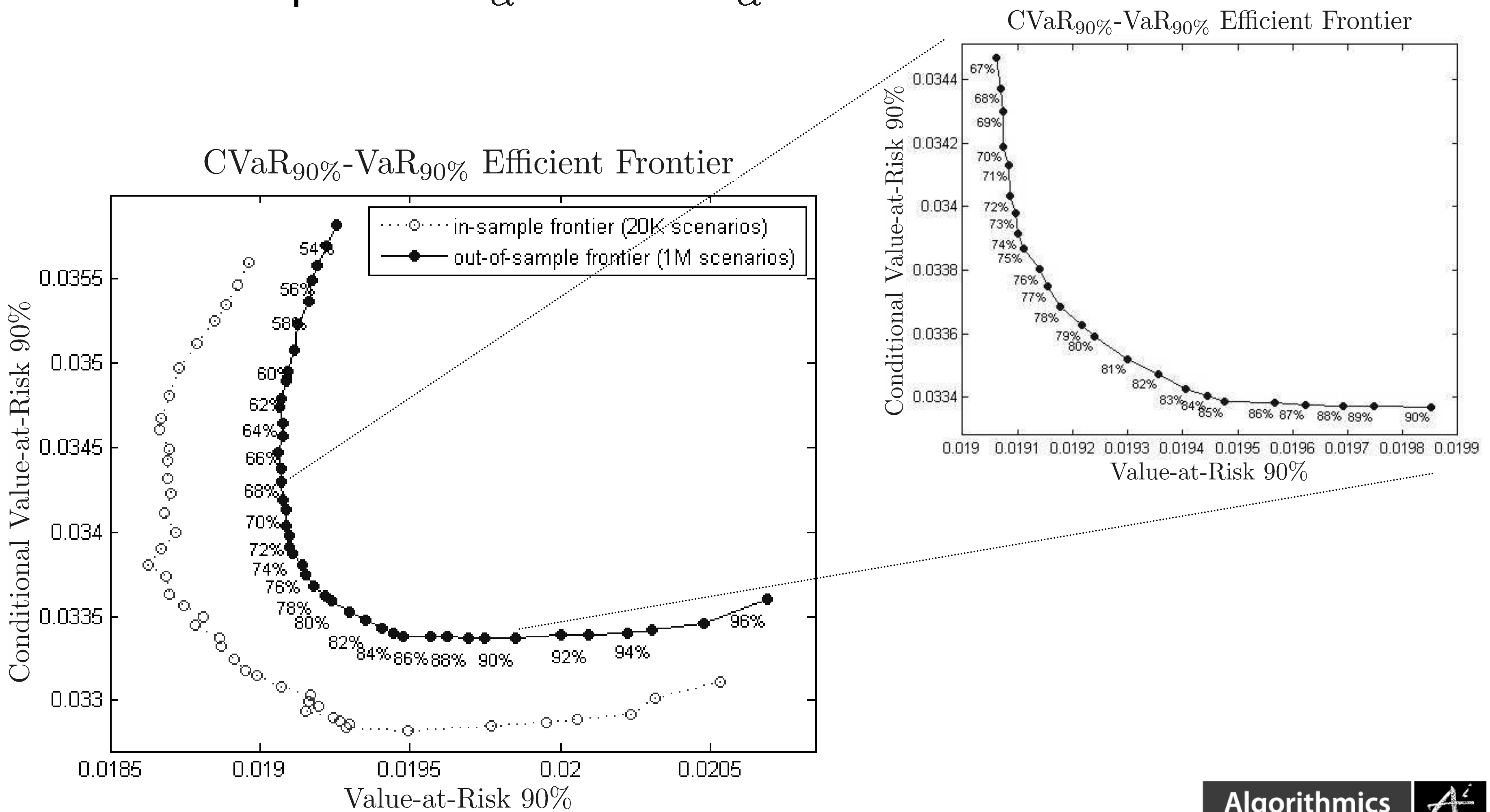
$$\begin{array}{ll} \boxed{\begin{array}{ll} \min\limits_{x,u,\ell} & \ell + \dfrac{1}{S(1-\alpha)} \displaystyle\sum_{s=1}^{S} u_s \\ \text{s.t.} & u_s \geq -\mu_s^T x - \ell,\ u_s \geq 0,\ s = 1,\ldots,S \end{array}} \\ \qquad\qquad\quad \sum_i x_i = 1 \\ \qquad\qquad\quad x \geq 0 \end{array}$$

$$\min_{x,\ell}\ \ell + \frac{1}{S(1-\alpha)} \sum_{s=1}^{S} \left[-\mu_s^T x - \ell\right]^+$$

- Solve CVaR optimization problem at different quantile levels α to compute $\mathrm{VaR}_{\alpha^*} - \mathrm{CVaR}_{\alpha^*}$ trade-off

VaR Optimization via CVaR Optimization

- Solve CVaR optimization problem at different quantile levels α to compute $\mathrm{VaR}_{\alpha^*} - \mathrm{CVaR}_{\alpha^*}$ trade-off



Algorithmics

& Conclusions

Conclusions

- Benefits of multi-objective optimization: a wider range of alternatives is identified, and models tend to be more realistic if more objectives considered
- Multi-objective optimization problems can be formulated as series of single-objective optimization problems and solved efficiently
- Many optimization problems in finance and risk management are multi-objective in their nature
- Efficient frontiers provide a decision make with the complete picture of choices and allow to identify trade-offs
- Robustness measure can be incorporated into multi-objective optimization problem as the additional objective

---

- Investigate Value-at-Risk – Conditional Value-at-Risk efficient frontiers
- Identify if other efficient frontiers in addition to the mean-variance trade-off are useful for practical applications

Algorithmics

Questions?

Algorithmics