Preprocessing and Reduction for Degenerate Semidefinite Programs

Vris Cheung, Simon Schurr and Henry Wolkowicz

Combinatorics and Optimization
University of Waterloo

13th Midwest Optimization Meeting

Motivation: Loss of Slater's CQ

- *Primal-dual interior-point methods* assume Slater's CQ to hold.

- However, surprisingly many conic opt / SDP relaxations,
  instances arising from applications
  (QAP, GP, strengthened MC, SNL, Molecular Conformation)
  do not satisfy Slater's CQ

- Lack of Slater's CQ results in: unbounded dual solutions;
  theoretical and numerical difficulties

- Solution:
  - theoretical facial reduction (Borwein, Wolkowicz'81[1])
  - preprocess for regularized smaller problem (C.,Schurr, Wolkowicz'11[4])
  - backward stable when # facial reduction step = 1

1. Theory
   - Minimal Faces for Preprocessing
   - Theorem of Alternative to Slater's CQ

2. Algorithm
   - Backward Stability
   - Numerical Results

Semidefinite Program, SDP

$$
\begin{array}{lll}
\text{(P)} & v_{\mathrm{P}} = \sup\limits_{y\in\mathbb{R}^m} \ b^\top y & \text{s.t. } \mathcal{A}^* y - C \preceq 0 \\
\text{(D)} & v_{\mathrm{D}} = \inf\limits_{X\in\mathbb{S}^n} \ \langle C, X\rangle & \text{s.t. } \mathcal{A}(X) = b, \ X \succeq 0
\end{array}
$$

where

- $\mathbb{S}^n$: set of $n \times n$ symm. matrices,
  $\mathbb{S}^n_+ \subset \mathbb{S}^n$: PSD matrices, $\mathbb{S}^n_{++} \subset \mathbb{S}^n_+$: PD matrices;
- $C \in \mathbb{S}^n$, $b \in \mathbb{R}^m$;
- $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is a linear map, with adjoint $\mathcal{A}^*$;
- for $A, B \in \mathbb{S}^n$,
  $A \succeq B$ means $A - B \in \mathbb{S}^n_+$, and $A \succ B$ means $A - B \in \mathbb{S}^n_{++}$.

Semidefinite Program, SDP

$$\text{(P)} \quad v_{\mathrm{P}} = \sup_{y \in \mathbb{R}^m} \; b^\top y \quad \text{s.t.} \; \mathcal{A}^* y - C \preceq 0$$

$$\text{(D)} \quad v_{\mathrm{D}} = \inf_{X \in \mathbb{S}^n} \; \langle C, X \rangle \quad \text{s.t.} \; \mathcal{A}(X) = b, \; X \succeq 0$$

- primal feasible set:

$$\mathcal{F}_{\mathrm{P}}^{Z} := \{Z \succeq 0 : Z = C - \mathcal{A}^* y\}$$

- dual feasible set:

$$\mathcal{F}_{\mathrm{D}}^{X} := \{X \succeq 0 : \mathcal{A}(X) = b\}$$

Semidefinite Program, SDP

$$
\begin{array}{llll}
\text{(P)} & v_{\mathrm{P}} = \sup\limits_{y \in \mathbb{R}^m} & b^\top y & \text{s.t. } \mathcal{A}^* y - C \preceq 0 \\
\text{(D)} & v_{\mathrm{D}} = \inf\limits_{X \in \mathbb{S}^n} & \langle C, X \rangle & \text{s.t. } \mathcal{A}(X) = b, \ X \succeq 0
\end{array}
$$

Assumptions

- $\mathcal{A}$ is onto.
- (P) is feasible; i.e., wlog, $C \succeq 0$.

Semidefinite Program, SDP

$$\text{(P)} \quad v_{\mathrm{P}} = \sup_{y \in \mathbb{R}^m} \; b^\top y \quad \text{s.t.} \; \mathcal{A}^* y - C \preceq 0$$

$$\text{(D)} \quad v_{\mathrm{D}} = \inf_{X \in \mathbb{S}^n} \; \langle C, X \rangle \quad \text{s.t.} \; \mathcal{A}(X) = b, \; X \succeq 0$$

Slater's CQ for (P)

$$\exists\, \hat{y} \;\; \text{s.t.} \; Z = C - \mathcal{A}^* \hat{y} \succ 0$$

Strong Duality

If Slater's CQ holds for (P) and opt(P) is bounded above,
then opt(P) = opt(D) and opt(D) is attained.

Faces of Cones - for Characterization of Optimality

Face

Let F, K be convex cones and $F \subseteq K$.

- $F \trianglelefteq K$: F is a *face* of K, i.e.,

$$x, y \in K \text{ and } x + y \in F \implies x, y \in F.$$

- $F \triangleleft K$: F is a *proper face* of K, i.e.,

$$F \trianglelefteq K \text{ and } F \neq K.$$

Faces of $\mathbb{S}^n_+$

$$F \trianglelefteq \mathbb{S}^n_+ \iff F = Q\mathbb{S}^{\bar{n}}_+ Q^\top$$

for some $Q \in \mathbb{R}^{n \times \bar{n}}$ s.t. $Q^\top Q = I$.

Minimal Faces

Primal and Dual Minimal Faces

$$f_{\mathrm{P}} := \text{face}(\mathcal{F}_{\mathrm{P}}^{Z}) \trianglelefteq \mathbb{S}_{+}^{n} \qquad (\mathcal{F}_{\mathrm{P}}^{Z} : \text{primal feasible set})$$
$$f_{\mathrm{D}} := \text{face}(\mathcal{F}_{\mathrm{D}}^{X}) \trianglelefteq \mathbb{S}_{+}^{n} \qquad (\mathcal{F}_{\mathrm{D}}^{X} : \text{dual feasible set})$$

Why Are We Interested in Minimal Faces?

- Let $Z = C - \mathcal{A}^*y$. Then

$$Z \in \mathbb{S}_{+}^{n} \iff Z \in f_{\mathrm{P}}$$

- $f_{\mathrm{P}} = Q_{\mathrm{fin}} \mathbb{S}_{+}^{\bar{n}} Q_{\mathrm{fin}}^{\top}$ for some $Q_{\mathrm{fin}} \in \mathbb{R}^{n \times \bar{n}}$ satisfying $Q_{\mathrm{fin}}^{\top} Q_{\mathrm{fin}} = I$.
- Slater's CQ holds iff

$$f_{\mathrm{P}} = \mathbb{S}_{+}^{n}.$$

Regularizing (P) Using Minimal Face

Borwein-Wolkowicz'81 [1], $f_{\mathrm{P}} := \mathrm{face}(\mathcal{F}_{\mathrm{P}}^{Z})$

(P) is equivalent to the regularized SDP

$$(\mathrm{P}_{\mathrm{reg}}) \qquad v_{\mathrm{P}} = v_{\mathrm{RP}} := \sup_{y} \left\{ b^{\top} y : \mathcal{A}^{*} y - C \preceq_{f_{\mathrm{P}}} 0 \right\}$$

Lagrangian Dual of $(\mathrm{P}_{\mathrm{reg}})$ Satisfies Strong Duality:

Let

$$(\mathrm{D}_{\mathrm{reg}}) \quad v_{\mathrm{DRP}} := \inf_{X} \left\{ \langle C, X \rangle : \mathcal{A}(X) = b, \ X \succeq_{(f_P)^{*}} 0 \right\}.$$

Then

$$v_{\mathrm{DRP}} = v_{\mathrm{RP}} = v_{\mathrm{P}},$$

and v_{DRP} is attained.

Regularizing (P) Using Minimal Face

(P) is equivalent to $(\mathrm{P_{reg}})$, given by

$$(\mathrm{P_{reg}}) \qquad \sup_y \; b^\top y \text{ s.t. } g^{\prec}(y) \preceq 0, \; g^{=}(y) = 0,$$

where

$$g^{\prec}(y) := Q_{\mathrm{fin}}^\top (\mathcal{A}^* y - C) Q_{\mathrm{fin}},$$

$$g^{=}(y) := \begin{bmatrix} P_{\mathrm{fin}}^\top (\mathcal{A}^* y - C) P_{\mathrm{fin}} \\ P_{\mathrm{fin}}^\top (\mathcal{A}^* y - C) Q_{\mathrm{fin}} \end{bmatrix},$$

$\begin{bmatrix} P_{\mathrm{fin}} & Q_{\mathrm{fin}} \end{bmatrix} \in \mathbb{R}^{n \times n}$ is orthogonal.

Generalized Slater's CQ holds for $(\mathrm{P_{reg}})$:

$$\exists \, \hat{y} \text{ s.t. } g^{\prec}(\hat{y}) \prec 0 \text{ and } g^{=}(\hat{y}) = 0.$$

Slater's CQ and Theorem of Alternative

Slater's CQ for (P)

$$\exists\, \hat{y} \;\text{ s.t. }\; Z = C - \mathcal{A}^*\hat{y} \succ 0$$

Theorem of the Alternative

Assume that $\exists\, \tilde{y}$ s.t. $C - \mathcal{A}^*\tilde{y} \succeq 0$.
Then Slater's CQ holds iff

$$\mathcal{A}(D) = 0,\ \langle C, D\rangle = 0,\ D \succeq 0 \implies D = 0. \qquad (*)$$

Theorem of Alternative and Primal Minimal Face

Alternative to Slater's CQ

$$\mathcal{A}(D) = 0,\ \langle C, D\rangle = 0,\ 0 \neq D \succeq 0 \qquad (*)$$

Determining a proper face $f \lhd \mathbb{S}^n_+$ containing $f_{\rm P}$

- Let D^* solve $(*)$. Then

$$C - \mathcal{A}^* y \succeq 0 \implies \langle C - \mathcal{A}^* y, D^*\rangle = 0,$$

  so $\mathcal{F}^Z_{\rm P} \subseteq \mathbb{S}^n_+ \cap \{D^*\}^\perp$.
- Wlog suppose $\mathrm{rank}\,(D^*) = n - \bar{n} < n$. Write $D^* = PD_+P^\top$, with $D_+ \succ 0$, and $[P\ Q] \in \mathbb{R}^{n\times n}$ orthogonal.

  Then

$$Z = C - \mathcal{A}^* y \succeq 0 \implies \begin{bmatrix} P^\top ZP & P^\top ZQ \\ Q^\top ZP & Q^\top ZQ \end{bmatrix} \in \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}^{\bar{n}}_+ \end{bmatrix},$$

  so $\mathcal{F}^Z_{\rm P} \subseteq Q\mathbb{S}^{\bar{n}}_+ Q^\top \lhd \mathbb{S}^n_+$.

Theorem of Alternative and Primal Minimal Face

Alternative to Slater's CQ

$$\mathcal{A}(D) = 0,\ \langle C, D\rangle = 0,\ 0 \neq D \succeq 0 \qquad (*)$$

Two Forms of Reduced Problem

Let $D^* = PD_+P^\top$ solve $(*)$, with
$D_+ \in \mathbb{S}_+^{n-\bar{n}}$ $(\bar{n} > 0)$, and $[P\ Q] \in \mathbb{R}^{n\times n}$ orthogonal.
Suppose $\mathcal{R}(Q \cdot Q^\top) \cap \mathcal{R}(\mathcal{A}^*)$ is of dim $\bar{m} > 0$. Then (P) is equivalent to

$$\sup_y \left\{ b^\top y : Z = C - \mathcal{A}^* y,\ Q^\top ZQ \succeq 0, P^\top ZP = 0, P^\top ZQ = 0 \right\},$$

or
$$\sup_v \left\{ b^\top (\mathcal{P}v) : \bar{C} - Q^\top(\mathcal{A}^*\mathcal{P}v)Q \succeq_{\mathbb{S}_+^{\bar{n}}} 0 \right\},$$

where $\mathcal{P} : \mathbb{R}^{\bar{m}} \to \mathbb{R}^m$ is a one-one map satisfying

$$\mathcal{R}(\mathcal{A}^*\mathcal{P}) = \mathcal{R}(Q \cdot Q^\top) \cap \mathcal{R}(\mathcal{A}^*).$$

In particular, the linear map $Q^\top(\mathcal{A}^*\mathcal{P}(\cdot))Q$ is one-one.

Auxiliary Problem

Alternative to Slater's CQ

$$\mathcal{A}_C(D) := \begin{pmatrix} \mathcal{A}(D) \\ \langle C, D \rangle \end{pmatrix} = 0, \; 0 \neq D \succeq 0 \qquad (*)$$

How to find a solution D^* of $(*)$?

- use the auxiliary problem

$$(AP) \qquad \min_{\delta, D} \; \delta \; \text{ s.t. } \; \begin{aligned} &\|\mathcal{A}_C(D)\|_2 \leq \delta, \\ &\operatorname{trace}(D) = \sqrt{n}, \\ &D \succeq 0. \end{aligned}$$

- Both (AP) and its dual satisfy Slater's CQ.
- Suppose (δ^*, D^*) is an optimal solution to (AP).
  If $\delta^* = 0$, then D^* solves $(*)$.
  If $\delta^* > 0$, then Slater's CQ holds for (P).

Auxiliary Problem and Strict Complementarity

Auxiliary Problem and Reduced Problem

If $(0, PD_+P^\top)$ (with $D_+ \in \mathbb{S}_+^{n-\bar{n}}$ and $[P\ Q]$ orthogonal) solves

$$(AP) \qquad \min_{\delta, D} \ \delta \ \text{ s.t. } \ \|\mathcal{A}_C(D)\|_2 \le \delta, \quad \operatorname{trace}(D) = \sqrt{n}, \quad D \succeq 0,$$

then

$$v_{\mathrm{P}} = \sup_v \left\{ b^\top(\mathcal{P}v) : \bar{C} - Q^\top(\mathcal{A}^*\mathcal{P}v)Q \succeq_{\mathbb{S}_+^{\bar{n}}} 0 \right\}.$$

Strict Complementarity of (AP) with soln. $(0, PD_+P^\top)$

Slater's CQ holds for $\sup_v \left\{ b^\top(\mathcal{P}v) : \bar{C} - Q^\top(\mathcal{A}^*\mathcal{P}v)Q \succeq_{\mathbb{S}_+^{\bar{n}}} 0 \right\}$

if and only if

(AP) has a strictly complementary optimal p-d solution pair.

Facial Reduction Algorithm

- *iteratively reduces (P) to a smaller equivalent problem*
- *requires at most $n-1$ iterations*

One iteration of facial reduction

Input($\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, $b \in \mathbb{R}^m$, $C \in \mathbb{S}^n$);
Obtain an optimal solution (δ^*, D^*) of

$$(AP) \qquad \min_{\delta, D} \ \delta \ \text{ s.t. } \ \|\mathcal{A}_C(D)\|_2 \le \delta, \ \ \text{trace}(D) = \sqrt{n}, \ \ D \succeq 0;$$

if $\delta^* > 0$, **then**
  STOP; *Slater's CQ holds for* $(\mathcal{A}, b, c)$;
else
  if $D^* \succ 0$, **then**
    STOP; *the unique solution* y *of the equation* $C = \mathcal{A}^* y$ *is optimal*;
  else
    using $D^* = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} D_+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix}$, form the equivalent problem

$$\sup_v \left\{ b^\top(\mathcal{P}v) : \bar{C} - Q^\top(\mathcal{A}^*\mathcal{P}v)Q \succeq_{\mathbb{S}^{\bar{n}}_+} 0 \right\};$$

    update: $\mathcal{A}^* \leftarrow \bar{\mathcal{A}}^* := Q^\top(\mathcal{A}^*\mathcal{P}(\cdot))Q, \quad C \leftarrow \bar{C}, \quad b \leftarrow \bar{b} := \mathcal{P}^* b.$
  end if
end if

Nearby Solutions for Reduced Problem

Assumptions

Let (δ^*, D^*) solve (AP),

$$D^* = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} D_+ & 0 \\ 0 & D_\epsilon \end{bmatrix} \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix} \text{ with } D_+ \succ 0,$$

and $\mathcal{R}(Q \cdot Q^\top) \cap \mathcal{R}(\mathcal{A}^*) \neq \{0\}$.

Nearby Solutions for Reduced Problem

v feas $\Longrightarrow$ dist(v, original fea. region) is small

Given any v s.t. $\bar{C} - \bar{\mathcal{A}}^* v \succeq 0$,
there exists y s.t. $C - \mathcal{A}^* y \succeq 0$ and

$$\|y - \mathcal{P}v\| \leq \xi \cdot \|C - \mathcal{A}^* y\|_\cdot,$$

where $\xi > 0$ depends on $\mathcal{A}$, C, and D^*:

$$\xi := \frac{3\sqrt{2}}{\sigma_{\min}(\mathcal{A})} \left[\alpha(\mathcal{A}, C) \frac{\|D^*\|}{\lambda_{\min}(D_+)} \right]^{1/2},$$

$$\text{and} \quad \alpha(\mathcal{A}, c) := \begin{cases} \frac{\delta^*}{\sigma_{\min}(\mathcal{A})} & \text{if } C \in \mathcal{R}(\mathcal{A}^*), \\ \frac{\delta^*}{\sigma_{\min}(\mathcal{A}_C)} & \text{if } C \notin \mathcal{R}(\mathcal{A}^*). \end{cases}$$

Nearby Solutions for Original Problem

 # facial reduction steps $= 1$:
y feas $\implies$ dist(y, reduced fea. region) is small

Suppose $\exists\, \hat{v}$ s.t. $\delta_2^* := \lambda_{\min}(\bar{C} - \bar{\mathcal{A}}^* \hat{v}) > 0$. Given any y such that

$Z = C - \mathcal{A}^*(y + y_Q) \succeq 0$,
there exists v such that $\bar{C} - \bar{\mathcal{A}}^* v \succeq 0$ and

$$\|y - \mathcal{P}v\| \le \zeta \cdot \|Z\| \cdot \frac{\delta_2^* + \sigma_{\max}(\mathcal{A}^*)\|y - \mathcal{P}\hat{v}\|}{\delta_2^* + \zeta\sigma_{\max}(\mathcal{A}^*)\|Z\|},$$

where $\zeta > 0$ depends on $\mathcal{A}$, C, and D^*:

$$\zeta := \frac{2\sqrt{2}}{\sigma_{\min}(\mathcal{A}^*_{PQ})}\left[\alpha(\mathcal{A}, C)\frac{\|D^*\|}{\lambda_{\min}(D_+)}\right]^{1/2},$$

$$\mathcal{A}^*_{PQ} u := \sum_{i=1}^{m-\bar{m}} u_i (PP^\top \widehat{A}_{\bar{m}+i} PP^\top + PP^\top \widehat{A}_{\bar{m}+i} QQ^\top + QQ^\top \widehat{A}_{\bar{m}+i} PP^\top),$$

and $\mathcal{R}(\mathcal{A}^*) = \operatorname{span}(\widehat{A}_1, \ldots, \widehat{A}_m)$ with $\mathcal{R}(\mathcal{A}^*) \cap \mathcal{R}(Q \cdot Q^\top) = \operatorname{span}(\widehat{A}_1, \ldots, \widehat{A}_{\bar{m}})$.

Numerics With/Without Facial Reduction

Computational results

- obtained using SeDuMi on MATLAB 7.11,
- performed on a machine with Intel Duo Core and 4GB RAM.

- First set of results are from specially generated test problems.
- Second set of results are from randomly generated instances where
  - there is a positive duality gap
  - $v_{\mathrm{P}} = 0$

Numerics With/Without Facial Reduction

Name	n	m	Optval with facial reduction	Optval without facial reduction
Example 1	3	2	0	-6.30238e-016
Example 2	3	2	0	+0.570395
Example 3 ($v_P = 0, v_D = 1$)	3	4	0	+6.91452e-005
Example 4 (infea. dual)	3	3	0	+Inf
Example 5 (Slater's CQ holds)	10	5	+5.02950e+02	+5.02950e+02
Example 6	6	8	+1	+1
Example 7	5	3	0	-2.76307e-012
Example 9a	20	20	0	Inf
Example 9b	100	100	0	Inf

[*Solved using SeDuMi on MATLAB*]

Numerics With/Without Facial Reduction

Name	*n*	*m*	Optval with facial reduction	Optval without facial reduction
RandGen1	10	5	+1.5914e-015	+1.16729e-012
RandGen2	100	67	+1.1056e-010	NaN
RandGen3	200	140	+5.0557e-010	NaN
RandGen4	200	140	+1.02803e-009	NaN
RandGen5	120	45	-5.47393e-015	-1.63758e-015
RandGen6	320	140	+5.9077e-025	NaN
RandGen7	40	27	-5.2203e-029	+5.64118e-011
RandGen8	60	40	-2.03227e-029	NaN
RandGen9	60	40	+5.61602e-015	-3.52291e-012
RandGen10	180	100	+2.47204e-010	NaN
RandGen11	255	150	+7.71685e-010	NaN

[*Solved using SeDuMi on MATLAB*]

Conclusion

- Minimal representations of the data regularize (P); use min. face f_P (and/or implicit rank reduction)
- goal: a backwards stable preprocessing algorithm to handle (feasible) conic problems for which Slater's CQ (almost) fails

J.M. Borwein and H. Wolkowicz, *Characterization of optimality for the abstract convex program with finite-dimensional range*, J. Austral. Math. Soc. Ser. A **30** (1980/81), no. 4, 390–411. MR 83i:90156

______, *Facial reduction for a cone-convex programming problem*, J. Austral. Math. Soc. Ser. A **30** (1980/81), no. 3, 369–380. MR 83b:90121

F. Burkowski, Y-L. Cheung, and H. Wolkowicz, *Semidefinite programming and side chain positioning*, Tech. Report CORR 2011, in progress, University of Waterloo, Waterloo, Ontario, 2011.

Y-L. Cheung, S. Schurr, and H. Wolkowicz, *Preprocessing and reduction for degenerate semidefinite programs*, Tech. Report CORR 2011-02, University of Waterloo, Waterloo, Ontario, 2011.

- N. Krislock, F. Rendl, and H. Wolkowicz, *Noisy sensor network localization using semidefinite representations and facial reduction*, Tech. Report CORR 2010-01, University of Waterloo, Waterloo, Ontario, 2010.
- N. Krislock and H. Wolkowicz, *Explicit sensor network localization using semidefinite representations and facial reductions*, SIAM Journal on Optimization **20** (2010), no. 5, 2679–2708.
- N. Krislock and H. Wolkowicz, *Euclidean distance matrices and applications*, Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications, CORR, no. 2009-06, Springer-Verlag, Waterloo, Ontario, to appear.

Thanks for your attention!

Preprocessing and Reduction for Degenerate Semidefinite Programs

Vris Cheung, Simon Schurr and Henry Wolkowicz

Combinatorics and Optimization
University of Waterloo

13th Midwest Optimization Meeting