THE BUZZARD-DIAMOND-JARVIS CONJECTURE FOR UNITARY GROUPS

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I will discuss the proof of the weight part of Serre's conjecture for mod p Galois representations attached to rank two unitary groups in the unramified case — i.e., the Buzzard-Diamond-Jarvis conjecture for unitary groups. This is joint work with Toby Gee and Tong Liu.

Notation. We fix the following notation. Let $p > 2$ be a prime number, and F an imaginary CM field with the following properties:

- F/F^+ is unramified at all finite places, split at all places of F^+ above p,
- $[F^+ : \mathbb{Q}]$ is even.

For simplicity I will further assume that F^+ has a unique place v lying above p, and write $v = ww^c$ in F.

The conditions on F guarantee the existence of an outer form $G_{/F^+}$ of GL_2 that is quasi-split at all finite places, compact at infinity, and split over F.

Let k be the residue field of F_w. In this context a *Serre weight* is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(k)$.

Definition. We say that $\bar{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight σ if $\bar{r}$ occurs in a certain space of algebraic modular forms for G with coefficients in σ.

If $\bar{r}$ is modular, let $W(\bar{r})$ be the set of Serre weights for which $\bar{r}$ is modular. When p is unramified in F, BDJ describe a set of weights $W^{\mathrm{BDJ}}(\bar{r})$ (whose definition I will return to momentarily).

Then our main global theorem is as follows.

Theorem A. *With notation as above, suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular and that $\bar{r}|_{G_{F(\zeta_p)}}$ is irreducible (or if $p = 3, 5$ require $\bar{r}$ irreducible and $\bar{r}(G_{F(\zeta_p)})$ adequate). Assume that p is unramified in F. Then $W(\bar{r}) = W^{\mathrm{BDJ}}(\bar{r})$.*

Recall that the irreducible $\overline{\mathbb{F}}_p$-representations σ of $\mathrm{GL}_2(k)$ have the following shape:

$$\sigma = \bigotimes_{\kappa : k \hookrightarrow \overline{\mathbb{F}}_p} (\det{}^{a_\kappa} \otimes \mathrm{Sym}^{b_\kappa - a_\kappa} k^2) \otimes_{k,\kappa} \overline{\mathbb{F}}_p$$

with $0 \le a_\kappa \le p-1$ (but not all equal to $p-1$) and $0 \le b_\kappa - a_\kappa \le p-1$.

Assume for the rest of the talk that F is unramified at p. We will identify an embedding $\kappa : k \hookrightarrow \overline{\mathbb{F}}_p$ with its unique lifting $F_w \hookrightarrow \overline{\mathbb{Q}}_p$.

Suppose that $\rho : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is a crystalline representation, and let D be the associated filtered φ-module, which is a module over $F_w \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. Since $F_w \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \prod_{\kappa : F_w \to \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p$, there is a decomposition $D \cong \prod_\kappa D_\kappa$ and similarly for each $\mathrm{Fil}^i D$, so that each D_κ is a filtered $\overline{\mathbb{Q}}_p$-vector space of dimension 2.

Definition. The jumps in the filtration of D_κ are the *κ-labeled Hodge-Tate weights* of ρ. (We normalize the sign of HT weights so that all the HT weights of the cyclotomic character are 1.)

Let σ be a Serre weight. We say that ρ has *Hodge type* σ if the κ-labeled Hodge-Tate weights of ρ are $(a_\kappa, b_\kappa + 1)$ for all κ.

The above definition is justified by the following basic fact.

Proposition. *If $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight σ, then $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type σ.*

(Indeed, $\bar{r} \cong \bar{r}_\pi$ for some cuspidal automorphic π whose infinitesimal character is determined by σ, and $r_\pi|_{G_{F_w}}$ is the desired lift.)

Definition. Define $W^{\mathrm{cris}}(\bar{r})$ to be the set of Serre weights σ such that $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type σ. So, if $\bar{r}$ is modular, then essentially by definition we have a containment $W(\bar{r}) \subset W^{\mathrm{cris}}(\bar{r})$.

One has the following important progress in the reverse direction. (Note: no ramification hypothesis here!)

Theorem (Barnet-Lamb, Gee, Geraghty, 2011)**.** *Under the hypotheses of Theorem A, suppose that $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge Type σ that moreover is* potentially diagonalizable *(in the sense of BLGG+Taylor). Then $\sigma \in W(\bar{r})$.*

Our task, then, is to remove the potentially diagonalizable hypothesis.

Definition. We define $W^{\mathrm{BDJ}}(\bar{r})$ to be the set of weights σ such that $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type σ that is:

- reducible, if $\bar{r}|_{G_{F_w}}$ is reducible, or
- induced from the unramified quadratic extension of F_w, if $\bar{r}|_{G_{F_w}}$ is irreducible.

Lifts as above are potentially diagonalizable (this is all we will need to say about potential diagonalizability), so by BLGG we have $W^{\mathrm{BDJ}}(\bar{r}) \subset W(\bar{r})$. Note that the set $W^{\mathrm{BDJ}}(\bar{r})$ is much more explicit than $W^{\mathrm{cris}}(\bar{r})$ — completely explicit if $\bar{r}|_{G_{F_w}}$ is semisimple, and in terms of crystalline extension classes in the non-split reducible case. We prove the following theorem, our main local result.

Theorem B. *Suppose that $F_w/\mathbb{Q}_p$ and that $\overline{\rho} : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is reducible. If $\overline{\rho}$ has a crystalline lift of Hodge type σ, then $\overline{\rho}$ has a* reducible *crystalline lift of Hodge type σ.*

We deduce from Theorem B that $W^{\mathrm{BDJ}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$; this is immediate in the reducible case, while the irreducible case follows by restricting to a quadratic extension, applying the reducible case there, and descending. Theorem A follows.

Remark.
1. By twisting it suffices to consider weights σ for which $a_\kappa = 0$ for all κ, and in this case the crystalline representations of Theorem B have all their Hodge–Tate weights in $[0, p]$.
2. In the special case where the Hodge–Tate weights are in the interval $[0, p-2]$ rather than $[0, p]$, then Theorem B is a consequence of Fontaine–Laffaille theory. Indeed, Theorem A was proved by Gee (2006) in the case of *generic* (or *regular*) weights. The regularity hypothesis allowed Gee to avoid the difficulties that arise when dealing with Hodge–Tate weights outside the Fontaine–Laffaille range. Our contribution now is a method for addressing these difficulties.

(3) Theorem B is false in any wider range of Hodge–Tate weights. For instance, if $F_w = \mathbb{Q}_p$ then $\overline{\epsilon}\oplus\overline{\epsilon}$ has a crystalline lift with Hodge–Tate weights $(0, p+1)$, but no reducible such lift.
(4) If $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with Hodge–Tate weights in the range $[0, p]$, then $\overline{\rho}$ is reducible if and only if ρ is reducible. But this is definitely *not* the case if $F_w \neq \mathbb{Q}_p$ (i.e., $\overline{\rho}$ may be reducible for ρ irreducible)

Our proof of Theorem B is purely local. Perhaps the most direct approach to Theorem B would be to write down all the filtered φ-modules corresponding to crystalline representations ρ of the sort considered in the theorem, and attempt to compute each $\overline{\rho}$ explicity, e.g. using the theory of (ϕ, Γ)-modules. Some partial results have been obtained along these lines by Dousmanis and by Zhu (and if you look at what they do, they have to work very very hard), but the general case has so far been resistant to these methods. Instead our approach is somewhat indirect.

Write W for the ring of integers of F_w, fix a uniformizer π of W, and define

- $\mathfrak{S} = W[\![u]\!]$,
- $\varphi : \mathfrak{S} \to \mathfrak{S}$ has $\varphi(u) = u^p$ and acts as Frobenius on W,
- $F_{w,\infty} = F_w(\pi^{1/p^\infty})$, and
- $G_\infty = \mathrm{Gal}(\overline{F}_w/F_{w,\infty})$.

A *Kisin module* of height r is a pair $(\mathfrak{M}, \varphi)$ where: $\mathfrak{M}$ is a finite rank free $\mathfrak{S}$-module, $\varphi : \mathfrak{M} \to \mathfrak{M}$ is φ-semilinear, and the cokernel of $\mathfrak{S} \otimes_\varphi \mathfrak{M} \to \mathfrak{M}$ is killed by $(u-\pi)^r$.

Theorem (Kisin)**.** *There is an exact and fully faithful functor $T : \{\text{Kisin modules}\} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$. If L is a G_{F_w}-stable lattice in a semistable representation of G_{F_w} with Hodge–Tate weights in $[0, r]$, then $L|_{G_\infty} \simeq T(\mathfrak{M})$ for some (unique) Kisin module $\mathfrak{M}$ of height r. The functor T is compatible with reduction mod p.*

The key technical result that we prove is the following structure theorem for Kisin modules of height p.

Theorem C. *Assume that $F_w/\mathbb{Q}_p$ is unramified and $p \geq 3$. Suppose that L is a G_{F_w}-stable lattice in a d-dimensional crystalline representation with Hodge–Tate weights $r_1, \ldots, r_d$ in the range $[0, p]$, and let $\mathfrak{M}$ be the corresponding Kisin module. Then there exists a basis of $\mathfrak{M}$ in which the matrix of φ is $X[(u-\pi)^{r_1}, \cdots, (u-\pi)^{r_d}]Y$ where X and Y are invertible matrices and Y is congruent to the identity modulo p.*

(To be precise, we prove a version with coefficients, but I will ignore coefficients here.) This structure theorem has other applications. As an example, we can prove the following (which I state with $F_w = \mathbb{Q}_p$ just for the sake of simplicity, but we can prove the analogous statement for F_w unramified):

Theorem D. *Suppose that $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ is a crystalline representation with Hodge–Tate weights $r_1, \ldots, r_d$ in the range $[0, p]$. If $\overline{\rho}^{\mathrm{ss}}$ is a sum of characters, then $\overline{\rho}|_{I_p}^{\mathrm{ss}} \simeq \overline{\epsilon}^{r_1} \oplus \cdots \overline{\epsilon}^{r_d}$.*

This extends a theorem of Fontaine–Laffaille (in the HT weight range $[0, p-1]$), and would be false in the wider range $[0, p+1]$.

Sketch of proof of Theorem C. Suppose that L is any semistable representation. Write S for the p-adic completion of the divided power envelope of $W[u]$ with respect to $(u-\pi)$. Then

$$\mathcal{D} := S[1/p] \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \simeq S[1/p] \otimes_{F_w} D$$

where D is the filtered (φ, N)-module associated to $L[1/p]$. In particular there is an operator N on $\mathcal{D}$. Let $\tilde{S} = W[\![u^p, u^p/p]\!]$. A delicate calculation shows that if L is *crystalline* then with respect to any basis of $\mathfrak{M}$, the matrix of the operator N lies in $u^p \mathrm{M}_d(\tilde{S}[1/p] \cap S)$. (So far, don't need any unramified hypothesis either.)

Now write $\mathfrak{M}^* = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset \mathcal{D}$ and set $\mathrm{Fil}^i \mathfrak{M}^* = \mathfrak{M}^* \cap \mathrm{Fil}^i \mathcal{D}$. Let $M \subset D$ be the image of $\mathfrak{M}^*$ under the map $u \mapsto \pi$, and $\mathrm{Fil}^i M = M \cap \mathrm{Fil}^i D$. Then one proves by induction on i that the map $\mathrm{Fil}^i \mathfrak{M}^* \to \mathrm{Fil}^i M$ sending $u \to \pi$ is surjective for all i; this uses in an essential way the shape of the operator N, the fact that F_w is unramified, and the hypothesis that $\mathrm{Fil}^i D = 0$ for $i > p$. □

Suppose L is a lattice in a crystalline representation of Hodge type σ with L/pL reducible, and $\mathfrak{M}$ the corresponding Kisin module. Then $\mathfrak{M}/p\mathfrak{M}$ is an extension of rank one φ-modules over $k[\![u]\!]$, and Theorem C places strong restrictions on the extension classes that can arise in this manner.

We would like to see that each of the extensions that could arise in this manner (according to Theorem C) can be lifted to a Kisin module corresponding to a reducible crystalline representation of Hodge type σ. But to deduce Theorem B it is essential for us to be able to see the whole representation with the functor T (not just the restriction to G_∞).

To that end we make crucial use of Liu's theory of $(\varphi, \hat{G})$-modules. A $(\varphi, \hat{G})$-module is a Kisin module together with some extra structure; I won't have time to be completely precise about this, but let me at least say the following. Liu constructs a certain $\mathfrak{S}$-algebra $\hat{\mathcal{R}} \subset W(R)$ where R is the usual ring $\varprojlim \mathcal{O}_{\mathbb{C}_p}/p$. Set $\hat{\mathfrak{M}} = \hat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then a $(\varphi, \hat{G})$-module is $(\mathfrak{M}, \varphi)$ together with an action of the group $\hat{G} = \mathrm{Gal}(F_{w,\infty}(\mu_{p^\infty})/F_w)$ satisfying certain compatibilities, one upshot of which is that there is a map $\tau : \mathfrak{M} \to \hat{\mathfrak{M}}$ satisfying certain compatibilities. (The element τ corresponds to a choice of generator in $\mathrm{Gal}((F_{w,\infty}(\mu_{p^\infty})/F_w(\mu_{p^\infty}))$.) Liu has proved:

Theorem (Liu)**.** *There is an exact (anti-)equivalence of categories T between $(\varphi, \hat{G})$-modules of height r and Galois-stable lattices in semi-stable representations with Hodge–Tate weights in $[0, r]$. This is compatible with Kisin's T and with reduction mod p.*

We prove a structure result for $(\varphi, \hat{G})$-modules coming from crystalline (as opposed to semi-stable) reprsentations, namely that

$$\tau(x) - x \in u^p \varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$$

where $\mathfrak{t}$ is a certain element of $W(R)$. The proof makes another use of the shape of N as in the proof of Theorem C.

Using this, we show that each of the above extensions of characteristic p Kisin modules can be extended in at most one way to an extension of characteristic p $(\varphi, \hat{G})$-modules coming from the reduction mod p of a crystalline representation.[1] Now Theorem B follows by counting dimensions: the mod p $(\varphi, \hat{G})$-modules coming from reducible crystalline representations of Hodge type σ account for all of the ones we are left with.

[1] This is not quite true when $b_\kappa - a_\kappa = p - 1$ for all κ.