The Legendre-Fenchel conjugate for a product of positive definite quadratic forms

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The Legendre-Fenchel transform (or LF conjugate) of a function $f : X \in \mathbf{R} \cup \{+\infty\}$ is a function defined on the topological dual space of X as

$$y \in X^* \mapsto f^*(y) := \sup_{x \in X} \{\langle y, x \rangle - f(x)\}.$$

Given a function $h : \mathbf{R}^n \to \mathbf{R}$, its LF conjugate is defined as

$$h^*(x) = \sup_{y \in R^n} \{x^T y - h(y)\}.$$

Eg. If $A \succ 0$, let q_A denote the quadratic form

$$q_A(y) = \frac{1}{2} y^T A y.$$

Then

$$q_A^*(y) = \frac{1}{2} y^T A^{-1} y.$$

Hiriart-Urruty & Martinez-Legaz (HM 2003) found the LF conjugate of the following operations:

1. inverting a strictly monotone convex function;
2. post-composing an arbitrary function with a strictly monotone concave function;
3. multiplying two positively valued strictly monotone convex functions.

Remark Their formulas show that, even for one-dimensional functions, the conjugate of the product cannot be expressed in a simple way...

What is the LF conjugate of the product function

$$y \in \mathbf{R}^n \mapsto f(y) := q_A(y)q_B(y),$$

where $A, B \succ 0$?

In his review paper, Hiriart-Urruty (H 2007) posed it as an open question in the field of nonlinear analysis and optimization.

Theorem. Zhao (Z 2010 a): Let $A, B \succ 0$, and let $f(y) = q_A(y) q_B(y)$ be convex. Then at any point $y \in \mathbf{R}^n$, the value of the conjugate $f^*(x)$ is finite, and $f^*(x) = 0$ if $x = 0$; otherwise, if $x \neq 0$,

$$f^*(x) = 3\alpha^{1/3} \left(\frac{x^T (A + \alpha B)^{-1} x}{4} \right)^{2/3},$$

where $\alpha > 0$ satisfies

$$2 = \frac{x^T (A + \alpha B)^{-1} x}{x^T (A + \alpha B)^{-1} A (A + \alpha B)^{-1} x}.$$

Corollary. (Z 2010 a): If $x \neq 0$,

$$f^*(x) = \frac{3}{2}\alpha^{1/3}\left\{\left[\frac{2q_A^*(x) - \alpha x^T(AB^{-1}A + \alpha A)^{-1}x}{4}\right]^{2/3} + \left[\frac{2q_B^*(x) - \alpha x^T(B + \alpha BA^{-1}B)^{-1}x}{4\alpha}\right]^{2/3}\right\}.$$

Let $A_1, \ldots, A_m \succ 0$, $m \geq 2$, and let $g : \mathbb{R}^n \to \mathbb{R}$ be the product $q_{A_1} \ldots q_{A_m}$, i.e.,

$$g(y) = \prod_{i=1}^{m} \frac{1}{2} y^T A_i y := \prod_{i=1}^{m} q_{A_i}(y).$$

When is the $g(y)$ convex?

For $y \neq 0$, the gradient and the Hessian matrix of g are given by,

$$\nabla g(y) = 2g(y) \sum_{i=1}^{m} \frac{A_i y}{y^T A_i y};$$

$$\nabla^2 g(y) = 2g(y) \left(\sum_{i=1}^{m} \frac{A_i}{y^T A_i y} + 2 \sum_{i=1}^{m} \sum_{j \neq i} \frac{A_i y y^T A_j}{y^T A_i y y^T A_j y} \right).$$

Theorem. Zhao (Z 2010 b): Let $A_i \succ 0$, $i = 1, \cdots, m$. If

$$\kappa(A_j^{-\frac{1}{2}} A_i A_j^{-\frac{1}{2}}) \leq \frac{\sqrt{4m-2}+2}{\sqrt{4m-2}-2} \quad \text{for all} \quad i, j = 1, \cdots, m, i \neq j,$$

then the product of m quadratic forms $g = \prod_{i=1}^{m} q_{A_i}$ is convex.

Theorem. Lin & Sinnamon (LS 2011 a): Let $A_i \succ 0$, $i = 1, \cdots, m$. If

$$\kappa(A_j^{-\frac{1}{2}} A_i A_j^{-\frac{1}{2}}) \leq \left(\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} \right)^2 \quad \text{for all} \quad i, j = 1, \cdots, m, i \neq j,$$

then the product of m quadratic forms $g = \prod_{i=1}^{m} q_{A_i}$ is convex. If $m = 2$ the condition is also necessary for the convexity of g.

Remark For $m \geq 2$, $2\sqrt{2m-2} > \sqrt{4m-2}$ so

$$\left(\frac{\sqrt{2m-2}+1}{\sqrt{2m-2}-1} \right)^2 > \frac{\sqrt{4m-2}+2}{\sqrt{4m-2}-2}.$$

Lemma. (LS 2011 a): Suppose $A, B \succ 0$ and let $\kappa = \kappa(A^{-1/2}BA^{-1/2})$. Then for $x, y \in \mathbb{R}^n$, with $y \neq 0$, we have

$$2\frac{x^T Ay}{y^T Ay}\frac{x^T By}{y^T By} \geq -\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^2 \left(\frac{x^T Ax}{y^T Ay} + \frac{x^T Bx}{y^T By}\right).$$

The inequality is sharp.

A connection to generalized Wielandt inequality

Generalized Wielandt inequality Let A be an invertible $n \times n$ matrix. If $x, y \in \mathbb{C}^n$ and $\Phi, \Psi \in [0, \pi/2]$ satisfy

$$|y^* x| \le \|x\| \|y\| \cos \Phi \quad \text{and} \quad \cot(\Psi/2) = \kappa(A) \cot(\Phi/2),$$

then

$$|(Ay)^*(Ax)| \le \|Ax\| \|Ay\| \cos \Psi.$$

Suppose V is a non-trivial real or complex vector space. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be inner products on V and define m, M by,

$$m = \inf_{0\neq v\in V} \frac{\|v\|_2}{\|v\|_1},$$
$$M = \sup_{0\neq v\in V} \frac{\|v\|_2}{\|v\|_1}$$

Here, as usual, $\|v\|_1 = \sqrt{\langle v, v\rangle_1}$ and $\|v\|_2 = \sqrt{\langle v, v\rangle_2}$. If V is a complex space and $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are complex inner products the angles φ and ψ are defined by $0 \leq \varphi \leq \pi$, $0 \leq \psi \leq \pi$,

$$\cos\varphi = \frac{\mathrm{Re}\langle u, v\rangle_1}{\|u\|_1\|v\|_1}, \quad \cos\psi = \frac{\mathrm{Re}\langle u, v\rangle_2}{\|u\|_2\|v\|_2}.$$

Theorem. (LS 2011 b): With the notation given above, we have the following variant of generalized Wielandt inequality

$$\frac{-\chi + \cos\varphi}{1 - \chi\cos\varphi} \leq \cos\psi \leq \frac{\chi + \cos\varphi}{1 + \chi\cos\varphi},$$

where $\chi = (M^2 - m^2)/(M^2 + m^2)$.

Using our notation,

$$2\frac{x^T A y}{y^T A y}\frac{x^T B y}{y^T B y} \geq -\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^2\left(\frac{x^T A x}{y^T A y}+\frac{x^T B x}{y^T B y}\right).$$

can be reformulated as

$$2\cos\varphi\cos\psi \geq -\left(\frac{M-m}{M+m}\right)^2\left(\frac{\|x\|_1}{\|x\|_2}\frac{\|y\|_2}{\|y\|_1}+\frac{\|x\|_2}{\|x\|_1}\frac{\|y\|_1}{\|y\|_2}\right),$$

Corollary. (LS 2011 b)

$$\cos\varphi\cos\psi \geq -\left(\frac{M-m}{M+m}\right)^2.$$

If $\cos\varphi \geq 0$, then $\cos\psi\cos\varphi \geq \frac{-\chi+\cos\varphi}{1-\chi\cos\varphi}\cos\varphi$, otherwise, we have $\cos\psi\cos\varphi \geq \frac{\chi+\cos\varphi}{1+\chi\cos\varphi}\cos\varphi$. In any case, it is easy to verify that

$$\min\left\{\frac{-\chi+\cos\varphi}{1-\chi\cos\varphi}\cos\varphi, \frac{\chi+\cos\varphi}{1+\chi\cos\varphi}\cos\varphi\right\} \geq -\left(\frac{M-m}{M+m}\right)^2.$$

There are no solved problems, there are only more-or-less solved problems. –H. Poincaré

Query Can we find $g^*(x)$ under weaker assumption? When is the product of m positive definite quadratic forms quasi-convex?

Future consideration

1. The LF-conjugate of $\frac{1}{g(x)}$.
2. Characterization of the convexity of $g(x)$.

- J. B. Hiriart-Urruty, Potpourri of conjectures and open questions in nonlinear analysis and optimization, SIAM Review, 49 (2007), pp. 255-273.
- J. B. Hiriart-Urruty and J. E. Martinez-Legaz, New formulas for the LegendreFenchel transform, J. Math. Anal. Appl., 288 (2003), pp. 544555.
- M. Lin, G. Sinnamon, A condition for convexity of a product of positive definite qudaratidc forms, SIAM J. Matrix Anal. Appl., 32 (2011) pp. 457-462.
- M. Lin, G. Sinnamon, The generalized Wielandt inequality in inner product spaces, submitted.
- Y.B. Zhao, The Legendre-Fenchel conjugate of the product of two positive definite quadratic forms, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 1792-1811.

- Y.B. Zhao, Convexity Conditions and the Legendre-Fenchel Transform for the Product of Finitely Many Positive Definite Quadratic Forms , Appl. Math. Optim., 62, (2010), 411-434.
- Y.B. Zhao, Convexity Conditions of Kantorovich Function and Related Semi-infinite Linear Matrix Inequalities, J. Comput. Appl. Math., 235 (2011), pp. 43894403.