Portfolio Selection under Model Uncertainty

A Penalized Moment-Based Optimization Approach

Industrial-Academic Workshop
on Optimization in Finance and Risk Management

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University of Toronto

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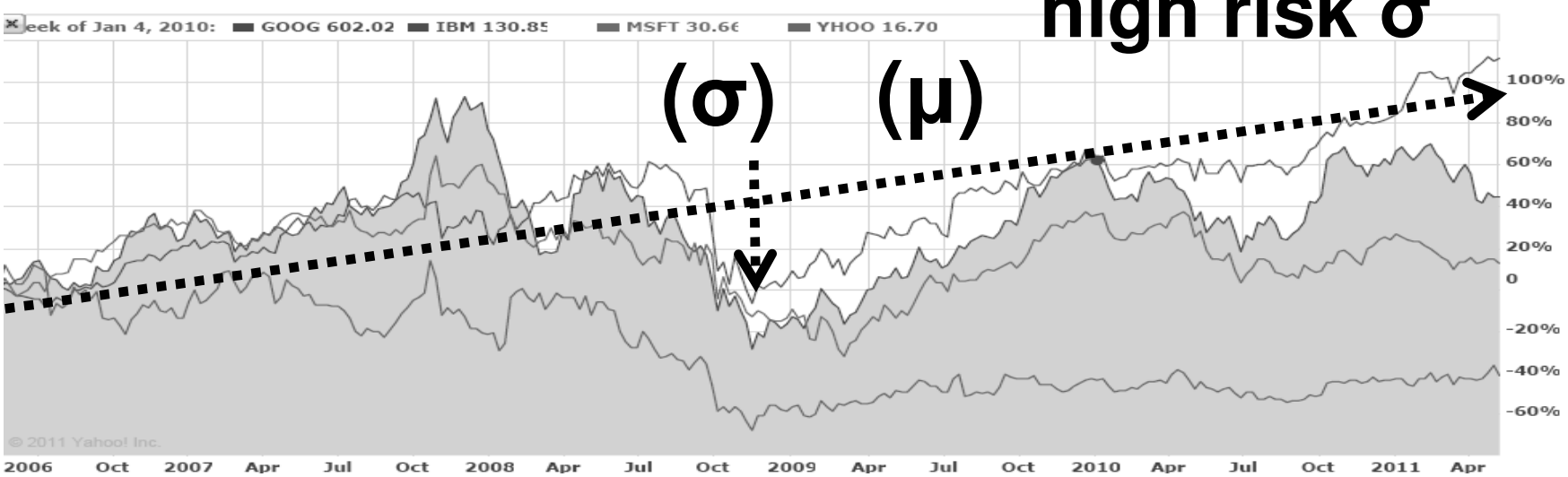
Modern portfolio Theory

Financial markets are volatile

high return μ
high risk σ



$$\max_x \mu^T x$$

$$s.t.\ \sigma := x^T \Sigma x \le b$$

$$e^T x = 1$$

- mean-variance model (Markowitz 1952)

Modern portfolio Theory

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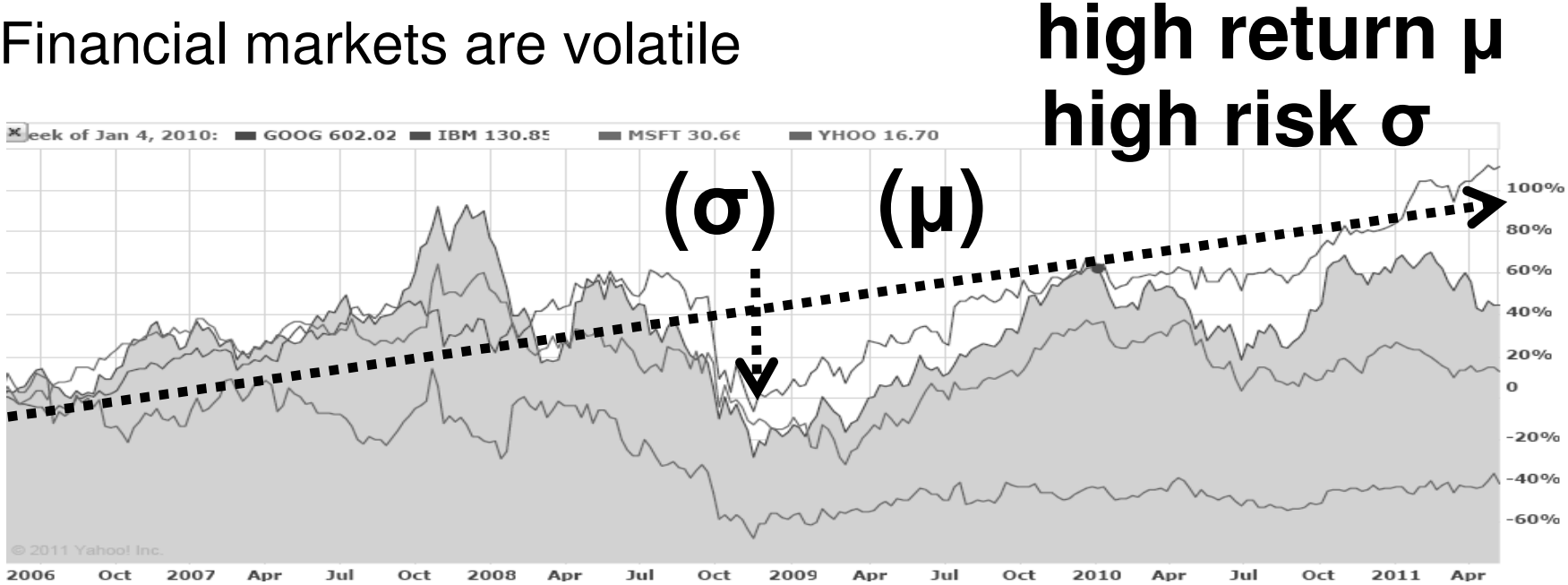

$$\max_x \mu^T x$$

$$s.t.\ \sigma := x^T \Sigma x \le b$$

$$e^T x = 1$$

- mean-variance model (Markowitz 1952)
- mean-absolute model (Konno 1998)
- mean-CVaR model (Rockafellar, Uryasev 2000)
- and many others…

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Practical Issue – unknown statistical model

In most existing models, it is assumed that

- return distributions are known, e.g. *Normal* distribution
- statistics, e.g. mean, variance, can be accurately estimated

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- statistics, e.g. mean, variance, can be accurately estimated

These assumptions fail in practice and could lead to catastrophic losses.

Issue #1: unknown distribution

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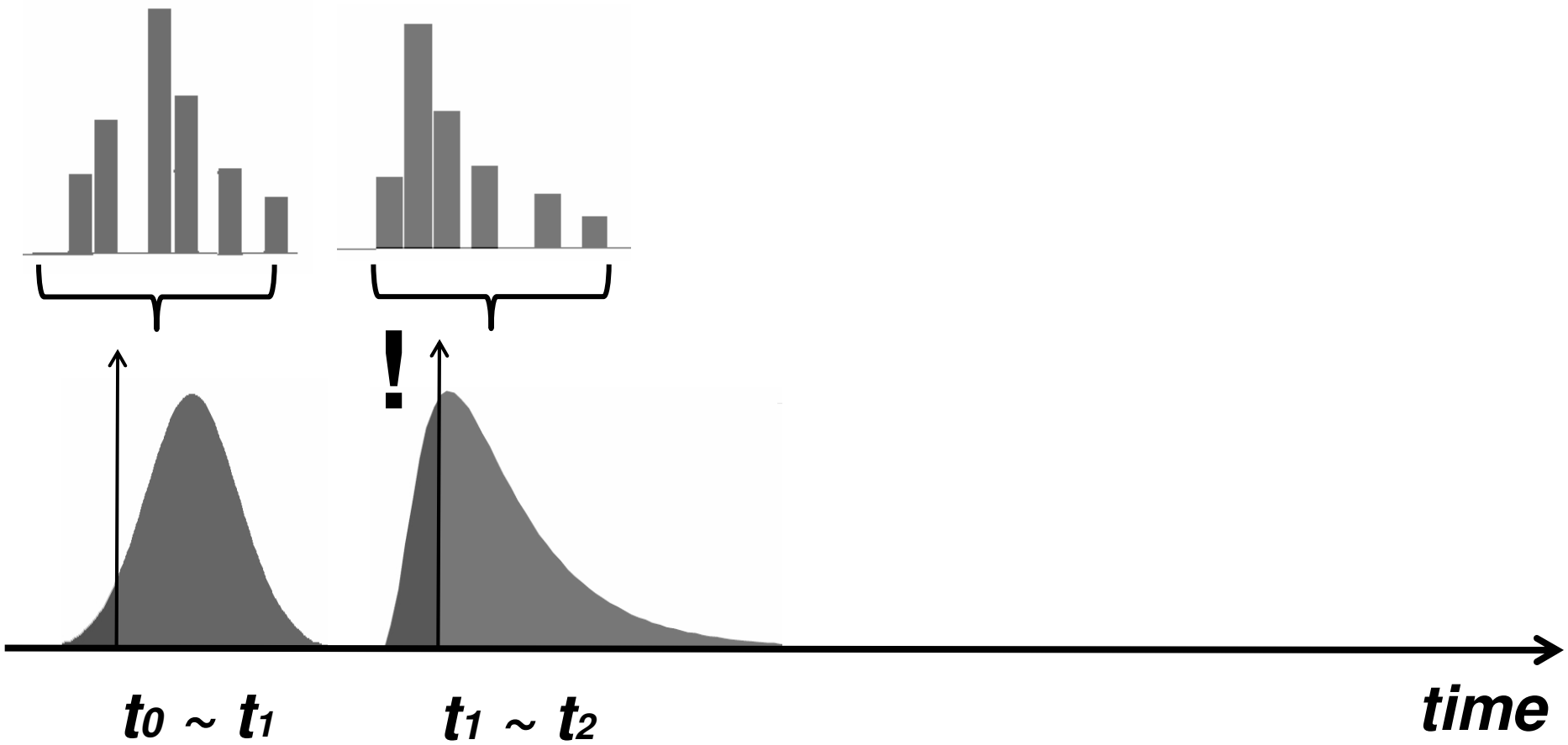
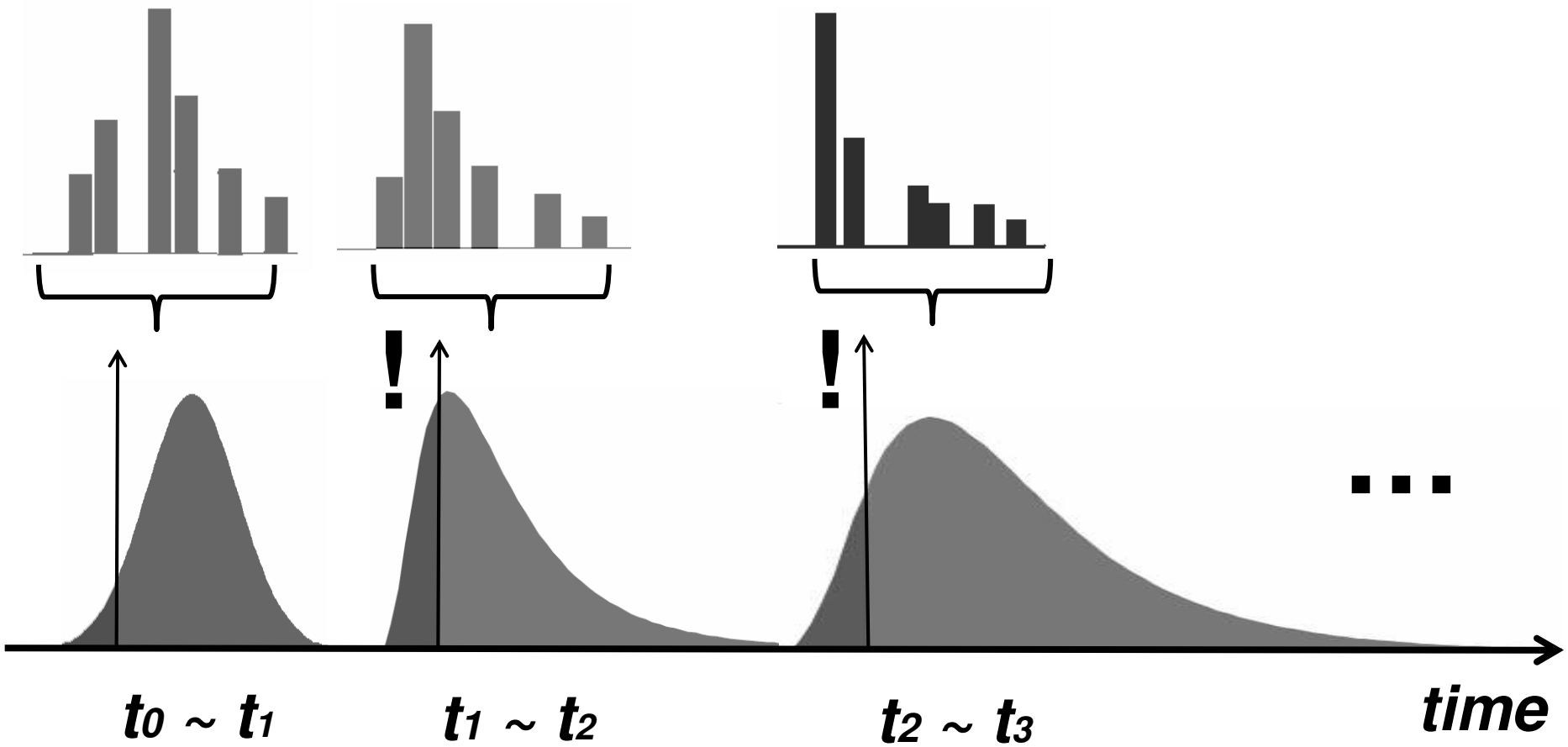
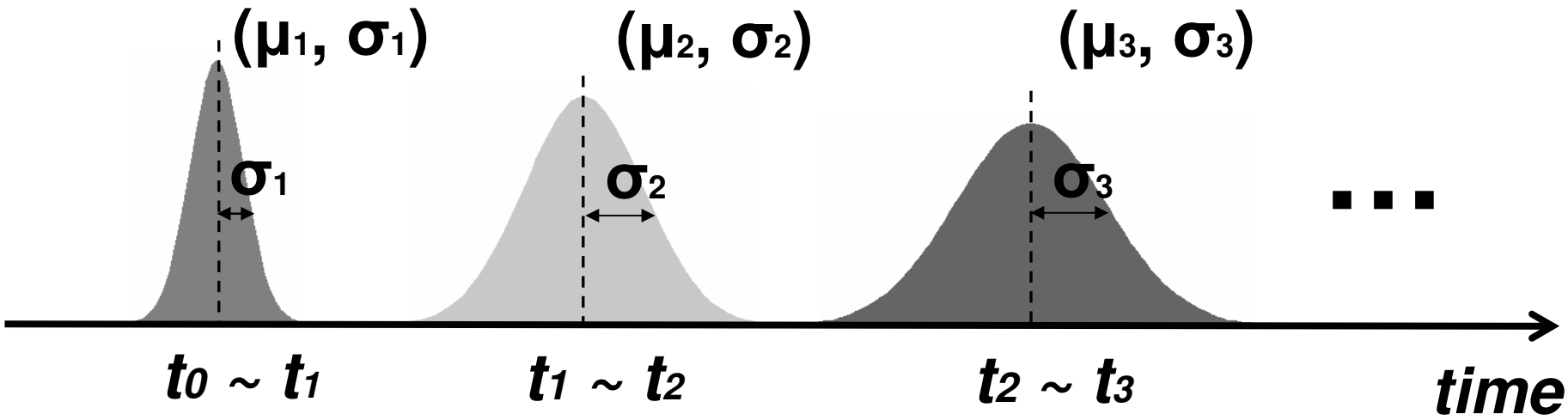


Issue #1: unknown distribution

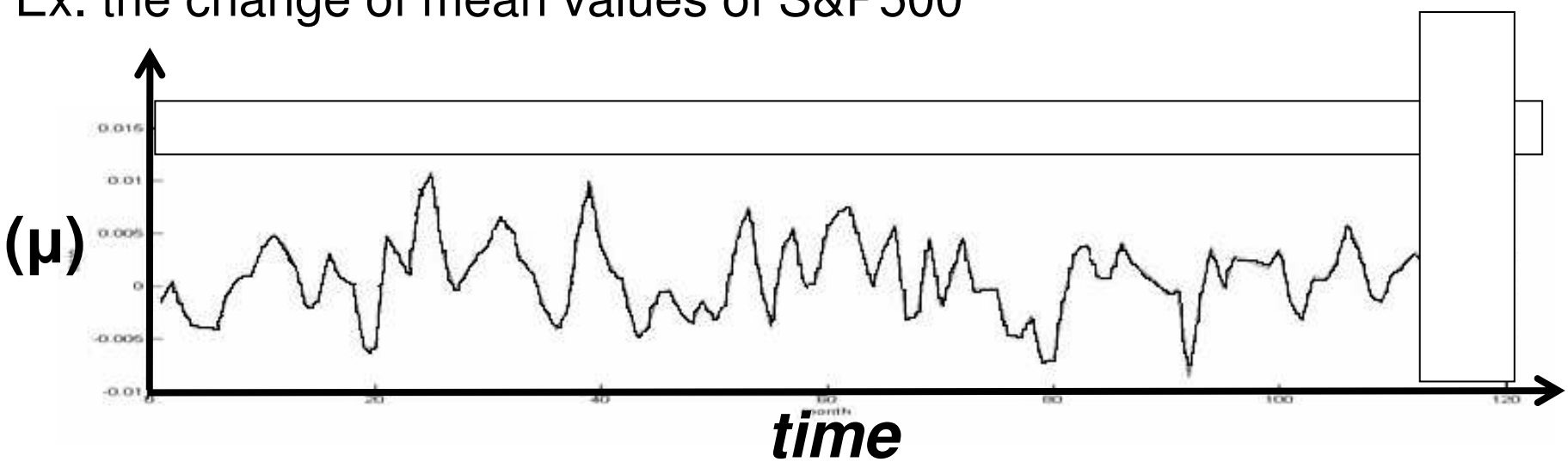


Even distributions with the same mean and variance,
the unexpected loss can be huge

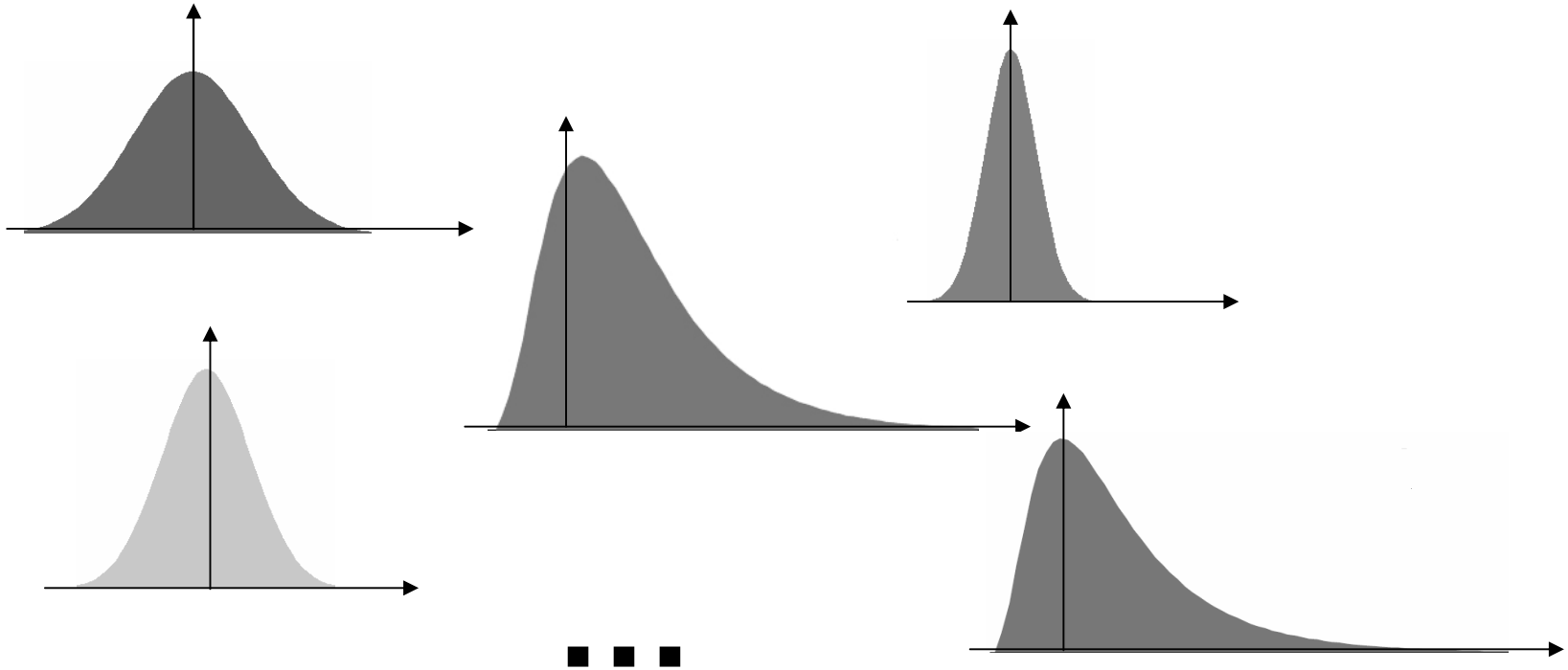
Issue #2: inaccurate estimation (limited data)



Ex. the change of mean values of S&P500



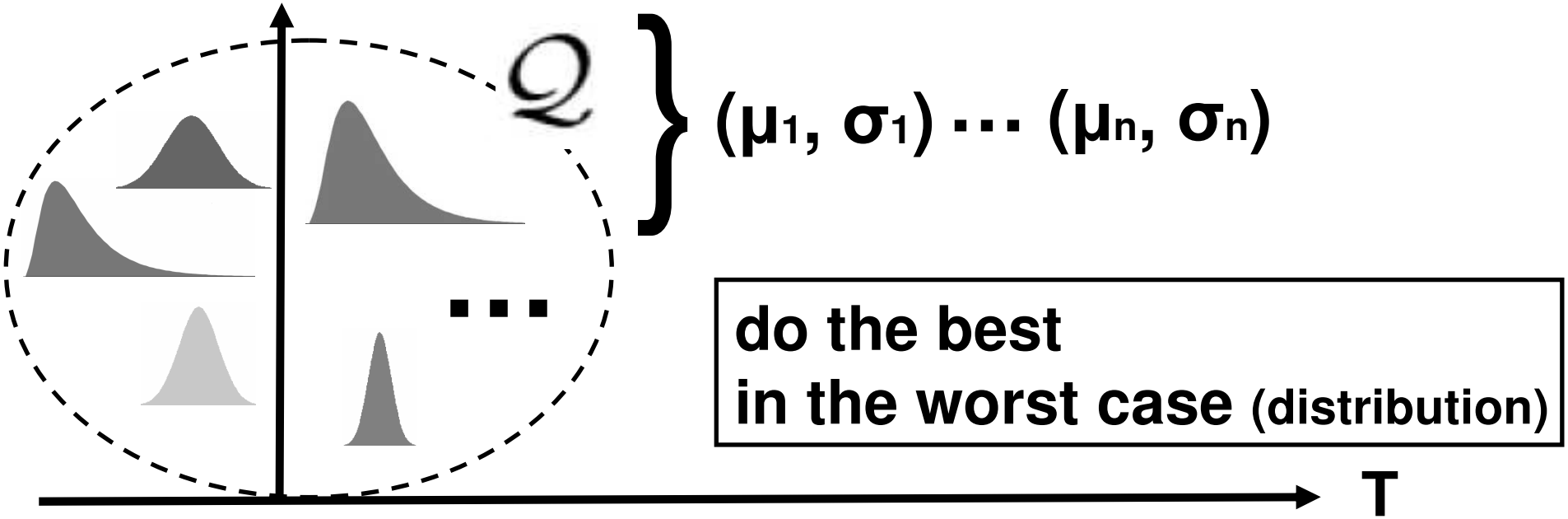
In a word: Model Uncertainty (Risk)



How to **protect** investors from model risk?

Modern Solution: Distributionally Robust Optimization

Consider all possible distributions at once



Idea: optimize portfolio in the worst case so that the optimal solution is robust

Modern Solution: Distributionally Robust Optimization

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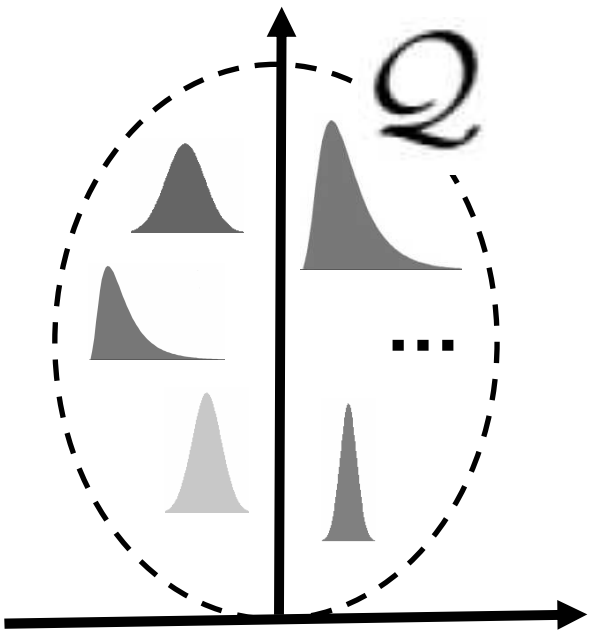

$$\inf_{x \in \mathcal{X}} \sup_{\mathbb{Q} \in \mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[\mathcal{G}(w^T x)]\}$$

$w \in \Re^n$: **random vector of asset returns,**

$\mathbb{Q}$: **associated probability distribution,**

$\mathcal{G}$: **convex measure function (risk&utility),**

x : **wealth allocation vector,**

$\mathcal{X} \subseteq \Re^n$: **convex feasible region.**

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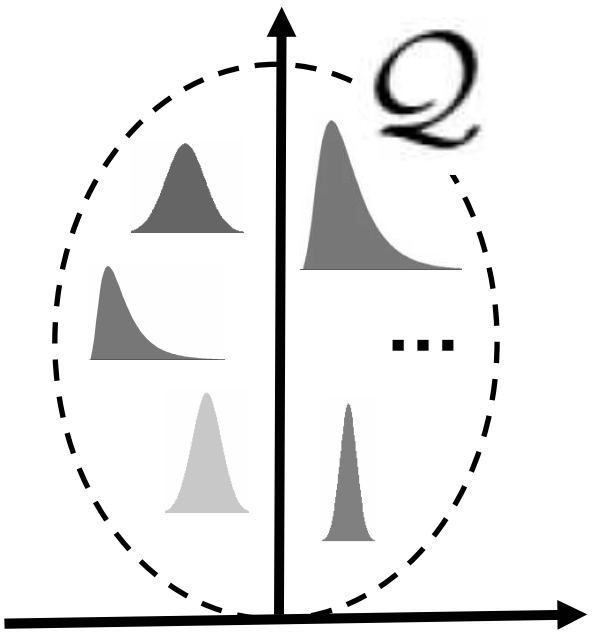

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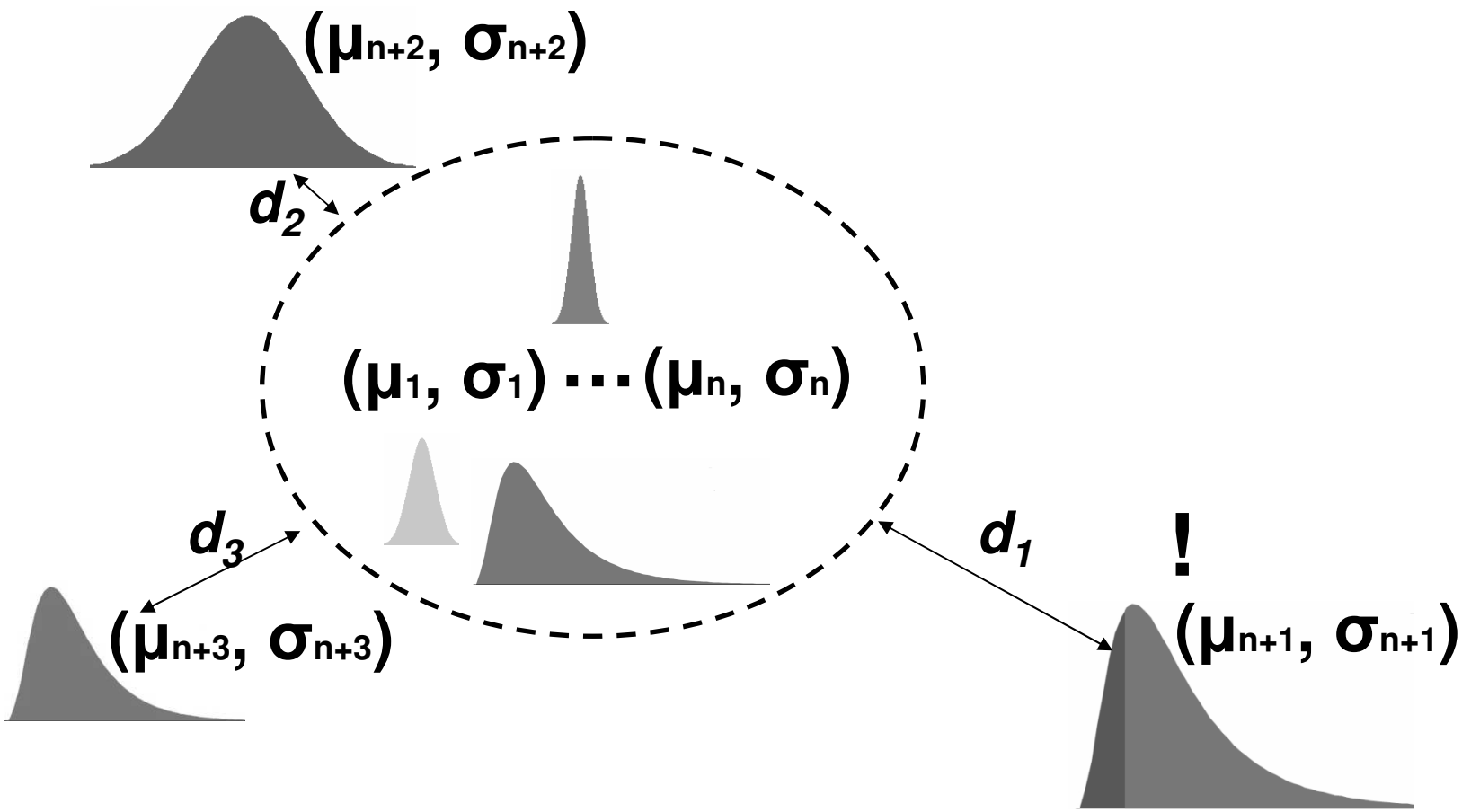
(Delage, Ye 2010)

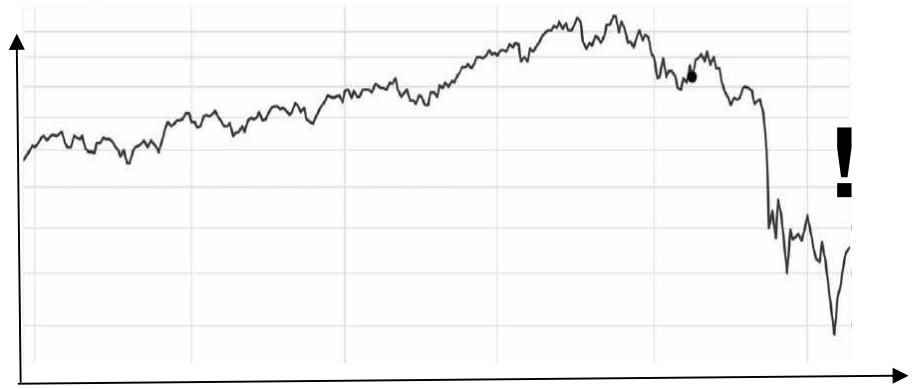
$$\mathcal{Q} = \left\{ \mathbb{Q} \mid \begin{array}{c} (\mathbb{E}_{\mathbb{Q}}[w] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}_{\mathbb{Q}}[w] - \mu_0) \leq \gamma_1 \\ \mathbb{E}_{\mathbb{Q}}[(w - \mu_0)(w - \mu_0)^T] \preceq (1 + \gamma_2)\Sigma_0 \end{array} \right\}$$

New Problem : how to deal with outliers?

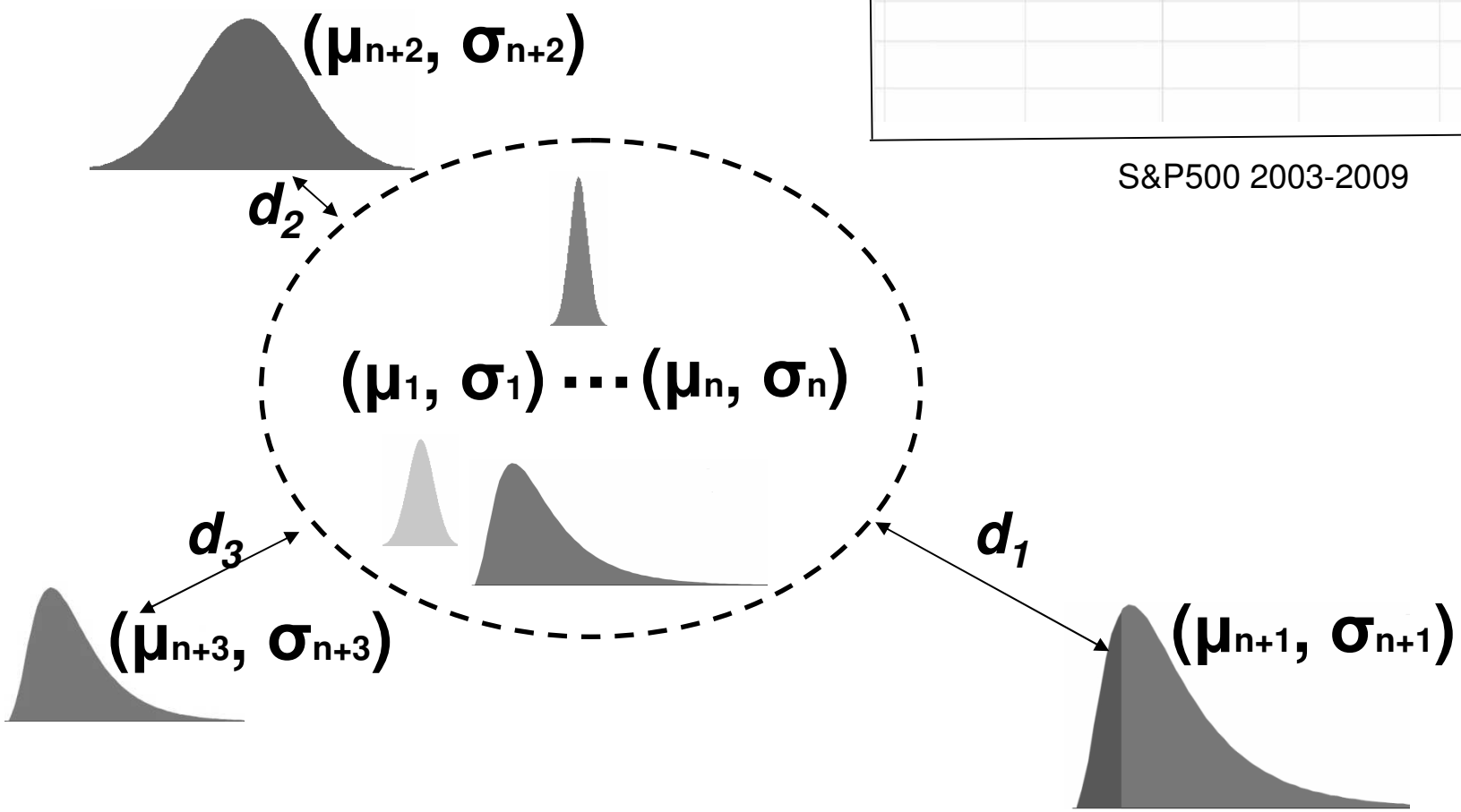


New Problem : how to deal with outliers?

e.g. 2008 financial crisis



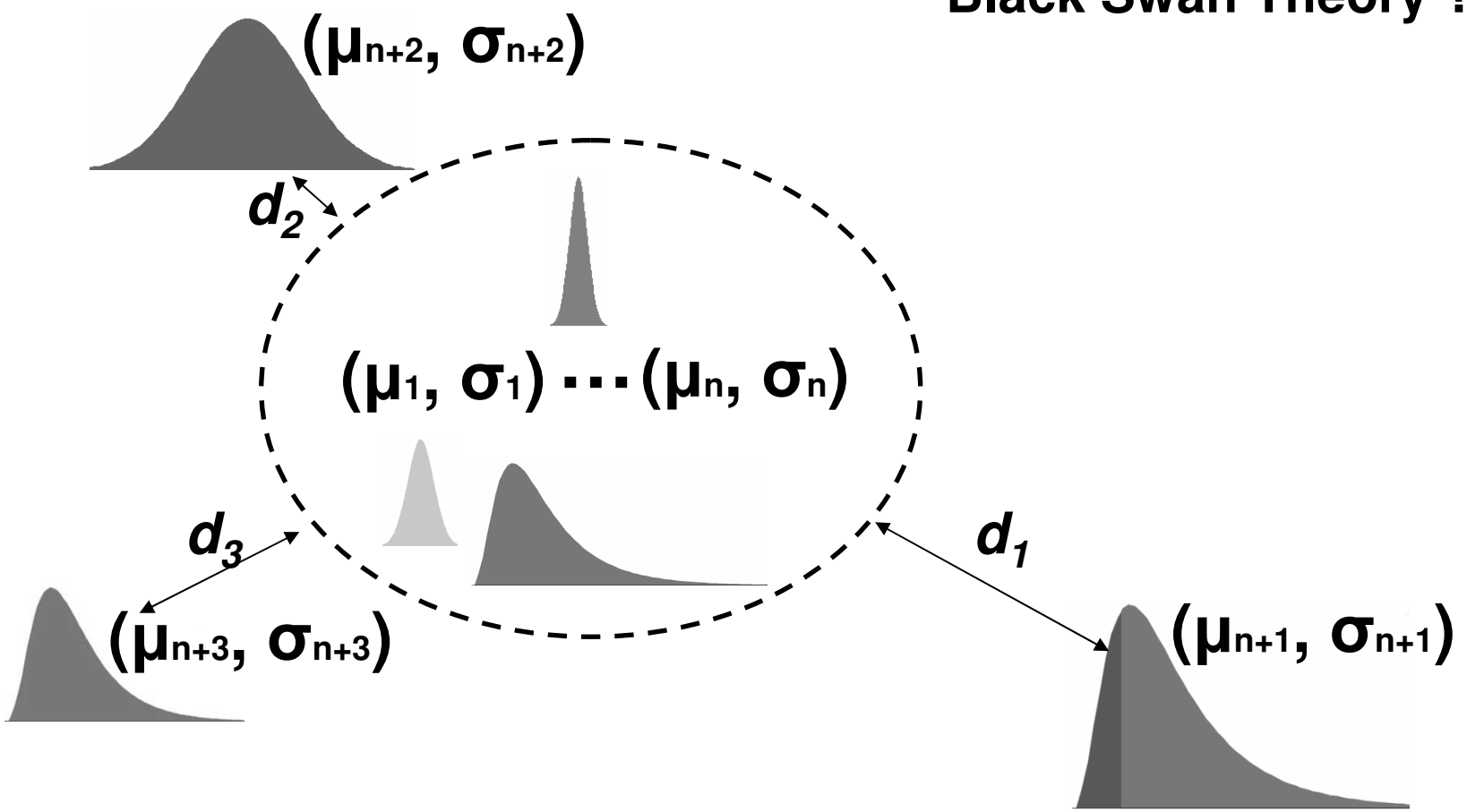
S&P500 2003-2009



New Problem : how to deal with outliers?

e.g. 2008 financial crisis

Black Swan Theory ??



The New York Times. 22 April 2007

The Black Swan: The Impact of the Highly Improbable (Taleb)

“ First, it (the Black Swan) is an *outlier*, as it lies outside the realm of regular expectations, because nothing in the past can convincingly point to its possibility. Second, it carries an extreme impact.”

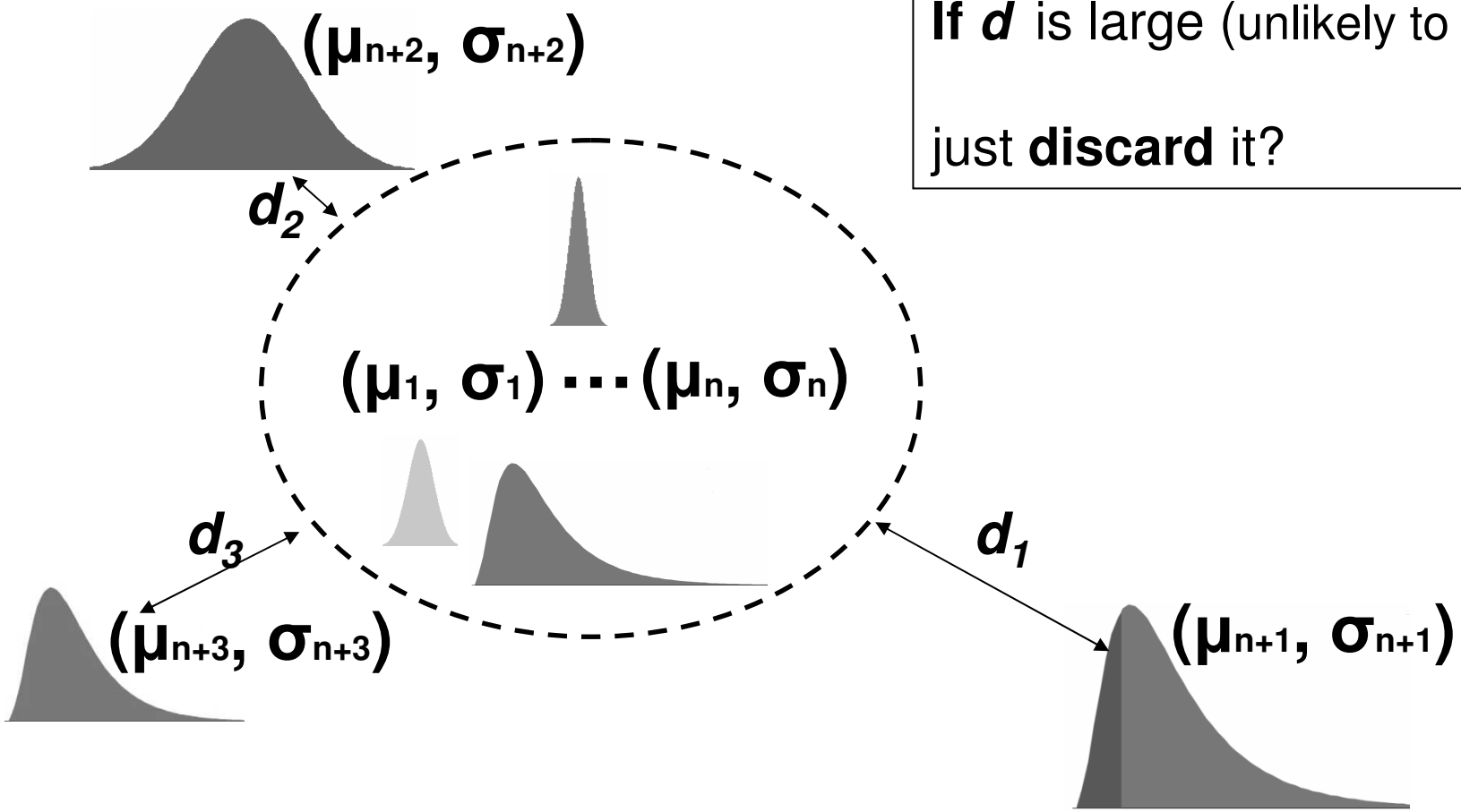
New Problem : how to deal with outliers?

e.g. 2008 financial crisis

Discard the outliers or not?

If ***d*** is large (unlikely to happen),

just **discard** it?



New Problem : how to deal with outliers?

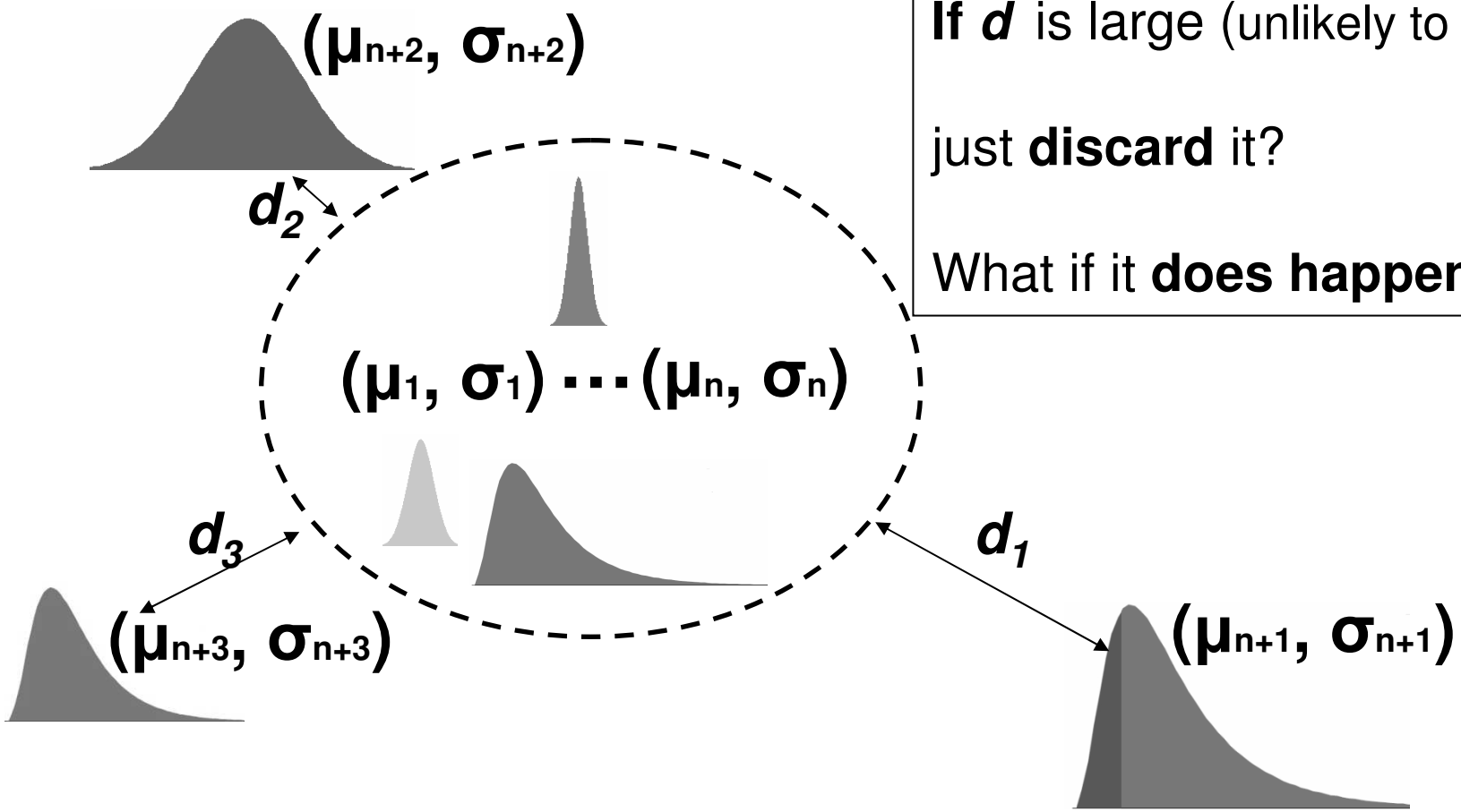
e.g. 2008 financial crisis

Discard the outliers or not?

If *d* is large (unlikely to happen),

just **discard** it?

What if it **does happen** ?



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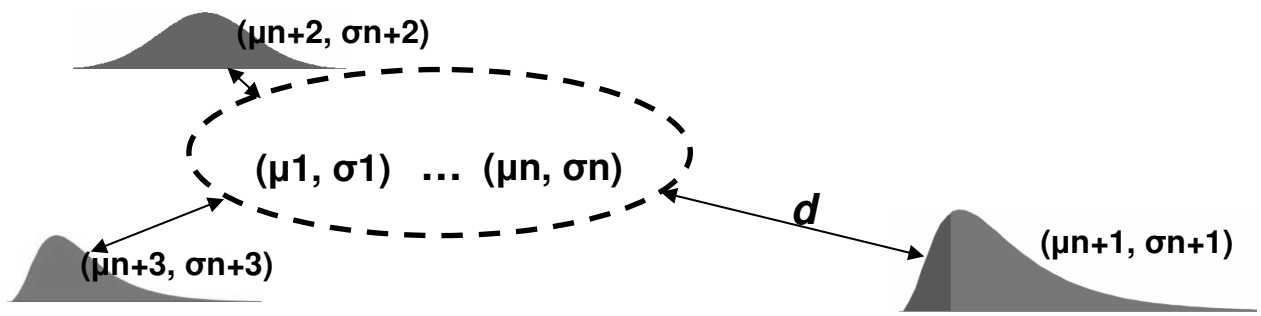
New Solution :

Penalized Distributionally Robust Optimization



Idea : we want to check how “harmful” it is before discarding it !

If “harmfulness” >> “unlikelihood” (*d)*

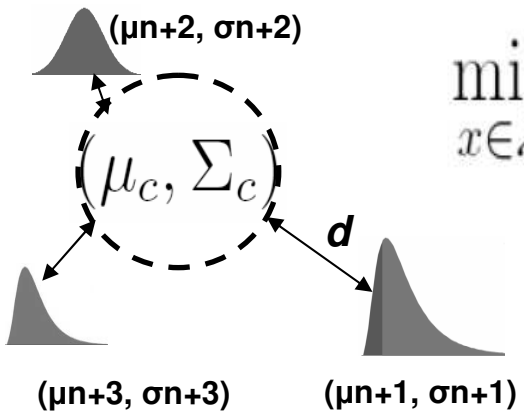
Don’t Discard it, Need to Protect.

If “harmfulness” << “unlikelihood” (*d*)

Not so Harmful, Discard it.

New Solution :

Penalized Distributionally Robust Optimization



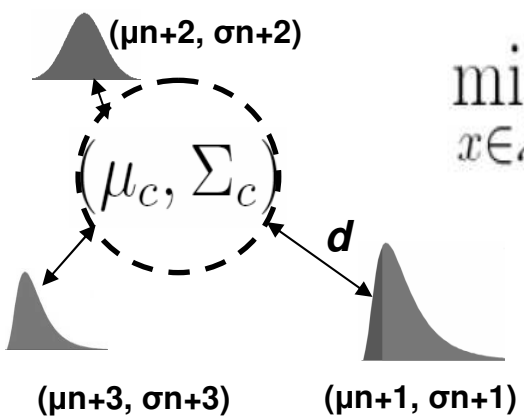
$$\min_{x \in \mathcal{X}} \quad \max_{\gamma, \mu, \Sigma, \mathbb{Q}(\cdot\ ;\ \mu, \Sigma)} \quad \int \mathcal{G}(w^T x) d\mathbb{Q}(w) - r_k(\gamma)$$

$$\text{subject to} \quad \inf_{\nu \in \mu_c} ||\mu - \nu|| \leq \gamma_1$$

$$\inf_{\sigma \in \Sigma_c} ||\Sigma - \sigma|| \leq \gamma_2$$

New Solution :

Penalized Distributionally Robust Optimization



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$\boxed{d}$

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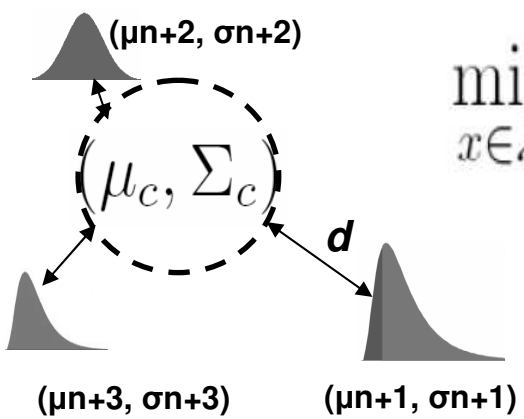

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d

Investors are now allowed to

1)define their confidence region of moments (μ_c, Σ_c)

2)have different levels of protection for instances falling in/outside (μ_c, Σ_c)

Benefit of using
Penalized Distributionally Robust Optimization

Provide full protection for those within the confidence region defined by investors

Provide flexible protection for outliers of mean/variance

Computational complexity

If μ_c and Σ_c are general convex sets, the complexity to obtain global optimal solution is equivalent to solving

$$(\mathcal{P}_\nu) \max_{0 \le \gamma \le a} \nu(\gamma) - r_k(\gamma)$$

$$\nu(\gamma) := \min_{x \in \mathcal{X}} \{ \max_{\mathbb{Q}(\cdot\ ;\ \mu, \Sigma)} \int \mathcal{G}(w^T x) d\mathbb{Q}(w) \Big| \begin{array}{l} \inf_{\nu \in \mu_c} ||\mu - \nu|| \le \gamma_1 \\ \inf_{\sigma \in \Sigma_c} ||\Sigma - \sigma|| \le \gamma_2 \end{array} \}$$

$\nu(\gamma)$ can be evaluated in polynomial time under mild condition. (via an ellipsoid method)

SDP reformulation

If μ_c, Σ_c, $r_k(\gamma)$, and $||\cdot||$ are semidefinite representable, the overall problem can be reformulated as a SDP problem

$$\begin{aligned} \min_{x,\lambda,\Lambda,r,s} \quad & r + s - \Lambda \bullet \mu_0 \mu_0^T \\ \text{subject to} \quad & (\mathcal{P}_s) \leq r \\ & \begin{pmatrix} \Lambda & \frac{1}{2}(\lambda - 2\Lambda\mu_c + a_k x) \\ \frac{1}{2}(\lambda - 2\Lambda\mu_c + a_k x) & s + b_k \end{pmatrix} \succeq 0, \end{aligned}$$

where $(\mathcal{P}_s)$ *denotes the following problem*

$$\begin{aligned} \max_{\gamma,\mu,\Sigma,\nu,\sigma} \quad & \lambda^T \mu + \Lambda \bullet \Sigma - k^T t \\ \text{subject to} \quad & ||\mu - \nu|| \leq \gamma_1, ||\Sigma - \sigma|| \leq \gamma_2, \nu \in \mu_c, \sigma \in \Sigma_c, r_k(\gamma_1, \gamma_2) \leq t, 0 \leq \gamma \leq a. \end{aligned}$$

Computational Experiment with historical data

Randomly choose 4 stocks out of 50 stocks and optimize the portfolio over (1997-2003) and (2004-2010)

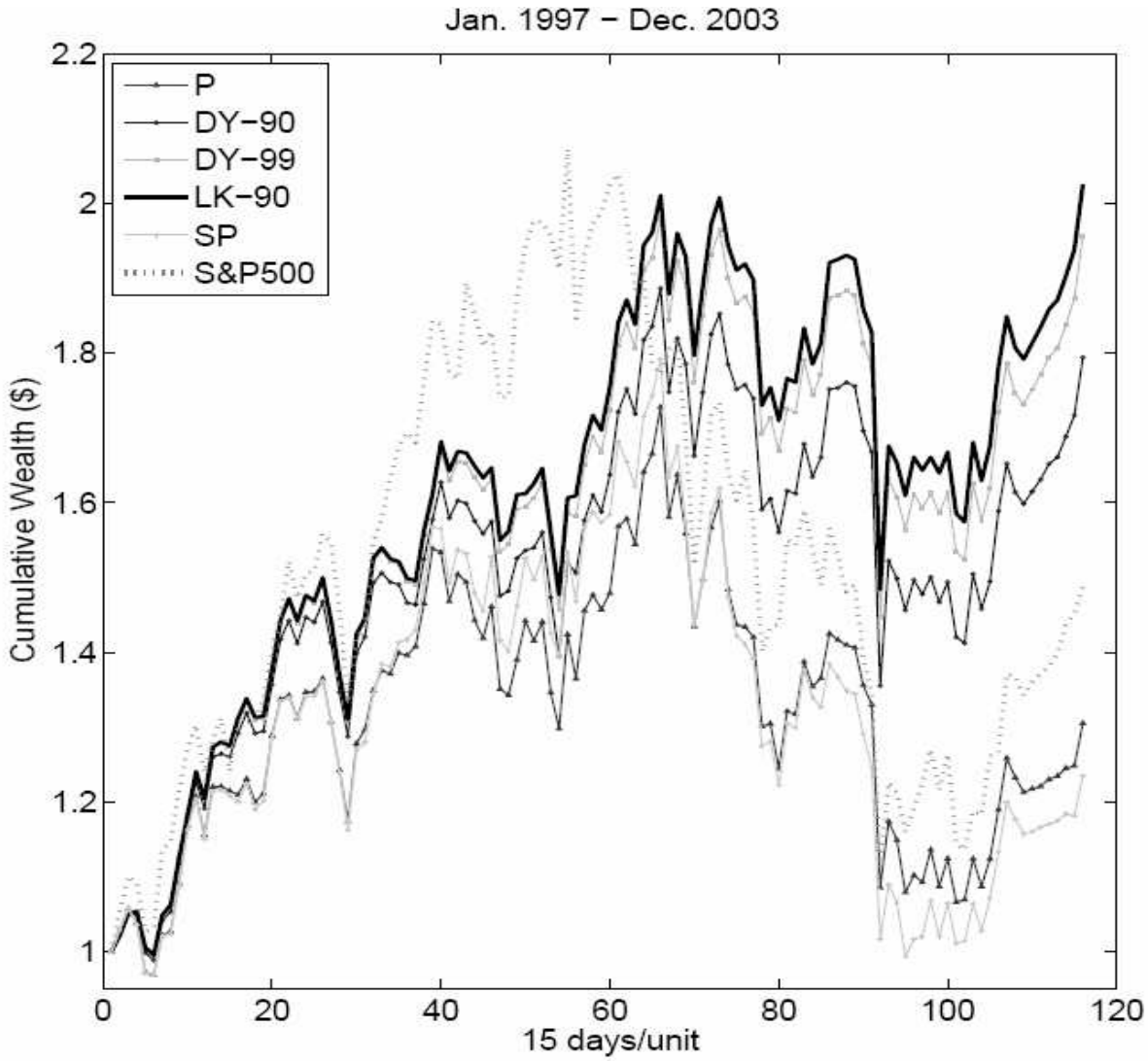
- Rebalance the portfolio every 15 days
- Estimate sampled mean/variance based on past 30 days data
- Estimate confidence region based the data over (1992-1996), (1992-2003)

We compare the following approaches

- Our penlized moment-based approach
- DRO with bounded moment uncertainty
- DRO with fixed moments
- Sample-based stochastic programming approach (SP)

Computational Result

Average wealth evolution in 300 experiments for 1997-2003



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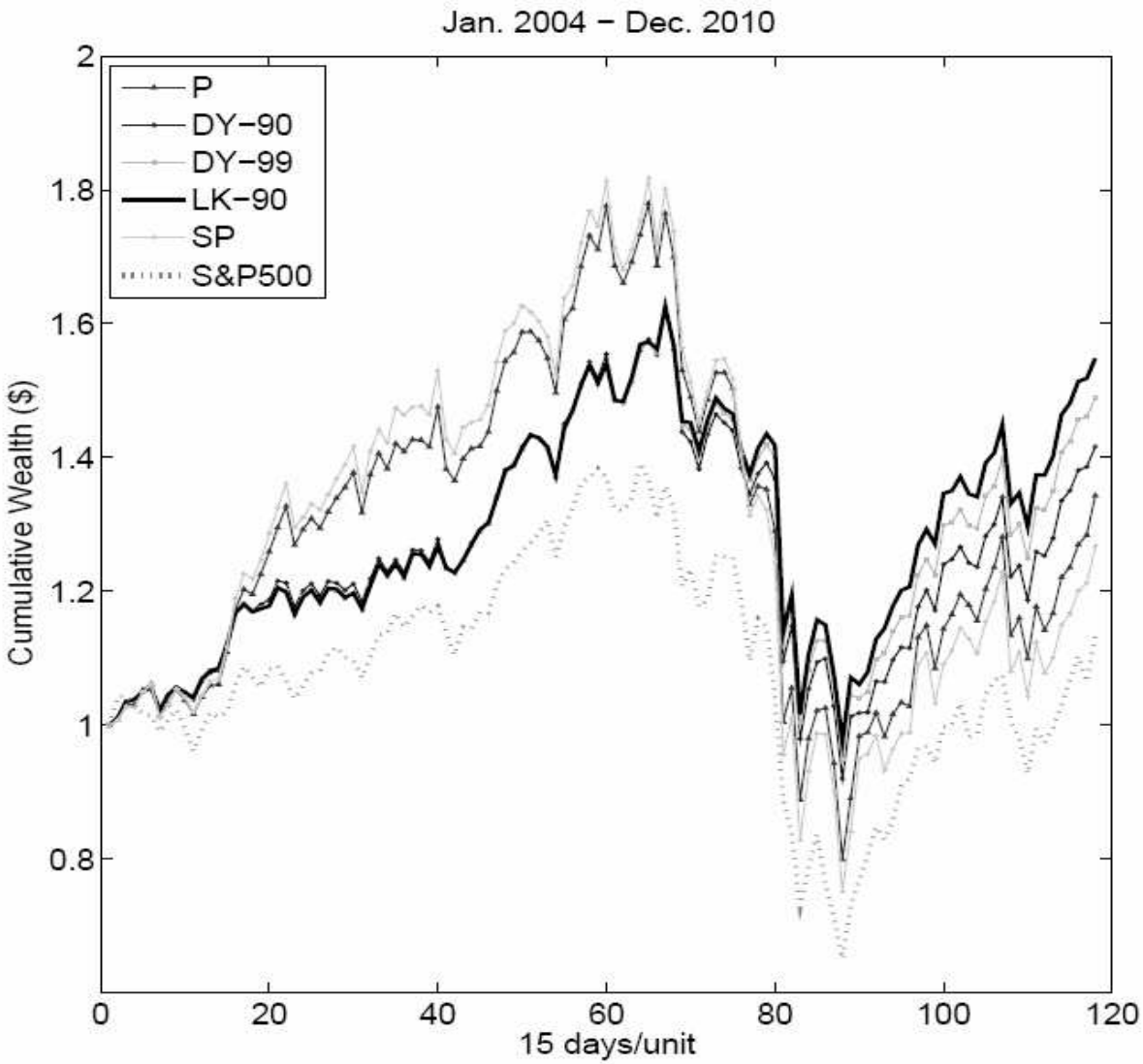

Our contribution

1. Design a portfolio selection model that helps investor to hedge against extreme model uncertainty

2. Show the tractability of the model and provide computational methods that solve the problem efficiently;

3. Show the promising performance of our approach in a volatile market

4. Show many possible extensions of our model to accommodate other practical needs, e.g. factor models.

In many real world applications, outliers are almost unavoidable, and we encourage taking that into account within portfolio optimization.

Future works

A Non-Parametric Penalized Approach

Step 1 : Define a class of ``reference" distributions by

$$\boxed{P := \{P \mid E_P[M(w)] = \rho_c\}}$$

that decision makers are <u>confident of but not fully certain about</u>

Future works

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that decision makers are <u>confident of but not fully certain about</u>

Step 2: Consider instead the optimization problem:

$$\min_{x \in \chi} \max_{Q \in \Theta} E_Q[f(x,w)] - \mathrm{P}_\kappa(\rho \mid \rho_c),$$

where

$$\boxed{\Theta := \{Q \mid E_Q[M(w)] = \rho\}}$$

Note that ρ_c can be either a singleton or a set (confidence region).

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where

$$\mathrm{P}_\kappa(\rho \mid \rho_c) > 0, \ \ if \ \rho \notin \rho_c$$

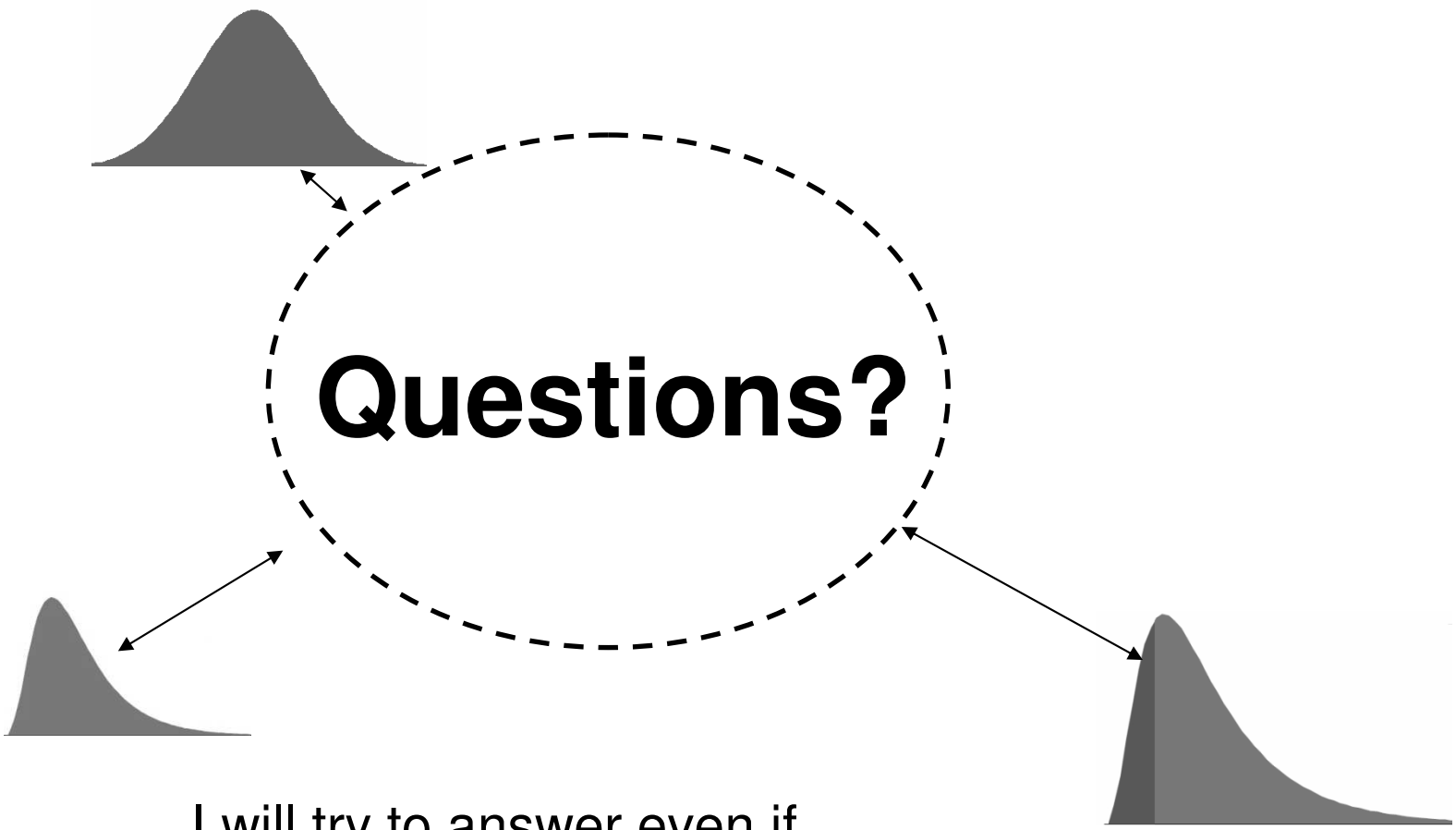
$$\mathrm{P}_\kappa(\rho \mid \rho_c) = 0, \ \ if \ \rho \in \rho_c$$

And the further ρ is "away" from ρ_c, the larger P_κ value is.

Potential applications

- Model uncertainty in derivative pricing (Li and Kwon 2011)

- Many others…



I will try to answer even if it is beyond the scope of my knowledge!