Identifiable Sets in Optimization

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Goals

1. Intuitive notion of identifiable sets.
2. Characterizations.
3. Existence, calculus.
4. Connection to previous work (partial smoothness, prox-regularity).
5. Generic existence (semi-algebraic setting).

Preliminaries

For a function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$, a vector v is a Frechét subgradient at $\bar{x}$, denoted $v \in \hat{\partial} f(x)$, if

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$$\partial f(\bar{x}) = \{ \lim_{i \to \infty} v_i : v_i \in \hat{\partial} f(x_i), x_i \to \bar{x}, f(x_i) \to f(\bar{x}) \}.$$

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- For convex f, critical points are global minimizers.
- If f is $\mathbf{C}^1$-smooth, criticality reduces to the classical condition $\nabla f(x) = 0$.

Sensitivity Analysis

Suppose $\bar{v} \in \partial f(\bar{x})$ for a function $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$. Consider the perturbed functions

$$f_v(x) = f(x) - \langle v, x \rangle.$$

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$$0 \in \partial f_v(x) \Longleftrightarrow v \in \partial f(x).$$

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$$0 \in \partial f_v(x) \iff v \in \partial f(x).$$

Thus given $\bar{v} \in \partial f(\bar{x})$, we want to understand how solutions x_v of

$$v \in \partial f(x),$$

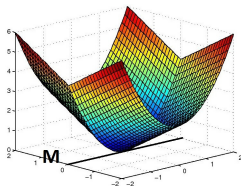vary, as we perturb v near $\bar{v}$.

Motivating example

Figure: $f(x, y) = x^2 + |y|$, $M = \{(t, 0) : -1 < t < 1\}$

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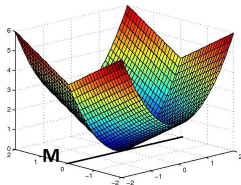

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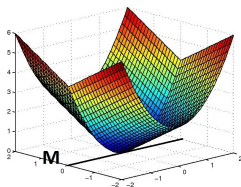- Observe $(0, 0) \in \partial f(0, 0)$.

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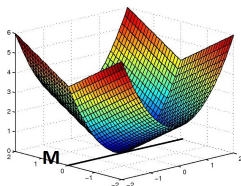## Motivating example

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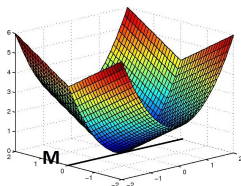
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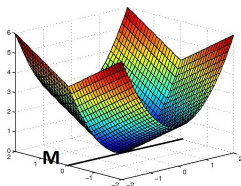
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All perturbed solutions x_v of $v \in \partial f(x)$ lie on $M \implies M$ captures all the sensitivity information!

- Only the restriction $f|_M$ matters!
- Goal: Look for small, well-behaved sets capturing only the essential information.

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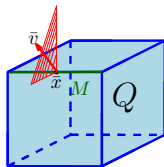where $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and $\bar{v} \in \partial f(\bar{x})$.

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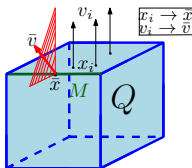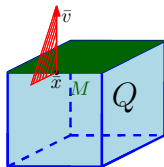where $f \colon \mathbf{R}^n \to \overline{\mathbf{R}}$ and $\bar{v} \in \partial f(\bar{x})$.

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Proposition (Uniform projections)

Suppose M is an identifiable set at $\bar{x}$ for $\bar{v}$.

f is prox-regular at $\bar{x}$ for $\bar{v}$ $\iff$ $f\big|_M$ is prox-regular at $\bar{x}$ for $\bar{v}$.

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Definition

An identifiable set M at $\bar{x}$ for $\bar{v}$ is locally minimal if

$$M' \text{ identifiable at } \bar{x} \text{ for } \bar{v} \Longrightarrow M \subset M', \text{ locally near } \bar{x}.$$

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A strong chain rule is available.

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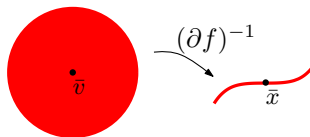This will follow from the following observation.

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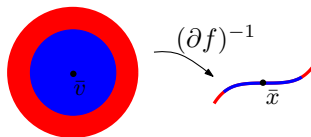This will follow from the following observation.

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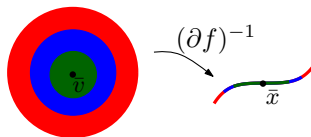This will follow from the following observation.

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Existence and calculus

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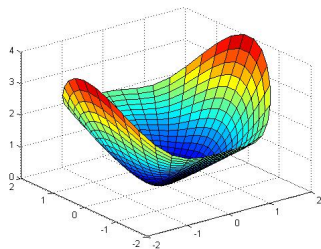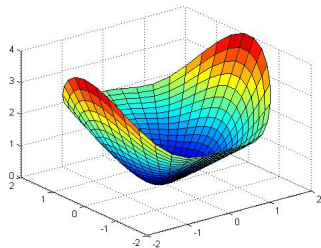
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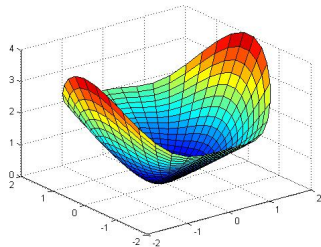Convex functions may fail to admit locally minimal identifiable sets!

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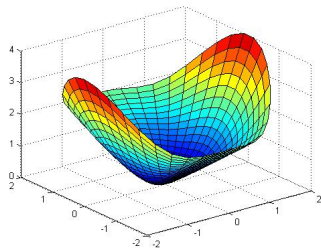
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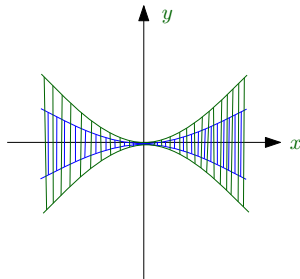Figure: $f(x, y) = \sqrt{x^4 + y^2}$.

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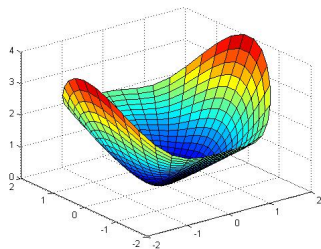
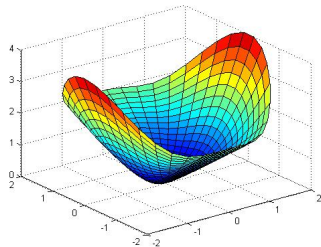Figure: $f(x, y) = \sqrt{x^4 + y^2}$.

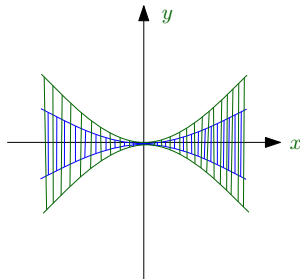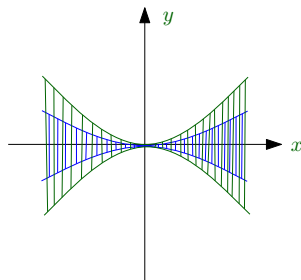Figure: Level sets: $[|\nabla f| < \epsilon]$.

Level sets of $|\nabla f|$ get "pinched". There is no locally minimal identifiable set at $\bar{x} = (0, 0)$ for $\bar{v} = (0, 0)$.

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- Many algorithms would generate iterates that eventually lie on a distinguished subset of Q (subgradient projection Calamai-Moré '87, Newton-like methods Burke-Moré '88, stochastic gradient methods Wright '11).
- One may try to exploit a nice identifiable set, if one exists; perhaps a $\mathbf{C}^2$-manifold.

For simplicity, we work with normal cones to a set $Q \subset \mathbf{R}^n$, that is

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Proposition (Identifiable manifolds)

Identifiable manifolds $M \subset Q$ are automatically locally minimal.

Partly smooth manifolds

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1. **(regularity)** $V \cap N_Q(x) \subset \hat{N}_Q(x)$ for each $x \in U \cap M$,
2. **(sharpness)** $\text{span } \hat{N}_Q(\bar{x}) = N_M(\bar{x})$.
3. **(continuity)** The mapping $x \mapsto V \cap N_Q(x)$ is continuous on M at $\bar{x}$.

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- Implication $\Uparrow$ was mostly proven in Hare-Lewis '04.

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    - *Q is prox-regular at $\bar{x}$ for $\bar{v}$.*
    - *the strong inclusion $\bar{v} \in \mathrm{ri}\, \hat{N}_Q(x)$ holds.*

- Implication $\Uparrow$ was mostly proven in Hare-Lewis '04.
- Equivalence yields intuitive interpretation of partial smoothness, prox-regularity, and nondegeneracy (all sophisticated concepts).

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- This is a difficult question in general, but when an identifiable manifold exists, it is straightforward! Just need to consider curvature of M.

Smooth dependence

Theorem (Sensitivity)

Suppose M is an identifiable manifold at $\bar{x}$ for $\bar{v}$ and $\bar{x}$ is a local maximizer of $\langle \bar{v}, \cdot \rangle$ restricted to M. Then the following are equivalent.

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2. *Second-order decay: There exists $\rho > 0$ such that,*

$$\langle \bar{v}, \bar{x} \rangle \geq \langle \bar{v}, x \rangle + \rho |x - \bar{x}|^2, \text{ for all } x \in M \text{ near } \bar{x}.$$

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- We answer this question in the setting of semi-algebraic sets: represented as finite union of sets, each defined by finitely many polynomial inequalities.
- Large class of sets for which the word typical has a canonical meaning.

Theorem

Suppose $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is semi-algebraic. Consider the perturbed functions

$$f_v(x) := f(x) - \langle v, x \rangle.$$

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3. *every local minimizer x_v of f_v restricted to M_v is a strong local minimizer of f_v, that is*

$$f_v(x) > f_v(x_v) + \rho|x - x_v|^2, \text{ for all } x \text{ near } x_v.$$

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This extends a result of Bolte, Daniilidis, Lewis '11.
Second-order sufficient conditions for optimality are almost necessary (Spingarn-Rockafellar '79).

- Presented the intuitive notion of identifiable sets.
- Showed how these objects relate to previously developed concepts (Partial Smoothness, prox-regularity, etc).
- Existence of identifiable sets (manifolds) leads to significant insight about the problem.

- Could consider identifiability for more general perturbations

$$\min_{x \in \mathbf{R}^n} f(x, v),$$

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Example

Semi-definite cone $\mathbf{S}^n_+$ stratifies into partly smooth manifolds

$$\{X \in \mathbf{S}^n_+ : \operatorname{rank} X = k\}, \ \text{for } k = 0, \dots, n,$$

and is self-dual. Natural duality between these manifolds.

Thank you.