Facility Location with Economies of Scale and Congestion: Models and Column Generation Heuristics

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Outline

- Facility Location(FL) Models
  - FL with concave cost (economies of scale)
  - FL with convex cost (congestion)
  - Our model: with economies of scale and congestion
- Lagrangian Relaxation
  - Subproblems
  - Lagrangian bound
- Column Generation Heuristics
- Numerical Test
  - Data generation
  - Test results

Facility Location Problem

- How to allocate facilities such that:
  - Total cost is minimized
  - All customers' demands are satisfied within the capacities of operating facilities.

Facility Location(FL) Models

- FL with Concave Cost
  - Unit cost decreases as output increases due to economies of scale
    - Examples include concave production costs in Romeijin et al. 2010 and Cohen and Moon 1991, concave site dependent costs in Dupont 2008, concave transportation costs in Lin et al. 2006, concave operating costs as a function of the number of assigned clients in Hajiaghayi et al. 2003,and concave technology acquisition costs in Dasci and Verter 2001.
    - Solution Methodologies include branch-and-bound (Dupont 2008), piecewise linear approximation (Dasci and Verter 2001), Benders decomposition (Cohen and Moon 1991) and greedy heuristics (Hajiaghayi et al. 2003, Romeijin et al. 2010, Lin et al. 2006).

Facility Location(FL) Models

- FL with Convex Cost
  - Unit cost increases as output increases due to over-utilization of resources, overtime, and facility congestion.
    - Examples include location models that consider waiting times and congestion (Desrochers et al. 1995, Elhedhli 2006), strategic inventory-location (Benjaafar et al. 2004) and stochastic transportation problems (Holmberg 1995).
    - Solution methodologies include tangential piecewise approximations (Elhedhli 2006, Benjaafar et al. 2004, Holmberg 1995), column generation (Desrochers et al. 1995), and branch-and-bound (Holmberg 1995).

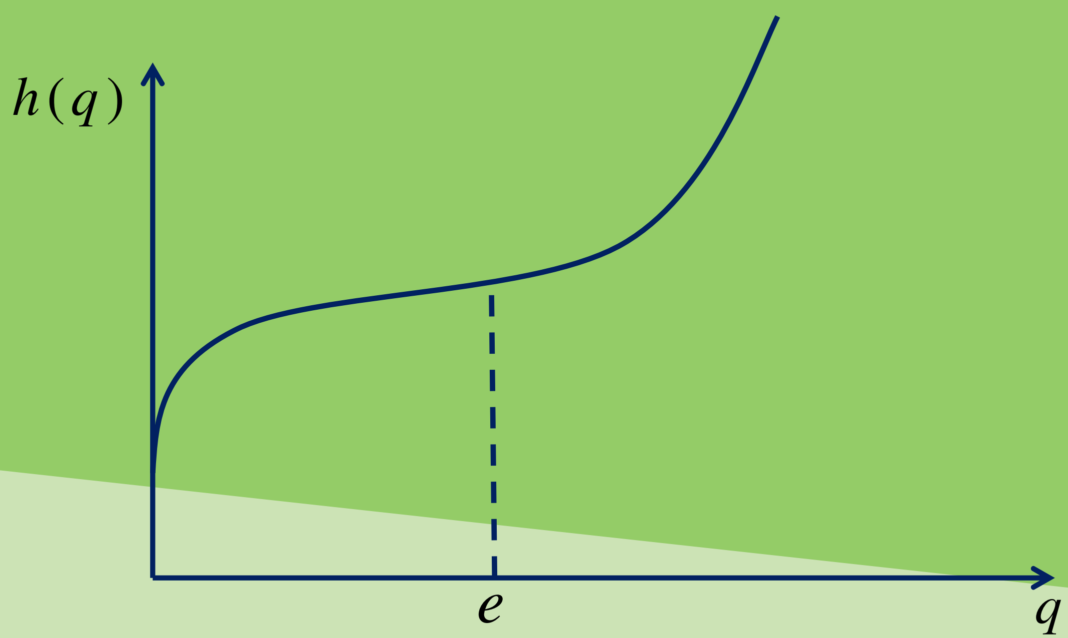
Facility Location(FL) Models

- FL with Both Convex and Concave Cost
  - Broek et al. 2006 studied a FL model with inverse S-shaped cost function. Schütz et al. 2008 extended the formers' work to a stochastic case considering both short-run and long-run scenarios. But both cost functions is linear after after deflection point capturing only economies of scale.
  - Solution approach is piecewise approximation and Lagrangian relaxation.

Facility Location(FL) Models

- A typical phenomenon in economics
  - E.g. a typical function often used by economists is cubic function as follows:

$$h(q) = aq^3 + bq^2 - cq + d$$



Facility Location(FL) Models

- Our model
  - Index
    - i : customers
    - j : facilities
  - Parameters:
    - d_i demand of customer i
    - K_j capacity at facility j
    - c_{ij} transportation cost from facility j to customer i
    - F_j fixed cost of facility j
    - e_j economic point at facility j

Facility Location(FL) Models

- Our model
  - Functions
    - $g_j(q)$ represents concave part of production cost
    - $f_j(q)$ represents convex part of production cost
  - Decision variables
    - x_{ij} quantity supplied by facility j to customer i
    - $y_j^e = \begin{cases} 1, & \text{if facility j operates under economic point} \\ 0, & \text{otherwise} \end{cases}$
    - $y_j^c = \begin{cases} 1, & \text{if facility j operates above economic point} \\ 0, & \text{otherwise} \end{cases}$

Our Model

$$\min \sum_j F_j(y_j^e + y_j^c) + \sum_j y_j^e g_j\left(\sum_i x_{ij}\right) + \sum_j y_j^c f_j\left(\sum_i x_{ij}\right) + \sum_i \sum_j c_{ij} x_{ij} \quad (1)$$

s.t.

$$\sum_j x_{ij} = d_i \qquad \forall i \quad (2)$$

$$e_j y_j^c \le \sum_i x_{ij} \le K_j y_j^c + e_j y_j^e \qquad \forall j \quad (3)$$

$$y_j^e + y_j^c \le 1 \qquad \forall j \quad (4)$$

$$y_j^e,\ y_j^c \in \{0,1\} \qquad \forall j \quad (5)$$

$$x_{ij} \ge 0,\ x_{ij} \text{ are integers} \qquad \forall i, j \quad (6)$$

Our Model

- Advantages
  - No approximation on the production cost function
    - No need to introduce extra binary variables
  - Decomposable in terms of (y_j^e, y_j^c)
    - $(y_j^e, y_j^c) = (0,0)$: facility j is closed
    - $(y_j^e, y_j^c) = (1,0)$: facility j is producing under e_j
    - $(y_j^e, y_j^c) = (0,1)$: facility j is producing above e_j

Lagrangian Relaxation

- Lagrangian Subproblems
  - Relax on constrain set (2): $\sum_j x_{ij} = d_i$
    - Resulted in the following relaxed problem:

$$\mathrm{LR}(\mu) = \min \sum_j F_j(y_j^e + y_j^c) + \sum_j y_j^e g_j(\sum_i x_{ij}) + \sum_j y_j^c f_j(\sum_i x_{ij}) + \sum_i \sum_j c_{ij} x_{ij} + \sum_i \mu_i (d_i - \sum_j x_{ij})$$

    - Subject to constrain set (3), (4), (5), (6)
  - Given $\bar{\mu}$, the problem decomposes in terms of facilities
    - $\mathrm{SP}(\bar{\mu})_j = \min F_j(y_j^e + y_j^c) + y_j^e g_j(\sum_i x_{ij}) + y_j^c f_j(\sum_i x_{ij}) + \sum_i (c_{ij} - \bar{\mu}_i) x_{ij}$
    - Subject to constrain (3), (4), (5), (6)
    - Futher decompose in terms of (y_j^e, y_j^c)

Lagrangian Relaxation

$$\mathrm{SP}(\bar{\mu})_j$$

Case 1

$$(y_j^e, y_j^c) = (0,0)$$

$$\begin{aligned} &\min \sum_i (c_{ij} - \bar{\mu}_i) x_{ij} \\ &\text{s.t.} \quad 0 \le \sum_i x_{ij} \le 0 \\ &\qquad x_{ij} \ge 0 \end{aligned}$$

Case 2

$$(y_j^e, y_j^c) = (1,0)$$

$$\begin{aligned} &\min F_j + g_j\left(\sum_i x_{ij}\right) + \sum_i (c_{ij} - \bar{\mu}_i) x_{ij} \\ &\text{s.t.} \quad 0 < \sum_i x_{ij} \le e_j \\ &\qquad x_{ij} \ge 0, \text{ integers} \end{aligned}$$

Case 3

$$(y_j^e, y_j^c) = (0,1)$$

$$\begin{aligned} &\min F_j + f_j\left(\sum_i x_{ij}\right) + \sum_i (c_{ij} - \bar{\mu}_i) x_{ij} \\ &\text{s.t.} \quad \mathrm{e}_j < \sum_i x_{ij} \le K_j \\ &\qquad x_{ij} \ge 0, \text{ integers} \end{aligned}$$

- Case 1: $SP(\bar{\mu})_j = 0$
- Case 2 and 3 could dominate only if $SP(\bar{\mu})_j \le 0$

Lagrangian Relaxation

- Add cuts to subproblems
  - $x_{ij} \le d_i, \ \forall i, j$ is valid in subproblems, though redundant in original model.
  - Then Case 2 and Case 3 become:

Case 2: concave bounded knapsack problem

$$\begin{aligned}
&\min F_j + g_j(\textstyle\sum_i x_{ij}) + \sum_i (c_{ij} - \bar{\mu}_i) x_{ij} \\
&\text{s.t.} \qquad 0 < \textstyle\sum_i x_{ij} \le e_j \\
&\qquad\qquad x_{ij} \le d_i, \forall i \\
&\qquad\qquad x_{ij} \ge 0, \ \text{integers}
\end{aligned}$$

Case 3: convex bounded knapsack problem

$$\begin{aligned}
&\min F_j + f_j(\textstyle\sum_i x_{ij}) + \sum_i (c_{ij} - \bar{\mu}_i) x_{ij} \\
&\text{s.t.} \qquad \mathrm{e}_j < \textstyle\sum_i x_{ij} \le K_j \\
&\qquad\qquad x_{ij} \le d_i, \forall i \\
&\qquad\qquad x_{ij} \ge 0, \ \text{integers}
\end{aligned}$$

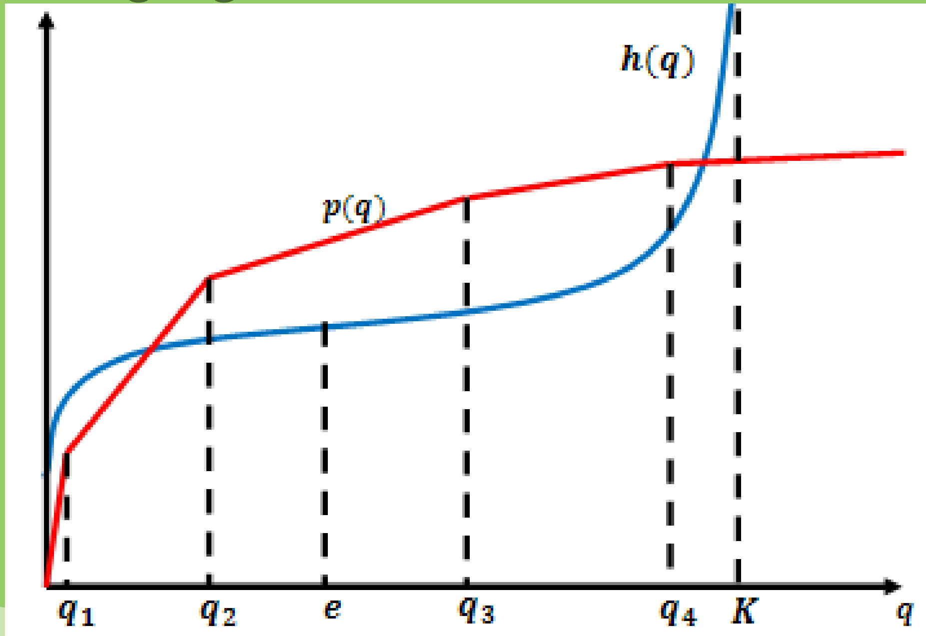
Lagrangian Relaxation

- How to solve the subproblems?
  - Let's drop index j, define $\bar{c}_i = \mu_i - c_i$.
  - Order $\bar{c}_i$ in descending order, denote the ordered $\bar{c}_i$ by $\bar{c}_{(i)}$.
  - Form a function:

$$p(q) = \begin{cases} 0 & \text{if } q = 0, \\ \bar{c}_{(1)} q & \text{if } 0 \le q \le d_{(1)}, \\ \bar{c}_{(1)} d_{(1)} + \bar{c}_{(2)}(q - d_{(1)}) & \text{if } d_{(1)} \le q \le d_{(1)} + d_{(2)}, \\ \ldots & \\ \sum_{t=1}^{i-1} \bar{c}_{(t)} d_{(t)} + \bar{c}_{(i)}(q - \sum_{t=1}^{i-1} d_{(t)}) & \text{if } \sum_{t=1}^{i-1} d_{(t)} \le q \le \sum_{t=1}^{i} d_{(t)}, \\ \ldots & \\ \sum_{t=1}^{m-1} \bar{c}_{(t)} d_{(t)} + \bar{c}_{(i)}(q - \sum_{t=1}^{m-1} d_{(t)}) & \text{if } \sum_{t=1}^{m-1} d_{(t)} \le q \le \sum_{t=1}^{m} d_{(t)}. \end{cases}$$

Lagrangian Relaxation

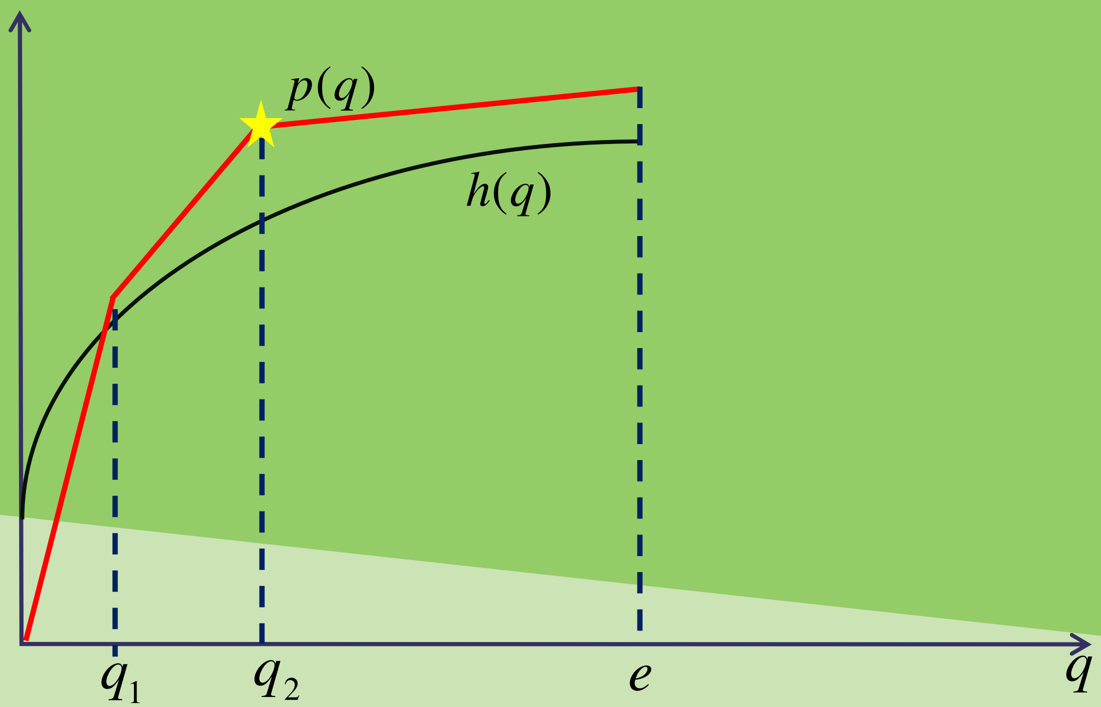
- For ease of notation, let's denote $\sum_{t=1}^{i} d_{(t)} = q_i,\ i = 1 \ldots m$, and denote $g(q) + f(q) = h(q)$
- The functions $p(q)$ and $h(q)$ are plotted in the following figure:



- Then all it takes to solve subproblems is to solve $\max\ p(q) - h(q)$

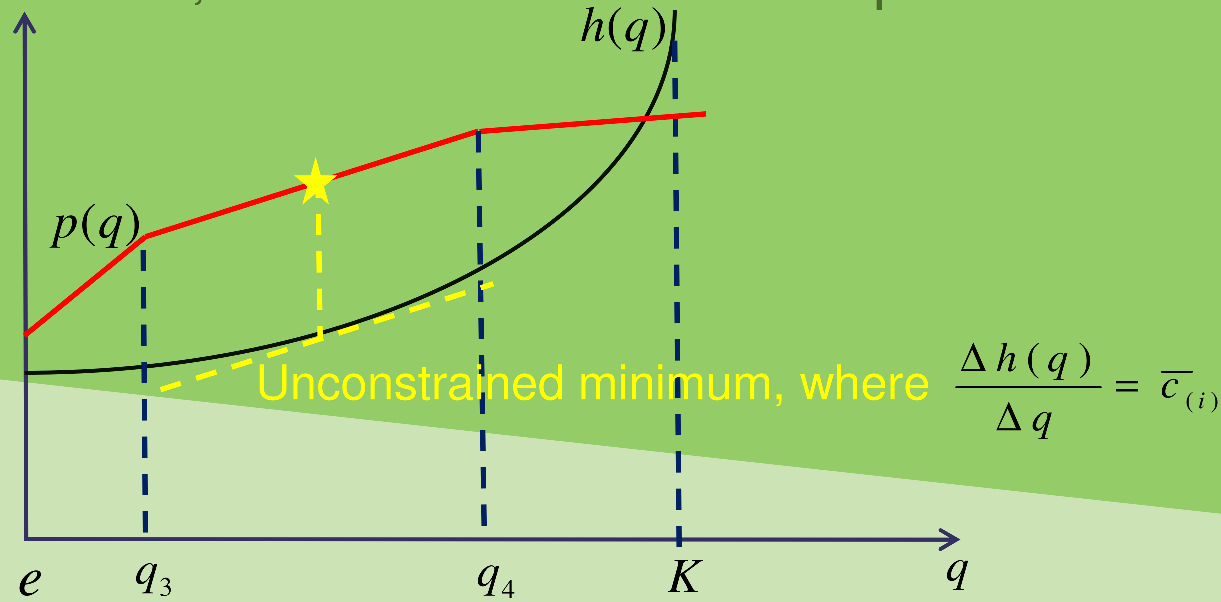
Lagrangian Relaxation

- Case 2: concave bounded knapsack problem
  - The subproblem is a piecewise concave minimization problem
  - Thus, one of the breakpoints is optimal.



Lagrangian Relaxation

- Case 3: convex bounded knapsack problem
  - The subproblem is a piecewise convex minimization problem, where the break-points are e, K and all the breakpoints q_i in between.
  - Since the function is convex, the optimal solution can either be the unconstrained minimum of the function, or the lower cost breakpoints.



Lagrangian Relaxation

- The previous algorithm gives a Lagrangian LB: $LR(\bar{\mu}) = \sum_j \mathrm{SP}(\bar{\mu})_j + \sum_i \bar{\mu}_i d_i$ for a given $\bar{\mu}$
- we need to choose the best μ that maximizes $LR(\mu)$
- To update μ , we create a Lagrangian dual master problem (DMP) and add a set of cuts using the solution from subproblems at each iteration.

Lagrangian Relaxation

- Lagrangian Dual Master Problem
  - Define

$$V_j = \left\{ \begin{array}{l} e_j y_j^c \le \sum_i x_{ij} \le K_j y_j^c + e_j y_j^e \\ x_{ij} \le d_i \\ y_j^e + y_j^c \le 1 \\ y_j^e,\ y_j^c \in \{0,1\} \\ x_{ij} \ge 0 \end{array} \right\}$$

where $|V_j| = H_j$

$$\theta_j = \min_{(x_{ij}^h, y_j^{e^h}, y_j^{c^h}) \in V_j} \{ F_j(y_j^{e^h} + y_j^{c^h}) + y_j^{e^h} g_j(\sum_i x_{ij}^h) + y_j^{c^h} f_j(\sum_i x_{ij}^h) + \sum_i (c_{ij} - \mu_i) x_{ij}^h \}$$

Lagrangian Relaxation

- Lagrangian Dual Master Problem:

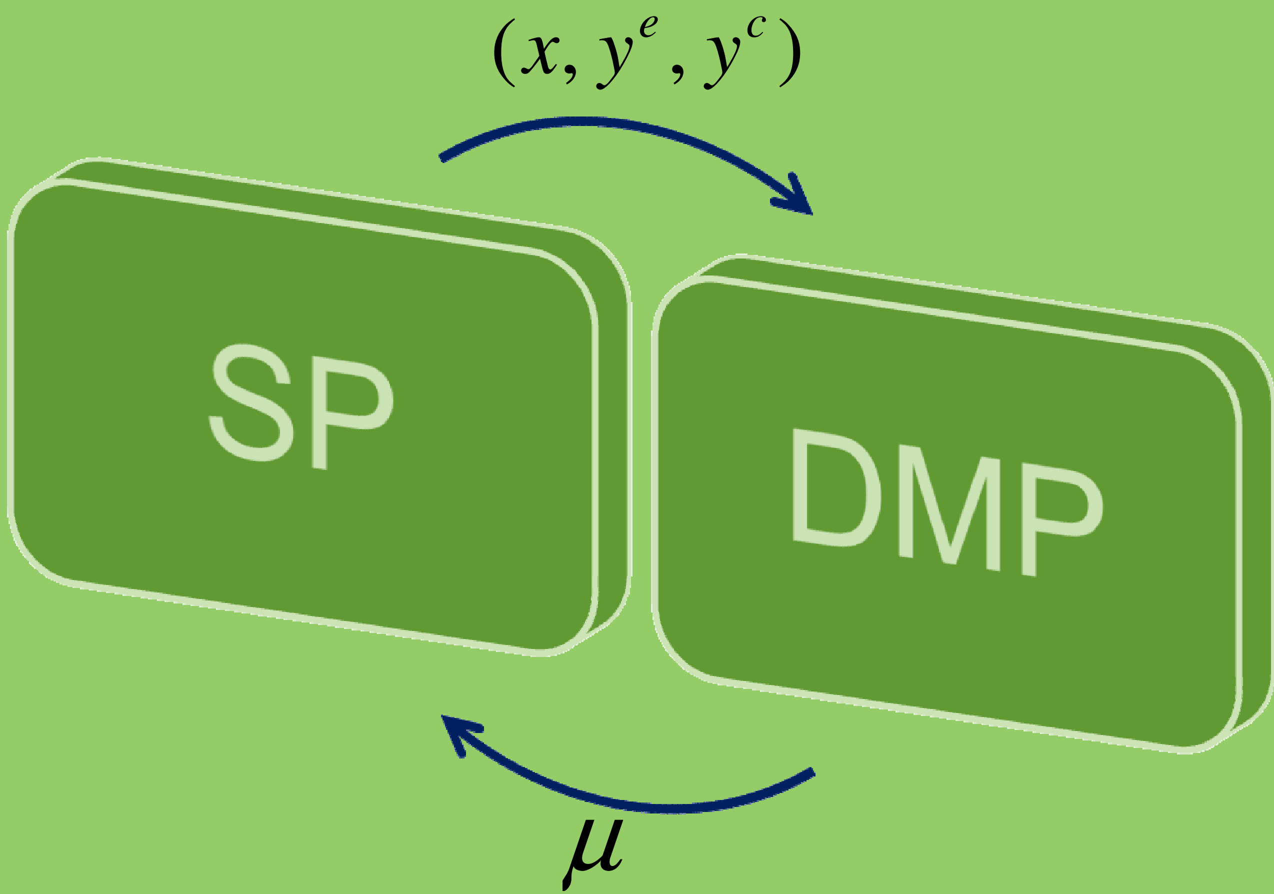
$$\max \sum_i \mu_i d_i + \sum_j \theta_j$$

$$\text{s.t.} \sum_i x_{ij}\mu_i + \theta_j \le F_j(y_j^e + y_j^c) + y_j^e g_j(\sum_i x_{ij}) + y_j^c f_j(\sum_i x_{ij}) + \sum_i c_{ij} x_{ij} \qquad \forall j$$

$$(x_{ij}, y_j^e, y_j^c) \in V_j \qquad \forall j$$

$$\mu_i \ge 0 \qquad \forall i$$

Solution Methodology



Column Generation Heuristics

- How to get a feasible solution
  - Resort to DW, i.e. the dual problem of DMP

$$\min \sum_{j=1}^{n} \sum_{h=1}^{H_j} b_j^h \lambda_j^h$$

$$s.t. \sum_{j=1}^{n} \sum_{h=1}^{H_j} x_{ij}^h \lambda_j^h \geq d_i, \quad \forall \mathrm{i{=}1,\dots,m}$$

$$\sum_{h=1}^{H_j} \lambda_j^h = 1, \quad \forall \mathrm{j{=}1,\dots,n}$$

$$\lambda_j^h \geq 0, \quad \forall j = 1,\dots,n; \forall h = 1,\dots,H_j$$

  - Construct a feasible solution $\overline{x}_{ij} = \sum_{h=1}^{H_j} x_{ij}^h \lambda_j^h$, and $(\overline{y}^e, \overline{y}^c)$ being set accordingly.

Column Generation Heuristics

- How to find a better feasible solution?
  - Embed the whole process into branching tree
  - The branching rule:

$$y_j^e + y_j^c = 1 \text{ or } 0$$

  - Note that the resulting branch-and-price do not guarantee an optimal solution due to
    - Concavity of function *g* in the objective function
    - Partial branching, i.e. no further branching on y^e or y^c

Column Generation Heuristics

- Three Heuristics based on branch-and-price
  - 1st heuristic (*Lagrangian heuristic*)
    - Solve DMP at the root node, construct a feasible solution and stop
  - 2nd heuristic (*column generation heuristic*)
    - branching is performed on open facilities that are operating under economies of scale based on the feasible solution obtained at each node
  - 3rd heuristic (*enhanced column generation heuristic*)
    - branching is performed on all open facilities based on the feasible solution obtained at each node
  - Branching is halted when:
    - Lagrangian lower bound exceeds incumbent
    - Closing any facility will result in an infeasible problem
    - All nodes can be created by the branching rule have been searched

Numerical test

- Test bed
  - A collection of 55 facility location instances of Holmberg et al. 1999.
  - Three types of function $h(q)$
  - 4 cost structures for each type of function (based on fixed costs, production costs, variable costs)
    - The 3 cost components are about the same percentage of total cost
    - Fixed costs dominate
    - Production costs dominate
    - Variable costs dominate

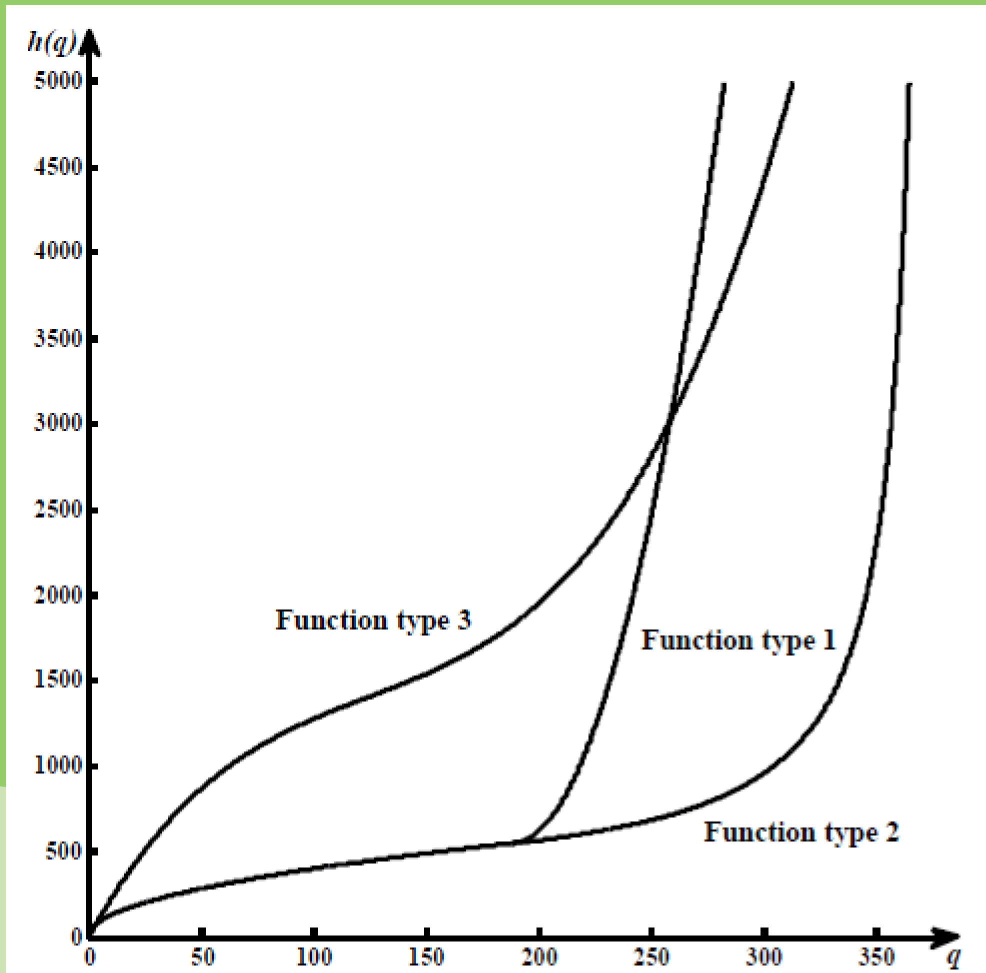
Numerical test

- A summary of instance features

Instance	n	m	K/D	Instance	n	m	K/D	Instance	n	m	K/D	Instance	n	m	K/D	Instance	n	m	K/D
p1	10	50	1.74	p12	10	50	2.06	p23	20	50	3.5	p34	30	150	4.04	p45	20	80	4.14
p2	10	50	1.74	p13	20	50	2.77	p24	20	50	3.5	p35	30	150	4.04	p46	30	70	7.1
p3	10	50	1.74	p14	20	50	2.77	p25	30	150	4.12	p36	30	150	4.04	p47	10	90	1.76
p4	10	50	1.74	p15	20	50	2.77	p26	30	150	4.12	p37	30	150	6.06	p48	20	80	4.06
p5	10	50	1.37	p16	20	50	2.77	p27	30	150	4.12	p38	30	150	6.06	p49	30	70	7.08
p6	10	50	1.37	p17	20	50	2.8	p28	30	150	4.12	p39	30	150	6.06	p50	10	100	1.89
p7	10	50	1.37	p18	20	50	2.8	p29	30	150	3.03	p40	30	150	6.06	p51	20	100	3.98
p8	10	50	1.37	p19	20	50	2.8	p30	30	150	3.03	p41	10	90	2.12	p52	10	100	1.6
p9	10	50	2.06	p20	20	50	2.8	p31	30	150	3.03	p42	20	80	4.99	p53	20	100	3.37
p10	10	50	2.06	p21	20	50	3.5	p32	30	150	3.03	p43	30	70	8.28	p54	10	100	1.52
p11	10	50	2.06	p22	20	50	3.5	p33	30	150	4.04	p44	10	90	1.76	p55	20	100	3.21

Numerical test

- A plot of three function type



Numerical test

- Results of function type 1

	Initial Lagrangean heuristic			Column Generation (CG) heuristic			Enhanced CG heuristic			Cost structure			Facility status		Capacity utilization		
	Gap	Time	Cols	Gap	Time	Cols	Gap	Time	Cols	Fixed	Prod	Var	Opn	ES	Umax	Umin	Uavg
Base Case																	
Max	23.01	390.86	1126	2.65	3205.79	11794	2.65	4988.04	61455	44.45	63.21	45.36	19	3	89.11	71.37	72.80
Avg	6.57	78.60	297.29	0.86	670	3389.69	0.66	516.73	12484.56	27.96	40.04	32.00	11.29	0.49	58.16	50.13	54.60
Min	0	1.51	79	0	2.70	130	0	7.22	558	13.14	30.05	18.77	5	0	51.07	27.46	47.17
Dominant fixed costs																	
Max	53.67	1236.43	1148	3.98	16872.55	29792	2.49	22098.06	1128727	80.77	28.96	21.57	17	1	94.25	78.92	80.89
Avg	12.62	175.59	417.44	0.78	1177.83	3802.67	0.59	1151.18	50375.69	69.91	18.48	11.60	9.69	0.07	64.37	57.60	60.51
Min	0	2.06	94	0	3.09	138	0	10.36	635	58.21	12.46	4.40	4	0	53.22	37.06	51.91
Dominant production costs																	
Max	15.86	797.70	1277	2.26	5085.60	17425	1.33	17864.77	917680	11.14	90.35	14.67	16	1	94.18	71.34	72.80
Avg	4.99	147.94	411.36	0.48	984.70	3895.35	0.42	1339.96	60428.17	6.43	84.01	9.56	9.62	0.02	65.69	58.36	60.86
Min	0	2.09	95	0	3.63	162	0	10.33	724	3.49	78.48	5.12	4	0	55.22	47.02	54.52
Dominant variable costs																	
Max	7.712	620.88	2097	1.067	6552.84	26985	0.64	1143.46	18645	22.73	24.85	87.80	19	16	100	61.39	72.80
Avg	0.860	80.47	347.44	0.232	807.01	4273.04	0.09	138.90	2580.08	10.58	14.51	74.91	13.71	5.42	67.28	27.68	49.55
Min	01	1.00	65	01	2.95	151	0	8.31	653	3.97	7.60	54.30	9	0	50.84	3.21	30.05

Numerical test

- Results of function type 2

	Initial Lagrangean heuristic			Column Generation (CG) heuristic			Enhanced CG heuristic			Cost structure			Facility status		Capacity utilization		
	Gap	Time	Cols	Gap	Time	Cols	Gap	Time	Cols	Fixed	Prod	Var	Opn	ES	Umax	Umin	Uavg
Base Case																	
Max	24.79	2629.58	2134	2.95	13707.28	18496	2.95	2587.74	29518	38.41	50.57	48.36	14	2	81.16	74.82	78.47
Avg	6.50	282.50	545.47	0.93	1485.21	3171.07	0.81	449.54	8123.53	24.16	38.53	37.32	8.60	0.27	75.20	59.30	69.66
Min	0	2.12	78	0	3.74	154	0	17.88	959	13.99	29.60	23.72	4	0	64.52	32.39	55.00
Dominant fixed costs																	
Max	46.63	4288.36	1862	11.67	36122.19	9328	6.26	29853.99	83636	79.36	26.69	29.10	12	1	92.85	92.77	92.81
Avg	12.37	466.15	558.76	2.67	2153.22	1957.98	2.19	1601.85	8949.47	64.13	18.70	17.18	6.73	0.07	87.32	74.18	84.48
Min	0.31	2.62	90	0	2.95	128	0.13	10.05	547	51.41	12.38	6.78	3	0	82.84	43.22	74.39
Dominant production costs																	
Max	10.09	2090.29	1650	2.72	7152.37	15056	1.56	1839.91	43850	10.11	90.50	18.06	15	1	73.63	72.18	72.80
Avg	3.17	322.57	555.31	0.53	810.04	1880.65	0.38	409.11	6494.53	5.75	83.89	10.37	8.60	0.13	67.87	60.87	66.07
Min	0	2.84	101	0	4.49	169	0	13.74	656	3.20	77.05	5.26	4	0	63.05	40.90	60.38
Dominant variable costs																	
Max	4.98	1042.90	2155	1.30	7021.46	25759	0.45	992.71	16493	22.79	22.78	88.82	18	15	91.58	67.57	81.49
Avg	0.60	183.28	520.64	0.15	791.68	3321.73	0.06	181.77	2857.03	10.37	12.64	76.99	12.55	5.65	78.47	23.95	54.50
Min	0	2.53	109	0	5.65	257	0	15.21	860	3.34	7.26	54.43	8	0	60.07	5.43	31.72

Numerical test

- Results of function type 3

	Initial Lagrangean heuristic			Column Generation (CG) heuristic			Enhanced CG heuristic			Cost structure			Facility status		Capacity utilization		
	Gap	Time	Cols	Gap	Time	Cols	Gap	Time	Cols	Fixed	Prod	Var	Opn	ES	Umax	Umin	Uavg
Base Case																	
Max	30.73	2289.22	1699	5.03	16680.61	8341	2.94	3098.23	29843	44.97	65.78	59.56	12	0	100	81.80	95.71
Avg	8.23	238.40	441.49	1.23	1796.57	2186.05	1.02	283.00	5040.42	25.38	33.62	40.99	8.35	0	93.38	70.58	85.41
Min	0	1.33	71	0	3.49	167	0	4.46	191	5.73	10.58	22.34	6	0	76.23	54.80	71.38
Dominant fixed costs																	
Max	61.83	3481.19	1711	21.20	15972.76	8776	10.04	2593.05	96583	88.67	43.47	29.62	10	1	100	99.47	99.89
Avg	17.86	394.71	461.87	4.08	1303.75	1693.09	3.40	464.54	7744.42	63.52	18.71	17.77	6.31	0.11	98.95	75.80	93.59
Min	0	1.28	64	0	2.20	92	0	2.11	92	37.70	2.23	7.53	3	0	85.50	4.57	76.14
Dominant production costs																	
Max	17.99	1309.99	1002	6.17	19674.19	13026	5.97	859.50	25459	37.81	85.17	49.29	13	1	100	93.54	98.76
Avg	7.63	182.17	376.82	1.18	1823.92	2159.22	0.92	172.81	5069.39	17.81	53.21	28.98	8.75	0.02	91.04	74.24	84.80
Min	0	1.68	79	0	5.58	238	0	13.21	748	2.47	19.48	10.23	6	0	75.61	38.66	71.38
Dominant variable costs																	
Max	4.84	1821.17	1709	1.68	5569.64	11960	0.79	661.55	15623	22.72	30.31	91.58	17	11	100	69.00	82.83
Avg	0.84	195.79	500.35	0.21	717.54	2912.24	0.10	135.59	4074.44	9.21	12.45	78.34	11.20	4.04	92.34	26.58	62.25
Min	0	1.11	82	0	4.35	289	0	9.81	764	2.24	2.48	61.35	8	0	71.09	5.43	42.65

Numerical test

- Performance of solution methodology
  - The average gaps for the Lagrangian, the CG, and the enhanced CG heuristics are 6.5%,1.11%, and .89% respectively.
  - The CG heuristic improves the optimality gap by an average of 5.7% at the expense of increasing the computational time and the number of iterations six fold.
  - The CG heuristic consumes on average one third of the CPU time (154.56s vs 570.41s) and one tenth of the number of columns generated.
  - The enhanced CG heuristic does not improve much over the CG heuristic.

Numerical test

- Observations of solution structure
  - The solution contains a number of facilities operating under economies of scale only when variable costs are dominant.
  - When production costs are dominant, the minimum, maximum, and average utilization is very close to one another.
  - Facilities are more congested when fixed costs dominate.

Thanks!