A stochastic collocation approach to constrained optimization for random data estimation problems

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Outline

Probabilistic or Stochastic models?

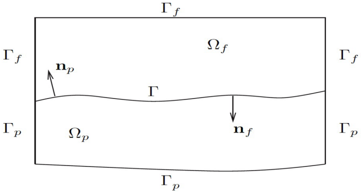
Predicting future events: Fighting the *curse of dimensionality*

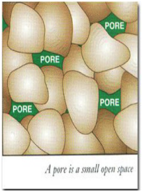
Many applications are affected by a relatively large amount of uncertainty in the input data such as model coefficients, forcing terms, boundary conditions, geometry, etc.

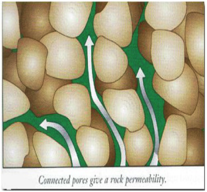
- A simple example includes financial markets where this may depend on the number of economic factors, number of underlying assets or the number of time points/time steps
- More complicated examples include environmental predictions, e.g. subsurface, combustion and turbulent flows, earthquake engineering, biomedical applications, etc.
- The model itself may contain an incomplete description of parameters, processes or fields (not possible or too costly to measure).
- There may be small, unresolved scales in the model that act as a kind of background noise (i.e. macro behavior from micro structure).

All these and many others introduce uncertainty in the model.

Stokes-Darcy problem







The fluid velocity and porous media piezometric head (Darcy pressure) satisfy

$$\begin{cases} u_t - \nu \triangle u + \nabla p = \mathbf{f}_f(x,t), \nabla \cdot u = 0, & \text{in} \quad \Omega_f, \\ S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p, & \text{in} \quad \Omega_p, \end{cases}$$

initial conditions $+$ coupling conditions across I.

<table>
<tr><td>K (m/s):</td><td>1</td><td>10^{-1}</td><td>10^{-2}</td><td>10^{-3}</td><td>10^{-4}</td><td>10^{-5}</td><td>10^{-6}</td><td>10^{-7}</td><td>10^{-8}</td><td>10^{-9}</td><td>10^{-10}</td><td>10^{-11}</td><td>10^{-12}</td></tr>
<tr><td>Permeability</td><td colspan="4">Pervious</td><td colspan="4">Semipervious</td><td colspan="5">Impervious</td></tr>
<tr><td rowspan="2">Soils</td><td colspan="2">Clean gravel</td><td colspan="3">Clean sand or sand and gravel</td><td colspan="4">Very fine sand, silt, loam</td><td colspan="4"></td></tr>
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<tr><td>Rocks</td><td colspan="4"></td><td colspan="3">Oil rocks</td><td colspan="2">Sandstone</td><td colspan="2">Good limestone, dolomite</td><td colspan="2">Breccia, granite</td></tr>
</table>

Typical values of hydraulic conductivity K. *Source: Bear (1979).*

Inverse problems in random media
Thermal, acoustic waves & reaction-diffusion problems

1. 1. Stochastic Optimal Control:

   The advantage of our novel approach over classical methods is that, considering random input data, we control statistical moments (mean value, variance, covariance, etc.) or even the whole probability distribution of physical quantities of interest

2. 2. Parameter Identification for SPDEs: Climate Modeling

   Given a set of measurements $\{\eta(\omega, x, t)\}$ corresponding to some quantity of interest $Q(u)$ (e.g. average temperature) that depends on the solution u of the SPDE, minimize the functional

$$\mathbb{E}\left[\left\|Q\big(u(\omega,\cdot)\big) - \eta(\omega,\cdot)\right\|^2\right],$$

   s.t. the stochastic solution u and the optimal stochastic coefficients satisfying the state system

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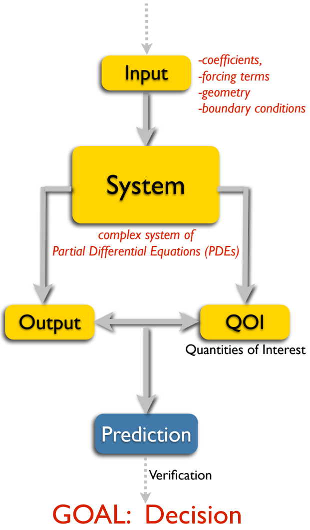
The Computational Stochastic PDE

Forward Problem: From Real World to Predictions to Decisions



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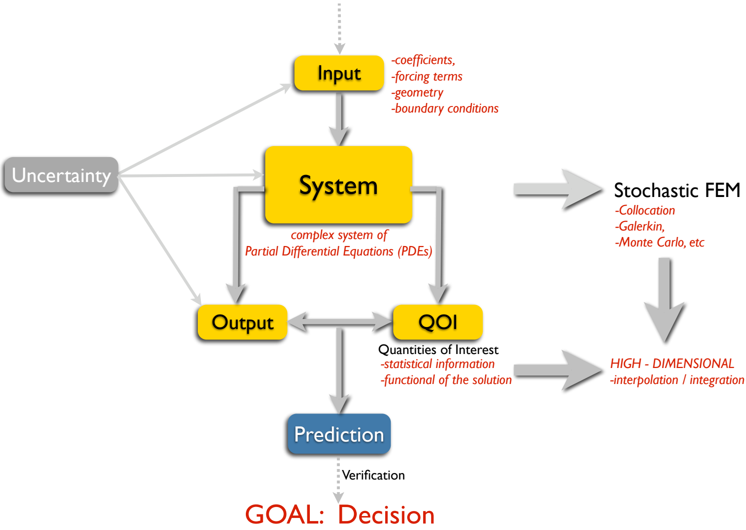

Stochastic formulation of uncertainty

A simplified general setting

Consider an operator $\mathcal{L}$, linear or nonlinear, on a domain $D \subset \mathbb{R}^d$, which depends on some coefficients $a(\omega, x)$ with $x \in D$, $\omega \in \Omega$ and $(\Omega, \mathcal{F}, P)$ a complete probability space. The forcing $f = f(\omega, x)$ and the solution $u = u(\omega, x)$ are random fields s.t.

$$\mathcal{L}(a)(u) = f \quad \text{a.e. in } D \tag{1}$$

equipped with suitable boundary conditions.

A_1. the solution to (1) has realizations in the Banach space $W(D)$, i.e. $u(\cdot, \omega) \in W(D)$ almost surely

$$\|u(\cdot, \omega)\|_{W(D)} \leq C \|f(\cdot, \omega)\|_{W^*(D)}$$

A_2. the forcing term $f \in L^2_P(\Omega; W^*(D))$ is such that the solution u is unique and bounded in $L^2_P(\Omega; W(D))$.

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Examples

Linear and Nonlinear Elliptic SPDEs

Example: The linear elliptic problem

$$\begin{cases} -\nabla\cdot(a(\omega,\cdot)\nabla u(\omega,\cdot)) &= f(\omega,\cdot) \quad \text{in } \Omega\times D,\\ u(\omega,\cdot) &= 0 \quad \text{on } \Omega\times\partial D,\end{cases}$$

with $a(\omega,\cdot)$ uniformly bounded and coercive and $f(\omega,\cdot)$ square integrable with respect to P, satisfies assumptions A_1 and A_2 with $W(D)=H_0^1(D)$.

Example: The nonlinear elliptic problem

Similarly, for $k\in\mathbb{N}^+$,

$$\begin{cases} -\nabla\cdot(a(\omega,\cdot)\nabla u(\omega,\cdot)) + u(\omega,\cdot)|u(\omega,\cdot)|^k &= f(\omega,\cdot) \quad \text{in } \Omega\times D,\\ u(\omega,\cdot) &= 0 \quad \text{on } \Omega\times\partial D,\end{cases}$$

satisfies assumptions A_1 and A_2 with $W(D)=H_0^1(D)\cap L^{k+2}(D)$

Goal of the computations
Stochastic QoI and Inverse problems

Forward Problem: to approximate u or some statistical QoI depending on u:

$$\mathbf{\Phi}_u = \langle \mathbf{\Phi}(u) \rangle := \mathbb{E}\left[\mathbf{\Phi}(u)\right] = \int_{\Omega} \int_D \Phi(u(\omega, x), \omega, x) dx \, \mathbb{P}(d\omega)$$

e.g. $\overline{u} = \mathbb{E}[u](x)$, OR $Var_u = \mathbb{E}[(\widetilde{u})^2](x)$, where $\widetilde{u} = u - \overline{u}$,

OR $\mathbb{P}\{u \geq u_0\} = \mathbb{P}\left[\{\omega \in \Omega \, : \, u(\omega) \geq u_0\}\right] = \mathbb{E}\left[\chi_{\{u \geq u_0\}}\right]$,

OR even statistics of functionals of u, i.e. $\phi(u) = \int_{\Sigma \subset D} u(\cdot, x) dx$

Inverse Problem: Given a set of measurements $\{\eta(\omega, x, t)\}$ corresponding to some statistical QoI $\mathbb{Q}(u)$ (e.g. expectation, inverse CDF, etc) depending on the solution u to the SPDE, minimize the functional

$$\int_D \left[\left\| \mathbb{Q}(u(\omega, \cdot)) - \eta(\omega, \cdot) \right\|^2 \right],$$

s.t. the random u and the optimal stochastic coefficients satisfy the state (1)

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Stochastic optimal control
Part 1. Theory applications to linear SPDEs

- Let $a_{\min}, a_{\max} > 0$ and denote $\mathcal{U}_{ad}$ the set of admissible coefficients s.t.

$$\mathcal{U}_{ad} = \{a \in L^{\infty}(\Omega; L^{\infty}(D)) : \mathbb{P}(a_{\min} \le a(\omega, x) \le a_{\max}, \text{ a.e. } x \in D) = 1\}$$

- Let $\overline{u}(\omega, x)$ and $a(\omega, x)$ be given target and coefficient random fields

The stochastic optimal control problems: minimize the functionals

$$J_1(f, u) = \mathbb{E}\left[\frac{1}{2}\|u(\omega, x) - \overline{u}(\omega, x)\|^2_{L^2(D)} + \frac{\alpha}{2}\|f(\omega, x)\|^2_{L^2(D)}\right] \qquad (P_1)$$

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over all $u \in L^2_P(\Omega; H^1_0(D) \cap H^2(D))$ and $f \in L^2_P(\Omega, L^2(D))$, subject to

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x) \quad \text{in } \Omega \times D$$

Remark: Can replace $\mathbb{E}[\cdot]$ with higher order statistics or even $\Phi^{-1}[\cdot]$, the inverse CDF

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Theorem [GTW11]

$(\widetilde{u}, \widetilde{f})$ is the unique optimal pair in problem (P_1) or (P_2) *if and only if* there exists an adjoint or co-state stochastic process $\xi \in L^2_P(\Omega; H^1_0(D))$ such that

$$\begin{cases} -\nabla \cdot (a(\omega,x)\nabla\xi(\omega,x)) & = \mathbb{F}\big(\widetilde{u}(\omega,x) - \overline{u}(\omega,x)\big) & \text{in } \Omega \times D, \\ \widetilde{f}(\omega,x) & = -\frac{1}{\alpha}\,\xi(\omega,x) & \text{a.e. in } \Omega \times D, \\ \xi(\omega,x) & = 0 & \text{on } \Omega \times \partial D, \end{cases}$$

where $\mathbb{F}[\cdot] = \mathcal{I}_d[\cdot]$ for problem (P_1) and $\mathbb{F}[\cdot] = \mathbb{E}[\cdot]$ for problem (P_2)

- The optimal control $\widetilde{f}$, the optimal state $\widetilde{u}$ and the optimal adjoint state $\xi(\omega,x)$ can be determined from solving the minimization problems (P_1) or (P_2) directly or by solving the system of *couple stochastic PDEs*:

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Stochastic parameter identification

Part 1. Theory for applications to linear SPDEs

- Given a perturbed stochastic observation $\overline{u}$ of the state, determine the coefficient a such that $u(a) = \overline{u}$ (or even $\mathbb{E}[u(a)] = \mathbb{E}[\overline{u}]$ if it exists)
- The diffusion coeff. can not be a Gaussian field (finite probability becomes negative). Use nonlinear transformations, e.g.
$$a(\omega, x) = a_{min} + \arctan(\gamma(\omega, x)), \qquad \gamma \sim N(\mu(\cdot), \varrho(\cdot,\cdot))$$

The stochastic identification problems: minimize the functionals

$$J_3(a,u) = \mathbb{E}\left[\frac{1}{2}\|u(\omega,x) - \overline{u}(\omega,x)\|^2_{L^2(D)} + \frac{\beta}{2}\|a(\omega,x)\|^2_{L^2(D)}\right] \qquad (P_3)$$

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Finite dimensional noise assumption

Transform SBVP to parameterized deterministic BVP

We assume that the random fields $a(\omega, x)$ and $f(\omega, x)$ depend on a finite number of random variables $\mathbf{Y}(\omega) = [Y_1(\omega), \dots, Y_N(\omega)] : \Omega \to \mathbb{R}^N$, namely

$$a_N(\omega, x) = a(\mathbf{Y}(\omega), x), f_N(\omega, x) = f(\mathbf{Y}(\omega), x) \Rightarrow u_N = u(\mathbf{Y}(\omega), x)$$

Finite-D noise: N terms of log-truncated Karhunen-Loève expansion

$$a(\omega, x) \approx a_N(\omega, x) = a_{min} + e^{b_0(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\, b_n(x) Y_n(\omega)}$$

- $(\lambda_n, b_n(x))$ eigenpairs of $T_{Cov[\log(a)]}$ and the random variables Y_n satisfy $\mathbb{E}[Y_n] = 0$, $Cov[Y_n, Y_m] = \delta_{nm}$.

Remark: u (and/or ξ) is in general analytic wrt $\mathbf{Y}$

- $\Gamma_n \equiv Y_n(\Omega) \subset \mathbb{R}$ and $\Gamma^N = \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$ where N is large.
- $[Y_1, Y_2, \dots, Y_N]$ have a joint probability density function $\rho : \Gamma^N \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma^N)$, i.e. for $\mathbf{y} \in \Gamma^N$ (transform the measure $\mathbb{P}$ to $\mathbb{R}^N$)

$$\mathbb{P}\left(Z \in \gamma \subset \Gamma^N\right) = \int_\gamma \rho(\mathbf{y})\, d\mathbf{y} \quad \Rightarrow \quad \mathbb{E}[u](x) = \int_{\Gamma^N} u(\mathbf{y}, x) \rho(\mathbf{y}) d\mathbf{y}$$

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Finite dimensional noise assumption

Transform SBVP to parameterized deterministic BVP

We assume that the random fields $a(\omega, x)$ and $f(\omega, x)$ depend on a finite number of random variables $\mathbf{Y}(\omega) = [Y_1(\omega), \ldots, Y_N(\omega)] : \Omega \to \mathbb{R}^N$, namely

$$a_N(\omega, x) = a(\mathbf{Y}(\omega), x), f_N(\omega, x) = f(\mathbf{Y}(\omega), x) \Rightarrow u_N = u(\mathbf{Y}(\omega), x)$$

Finite-D noise: N terms of log-truncated Karhunen-Loève expansion

$$a(\omega, x) \approx a_N(\omega, x) = a_{min} + e^{b_0(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\, b_n(x) Y_n(\omega)}$$

- $(\lambda_n, b_n(x))$ eigenpairs of $T_{Cov[\log(a)]}$ and the random variables Y_n satisfy $\mathbb{E}[Y_n] = 0$, $Cov[Y_n, Y_m] = \delta_{nm}$.

Remark: u (and/or ξ) is in general analytic wrt $\mathbf{Y}$

- $\Gamma_n \equiv Y_n(\Omega) \subset \mathbb{R}$ and $\Gamma^N = \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$ where N is large.
- $[Y_1, Y_2, \ldots, Y_N]$ have a joint probability density function $\rho : \Gamma^N \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma^N)$, i.e. for $\mathbf{y} \in \Gamma^N$ (transform the measure $\mathbb{P}$ to $\mathbb{R}^N$)

$$\mathbb{P}\left(Z \in \gamma \subset \Gamma^N\right) = \int_\gamma \rho(\mathbf{y})\, d\mathbf{y} \quad \Rightarrow \quad \mathbb{E}[u](x) = \int_{\Gamma^N} u(\mathbf{y}, x) \rho(\mathbf{y}) d\mathbf{y}$$

Applications to stochastic parameter iD

Parametrized equivalent (deterministic) formulation

- $\widehat{a}(\omega, x) = \max\{a_{\min}, \min\{-\frac{1}{\beta}\nabla\widetilde{u}(\omega, x)\nabla\xi(\omega, x), a_{\max}\}\}$ a.e. in $\Omega\times D$

Strong formulation: find $u(\mathbf{y}, x), \xi(\mathbf{y}, x) \in H_\rho = L^2_\rho(\Gamma^N; H^1_0(D))$ s.t.

$$\begin{cases} -\nabla\cdot(\widehat{a}(\mathbf{y}, x)\nabla u(\mathbf{y}, x)) = f(\mathbf{y}, x) & \text{for a.e. } x \in D, \\ -\nabla\cdot(\widehat{a}(\mathbf{y}, x)\nabla \xi(\mathbf{y}, x)) = \mathbb{E}\big(u(\mathbf{y}, x) - \overline{u}(\mathbf{y}, x)\big) & \text{for a.e. } x \in D, \end{cases}$$

where $\mathbf{y} \in \Gamma^N \subset \mathbb{R}^N$ and $x \in \overline{D}$

Weak formulation: find $u, \xi \in H_\rho$ s.t., $\forall v \in H_\rho$

$$\begin{cases} \int_{\Gamma^N} (\widehat{a}\,\nabla u, \nabla v)_{L^2(D)}\rho(\mathbf{y})\, d\mathbf{y} = \int_{\Gamma^N} (f, v)_{L^2(D)}\rho(\mathbf{y})\, d\mathbf{y} \\ \int_{\Gamma^N} (\widehat{a}\,\nabla \xi, \nabla v)_{L^2(D)}\rho(\mathbf{y})\, d\mathbf{y} = \int_{\Gamma^N} ((u - \overline{u}), v)_{L^2(D)}\rho(\mathbf{y})\, d\mathbf{y} \end{cases}$$

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Stochastic polynomial approximation

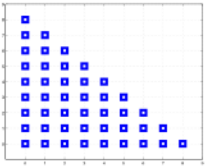
Approximating spaces: Let $\mathcal{T}_h$ be a triangulation of D and $\mathbf{p} = (p_1, \ldots, p_N)$

- $W^h(D) \subset W(D)$ contains cont. piecewise polynomials defined in $\mathcal{T}_h$
- $\mathcal{J}(p) \subset \mathbb{N}^N$ is an index set and define the multivariate polynomial space

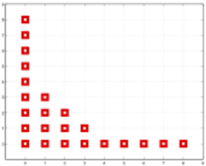
$$\mathbb{P}_{\mathcal{J}(p)}(\Gamma^N) = \text{span}\left\{\prod_{n=1}^N y_n^{p_n}, \quad \text{with } \mathbf{p} \in \mathcal{J}(\mathbf{p})\right\} \subset L^2_\rho(\Gamma^N)$$

E.g. Tensor products: $\max_n \alpha_n p_n \leq p$ (Intractable for large N),
Total degree: $\sum_{n=1}^N \alpha_n p_n \leq p$, Hyperbolic cross: $\prod_{n=1}^N (p_n + 1)^{\alpha_n} \leq p + 1$,
Smolyak space: $\sum_{n=1}^N \alpha_n f(p_n) \leq f(p)$ where $f(p) = \begin{cases} 0, & p = 0 \\ 1, & p = 1 \\ \lceil \log_2(p) \rceil, & p \geq 2 \end{cases}$

TD: $\sum_n p_n \leq p$



HC: $\prod_n (p_n + 1) \leq (p + 1)$



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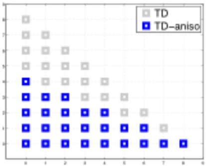
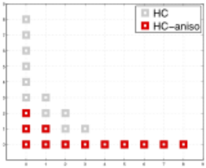
ATD: $\sum_n \alpha_n p_n \leq p$



AHC: $\prod_n (p_n+1)^{\alpha_n} \leq p+1$



Stochastic polynomial approximation
Fully discrete approximations

Fully discrete approximations $u_p \in \mathbb{P}_{\mathcal{J}(p)}(\Gamma^N) \otimes W^h(D)$:

$$u_p(\mathbf{y}, x) = \sum_{k=1}^{M} u_k(x)\psi_k(\mathbf{y}) \quad \text{with } u_k \in W^h(D)$$

- $M = \dim\left[\mathbb{P}_{\mathcal{J}(p)}(\Gamma^N)\right]$ and $\{\psi_k\}_{k=1}^M$ form a basis for $\mathbb{P}_{\mathcal{J}(p)}(\Gamma^N)$, e.g. multivariate Legendre, Hermite, etc.

Stochastic collocation FEM: $\Longleftrightarrow u^h(y_k) := \pi^h u(y_k) \in W^h(D)$, $y_k \in \Gamma^N$

However, we also want to recover $u : \Gamma^N \to \mathbb{R}$ approximated by the fully discrete SCFEM $u_p \in \mathbb{P}_{\mathcal{J}(p)}(\Gamma^N) \otimes W^h(D)$:

$$u_p(\mathbf{y}, \cdot) = \sum_{k \in \mathcal{K}} u^h(y_k, \cdot) l_k^{\mathbf{P}}(\mathbf{y}) \quad \left(= \mathcal{I}_p^{(N)}\left[u^h\right]\right).$$

$\Rightarrow$ Moments become simple *interpolatory* quadrature approx.:

$$\mathbb{E}[u] \approx \mathbb{E}[u_p] \approx \sum_{k \in \mathcal{K}} u^h(y_k, \cdot) \underbrace{\rho(y_k) \int_{\Gamma^N} l_k^{\mathbf{P}}(\mathbf{y}) d\mathbf{y}}_{w_k}$$

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The generalized SC (gSC) FEM

Sparse approximation for high-dimensional problems

1. Given a (mixed) sequence of $1d$ polynomial interpolant operators $\mathscr{U}_n^{m(i)} : C^0(\Gamma_n) \to \mathbb{P}_{m(i)-1}(\Gamma_n)$ with increasing number of points
   -The i-th interpolant uses $m(i)$ abscissas $\vartheta_n^i = \{y_{n,1}^i, \ldots, y_{n,m_i}^i\} \subset \Gamma_n$
   -Typical choice is $m(i) \approx 2i$ (double the points on each level)
2. Take differences of consecutive hierarchical operators:
$$\Delta_n^i = \mathscr{U}_n^{m(i)} - \mathscr{U}_n^{m(i-1)}, \qquad \mathscr{U}^{m(0)} = 0$$
3. For integers $p \in \mathbb{N}$ the *gSC* sparse approximation is given by:
$$u_p = \mathscr{A}^{m,g}(p, N)(u) = \sum_{\mathbf{i} \in \mathbb{N}^N : g(\mathbf{i}) \le p} \left(\Delta_1^{m(i_1)} \otimes \cdots \otimes \Delta_N^{m(i_N)}\right)(u)$$
   where $\mathbf{i} = (i_1, \ldots, i_N) \in \mathbb{N}_+^N$ is a multi-index and $g : \mathbb{N}^N \to \mathbb{N}$ a strictly increasing function

   Can build sparse approximations (grids) corresponding to any polynomial space $\mathbb{P}_{\mathcal{J}(p)}(\Gamma^N)$, e.g. total degree, hyperbolic cross, Smolyak, etc.

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The gSCFEM sparse approximation
Equivalent formulation

$$u_p = \mathscr{A}^{m,g}(p, N)(u) = \sum_{\mathbf{i}\in\mathbb{N}^N : g(\mathbf{i})\leq p} c(\mathbf{i}) \left(\mathscr{U}_1^{m(i_1)} \otimes \cdots \otimes \mathscr{U}_N^{m(i_N)} \right)(u) \qquad (2)$$

with $c(\mathbf{i}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^N \\ g(\mathbf{i}+\mathbf{j})\leq p}} (-1)^{|\mathbf{j}|_1}$

- linear combination of tensor product grids, with a relatively low number of points (but maintain the **asymptotic accuracy**)

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Example: Anisotropic Smolyak sparse grids [NTW08ab]. Let $m(i) = 2^{i+1} - 1$, $g(\mathbf{i}) = \sum_{n=1}^N \alpha_n (i_n - 1)$ and define the set

$$Y_{\boldsymbol{\alpha}}(p,N) := \left\{\mathbf{i}\in\mathbb{N}_+^N, \mathbf{i}\ge\mathbf{1} \;:\; g(\mathbf{i}) \le p\alpha_{\min}\right\}$$

Then from (2) for integers $p\in\mathbb{N}$ we get that

$$\mathscr{A}(p,N) = \sum_{\mathbf{i}\in Y_{\boldsymbol{\alpha}}(p,N)} c(\mathbf{i}) \left(\mathscr{U}_1^{m(i_1)} \otimes \cdots \otimes \mathscr{U}_N^{m(i_N)}\right)$$

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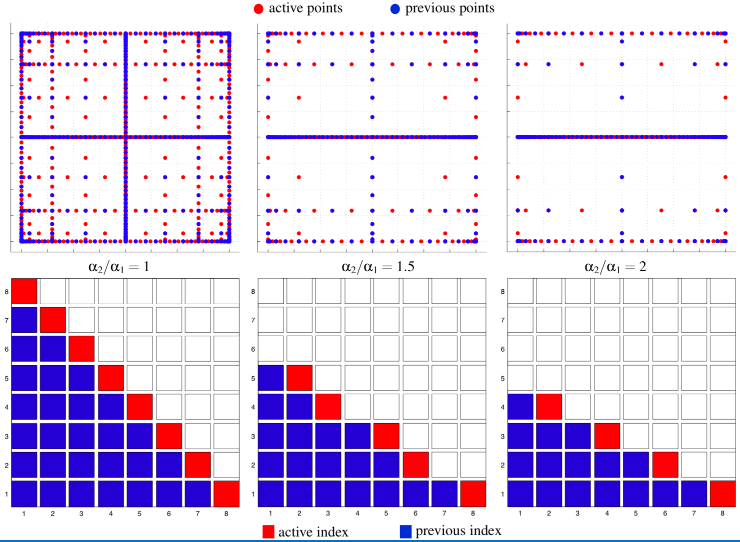
To compute the approximation u_p we interpolate on the "sparse grid"

$$\mathscr{H}_\alpha(p,N) = \bigcup_{\mathbf{i}\in Y_\alpha(p,N)} \left(\vartheta_1^{i_1} \times \cdots \times \vartheta_N^{i_N} \right)$$

Generated C-C anisotropic sparse grids

Correspondng indices $(i_1, i_2) \in Y_\alpha(7,2)$



Convergence of gSC approximation

Recall: convergence of sparse isotropic SC: $\varepsilon_{SG}(M) \approx \mathcal{O}\left(M^{-\sigma/(log(2N))}\right)$

Theorem [NTW08b]

For functions $u \in C^0(\Gamma^N; W(D))$, the **anisotropic** sparse approximation approach satisfies:

$$\varepsilon(M) = \|u - u_p\|_{L^2_{\rho,N}} \leq C(\alpha, N) M^{-\sigma/\mathscr{G}(\sigma,N)},$$

where $\mathscr{G}(\sigma, N) = \sum_{n=1}^{N} \sigma/\hat{\varrho}_n$

- An analogous result holds for the Gaussian abscissas
- $\sigma = r$ when $u \in \mathcal{W}_r^{(N)}$ (bdd mixed derivatives of order r)
- For highly isotropic problems $\mathscr{G}(\sigma, N) \approx \log(2N)$
- For highly anisotropic problems, i.e. the larger the ratio $\alpha_{max}/\alpha_{min}$ becomes, the smaller the constant $C_3(\alpha, N)$ and $\mathscr{G}(\sigma, N) << \log(2N)$
- $\lim_{N\to\infty} C(\alpha, N) = C^* < \infty$ **NO curse of dimensionality!**

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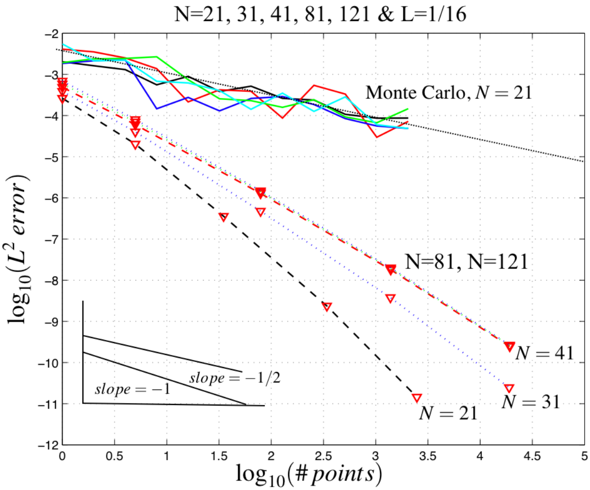
Convergence in $\mathbb{E}[u]$ for nonlinear SPDE

$N = 21, \ldots, 121, \ldots \infty$ random variables



Error estimates for (P_4)

Fully discrete stochastic identification approximation

If in the state equation we use conductivity coefficients of the following form

$$\frac{1}{2}(\kappa_{min} + \kappa_{max}) + \frac{1}{\pi}(\kappa_{min} - \kappa_{max})\arctan(\kappa(\omega, x)), \tag{3}$$

then the set $\mathcal{U}_{ad}$ of admissible conductivity coefficients becomes

$$\mathcal{U}_{ad} = L^\infty(\Omega; L^\infty(D))$$

and the optimality condition corresponding to the cost functional (J_4) subject to

$$-\nabla\cdot\Big(\{\frac{1}{2}(\kappa_{min} + \kappa_{max}) + \frac{1}{\pi}(\kappa_{min} - \kappa_{max})\arctan(\kappa(\omega, x))\}\nabla u(\omega, x)\Big) = f(\omega, x)$$

writes now

$$\frac{1}{1 + (\mathring{\kappa}(\omega, x))^2}\nabla\eta(\omega, x)\nabla\mathring{u}(\omega, x) + \beta\mathring{\kappa}(\omega, x) = 0 \qquad \text{a.e. in } \Omega\times D, \tag{4}$$

where $\eta \in L^2_P(\Omega; H^1_0(D))$ is the solution of

$$-\nabla\cdot\big(\{\frac{1}{2}(\kappa_{min}+\kappa_{max})+\frac{1}{\pi}(\kappa_{min}-\kappa_{max})\arctan(\mathring{\kappa}(\omega, x))\}\nabla\eta(\omega, x)\big) = \mathbb{E}\big(\mathring{u}(\cdot, x) - \overline{u}(\cdot, x)\big)$$

The nonlinear problems to be considered are of the type

$$F(\beta, \varphi) \equiv \varphi + TG(\beta, \varphi) = 0, \tag{5}$$

where $T \in \mathcal{L}(Y, X)$, G is a C^2 mapping from $\Lambda \times X$ into Y, and X, Y are Banach spaces, and Λ is a compact interval of $\mathbb{R}$.

Definition

We say that $(\lambda, \varphi(\lambda)) : \lambda \in \Lambda$ is a branch of solutions of (5) if $\lambda \to \varphi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda, \varphi(\lambda)) = 0$. The branch is called a *nonsingular branch* if we also have that $D_\varphi F(\lambda, \varphi)$ is an isomorphism from X into X for all $\lambda \in \Lambda$.

Approximations are defined by introducing an approximating operator $T^{N,h,M} \in \mathcal{L}(Y, X^{N,h,M})$, with $X^{N,h,M} \subset X$. Then we seek $\varphi^{N,h,M} \in X^{N,h,M}$ such that

$$\varphi^{N,h,M} + T^{N,h,M} G(\lambda, \varphi^{N,h,M}) = 0. \tag{6}$$

Suppose that (5) has a branch of nonsingular solutions $\{(\lambda, \varphi(\lambda)) : \lambda \in \Lambda\}$. We make the following assumptions. First, there is another Banach space Z contained in Y, with continuous imbedding, such that

$$D_\varphi G(\lambda, \varphi) \in \mathcal{L}(X, Z), \quad \forall \lambda \in \Lambda, \forall \varphi \in X. \tag{7}$$

Concerning the operator $T^{N,h,M}$, we assume that

$$\lim_{h \to 0, N, M \to \infty} \|(T^{N,h,M} - T)g\|_X = 0, \quad \forall g \in Y, \tag{8}$$

and

$$\lim_{h \to 0, N, M \to \infty} \|T^{N,h,M} - T\|_{L(Z,X)} = 0. \tag{9}$$

Note that if $Z \subset Y$ is compact embedding, then $D_\varphi F(\lambda, \varphi) \in \mathcal{L}(X, X)$ is compact.

Theorem

Let X, Y be Banach spaces and Λ a compact set of $\mathbb{R}$. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y, D^2G is bounded on all bounded subsets of $\Lambda \times X$. Assume that (7)-(9) hold and that $\{(\lambda, \varphi(\lambda)) : \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (5). Then there exists a neighborhood $\mathcal{O}$ of the origin in X and (for $h \le h_0$ small enough and N, M large enough) a unique C^2 function $\lambda \mapsto \varphi^{N,h,M} \in X$ such that

$$\{(\lambda, \varphi^{N,h,M}(\lambda))\} \text{ is a branch of nonsingular solutions of } (6), \tag{10}$$

$$\varphi^{N,h,M}(\lambda) - \varphi(\lambda) \in \mathcal{O}. \tag{11}$$

Moreover, there exists a constant C independent of h, p and λ such that

$$\|\varphi^{N,h,M}(\lambda) - \varphi(\lambda)\|_X \le C\|(T^{N,h,M} - T)G(\lambda, \varphi(\lambda))\|_X, \qquad \forall \lambda \in \Lambda. \tag{12}$$

We shall recast the optimality system corresponding to (P_4) and its discretization (16) into a form that fits the above framework. We note that the theorem holds without major modification.

Let

$$W(D) := W_0^{1,6}(D), \qquad W'(D) := W^{-1,\frac{6}{5}}(D),$$

and define the spaces

$$X = \left(L_P^6(\Omega; W(D))\right)^2, \quad Y = \left(L_P^6(\Omega; W'(D))\right)^4, \quad Z = \left(L_P^6(\Omega; L^2(D))\right)^4.$$

We also introduce the following approximating spaces

$$X^N := \left(L_\rho^6(\Gamma^N; W(D))\right)^2, \qquad X^{N,h} := \left(L_\rho^6(\Gamma^N; W_h(D))\right)^2,$$

$$X^{N,h,M} := \left\{ v \in C^0(\Gamma^N; W_h(D)) : v(\gamma, x) = \sum_{m=1}^{M} v_m(x)\ell_m(y), \{v_m\}_{m=1}^{M} \in W_h(D) \right\}^2.$$

Let the operator $T \in \mathcal{L}(Y, X)$ be defined as follows: $T(\theta, \Theta, f, \tau) = (v, \zeta)$ iff

$$\int_\Omega \int_D \frac{1}{2}(\kappa_{min} + \kappa_{max}) \nabla v(\omega) \cdot \nabla z \, dx dP = \int_\Omega \int_D -\frac{1}{\pi}(\kappa_{min} - \kappa_{max}) \theta(\omega) \cdot \nabla z + f(\omega) z \, dx dP,$$

$$\int_\Omega \int_D \frac{1}{2}(\kappa_{min} + \kappa_{max}) \nabla \zeta(\omega) \cdot \nabla z \, dx dP = \int_\Omega \int_D -\frac{1}{\pi}(\kappa_{min} - \kappa_{max}) \Theta(\omega) \cdot \nabla z + \tau(\omega) z \, dx dP,$$

for all $z \in W(D)$.

We denote now by $\mathcal{R}(\phi)$ the unique real root of

$$\mathcal{R}(1+\mathcal{R}^2) = \phi,$$

namely

$$\mathcal{R}(\phi) = \sqrt[3]{\frac{\phi}{2} + \sqrt{\frac{\phi^2}{4} + \frac{1}{27}}} + \sqrt[3]{\frac{\phi}{2} - \sqrt{\frac{\phi^2}{4} + \frac{1}{27}}}$$

and remark that the optimality condition writes as

$$\kappa(\omega, x) = \mathcal{R}(-\frac{1}{\beta}\nabla u(\omega, x)\nabla\eta(\omega, x)) \qquad \text{a.e. in } \Omega \times D. \tag{13}$$

Next we define the nonlinear mapping $G : \Lambda \times X \to Y$ as follows:

$$G(\lambda, (v, \zeta)) = (-\arctan(\mathcal{R}(-\frac{1}{\beta}\nabla v \nabla\zeta))\nabla v, -\arctan(\mathcal{R}(-\frac{1}{\beta}\nabla v \nabla\zeta))\nabla\zeta, f, \mathbb{E}(v - \overline{u})),$$

where $f \in L^2_P(\Omega; L^2(D))$ is the given data.
Clearly the solution to the optimality system is equivalent to

$$(u, \eta) + TG(\lambda, (u, \eta)) = 0.$$

The discrete operator $T^{N,h,M} \in \mathcal{L}(Y, X^{N,h,M})$ is defined a similar manner, by $T^{N,h,M}(\theta, \Theta, f, \tau) = (v^{N,h,M}, \zeta^{N,h,M})$, where $(v^{N,h,M}, \zeta^{N,h,M})$ satisfies (16). First let denote by $(v^N, \zeta^N) \in X^N$ the Karhunen-Loève truncation solution, corresponding to $(\theta^N, \Theta^N, f^N, \tau^N)$, the K-L truncation of the $(\theta, \Theta, f, \tau)$:

$$\int_\Omega\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla v^N(\omega)\cdot\nabla z\,dxdP = \int_\Omega\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\theta^N(\omega)\cdot\nabla z + f^N(\omega) z$$
$$\int_\Omega\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla \zeta^N(\omega)\cdot\nabla z\,dxdP = \int_\Omega\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\Theta^N(\omega)\cdot\nabla z + \tau^N(\omega) z \tag{14}$$

and equivalently

$$\begin{aligned}
&\int_{\Gamma^N}\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla v^N(\gamma)\cdot\nabla z\,dxd\rho \\
&=\int_{\Gamma^N}\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\theta^N(\gamma)\cdot\nabla z + f^N(\gamma) z\,dxd\rho, \\
&\int_{\Gamma^N}\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla \zeta^N(\gamma)\cdot\nabla z\,dxd\rho \\
&=\int_{\Gamma^N}\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\Theta^N(\gamma)\cdot\nabla z + \tau^N(\gamma) z\,dxd\rho,
\end{aligned}
\qquad \text{for all } z \in W(D).$$

Secondly, let denote by $(v^{N,h}, \zeta^{N,h}) \in X^{N,h}$ the solution to the following problem

$$
\begin{aligned}
&\int_{\Gamma^N}\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla v^{N,h}(\gamma)\cdot\nabla z_h\, dxd\rho \\
&\quad = \int_{\Gamma^N}\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\theta^N(\gamma)\cdot\nabla z_h + f^N(\gamma) z_h\, dxd\rho, \\
&\int_{\Gamma^N}\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla \zeta^{N,h}(\gamma)\cdot\nabla z_h\, dxd\rho \\
&\quad = \int_{\Gamma^N}\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\Theta^N(\gamma)\cdot\nabla z_h + \tau^N(\gamma) z_h\, dxd\rho,
\end{aligned}
\tag{15}
$$

for all $z_h \in W_h(D)$.

Thirdly, we use Lagrange interpolation to approximate the solution to (15) in Γ^N

$$v^{N,h,M}(\gamma,x) \approx \sum_{m=1}^{M} v^{N,h}(\gamma^m,x)\ell_m(\gamma), \qquad \zeta^{N,h,M}(\gamma,x) \approx \sum_{m=1}^{M} \zeta^{N,h}(\gamma^m,x)\ell_m(\gamma),$$

for any interpolating nodes, and multi-index m.
Finally, the discrete operator $T^{N,h,M}(\theta,\Theta,f,\tau) = (v^{N,h,M},\zeta^{N,h,M})$ is defined as

$$\begin{aligned}
&\sum_{m=1}^{M}\Big(\int_{\Gamma^N}\ell_m(\gamma)d\rho\Big)\Big(\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla v^{N,h,M}(\gamma^m,x)\cdot\nabla z_h\,dx\Big)\\
&\quad=\sum_{m=1}^{M}\Big(\int_{\Gamma^N}\ell_m(\gamma)d\rho\Big)\Big(\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\theta^N(\gamma^m,x)\cdot\nabla z_h + f^N(\gamma^m,x)z_h\,dx\Big),\\
&\sum_{m=1}^{M}\Big(\int_{\Gamma^N}\ell_m(\gamma)d\rho\Big)\Big(\int_D \tfrac{1}{2}(\kappa_{min}+\kappa_{max})\nabla \zeta^{N,h,M}(\gamma^m,x)\cdot\nabla z_h\,dx\Big)\\
&\quad=\sum_{m=1}^{M}\Big(\int_{\Gamma^N}\ell_m(\gamma)d\rho\Big)\Big(\int_D -\tfrac{1}{\pi}(\kappa_{min}-\kappa_{max})\Theta^N(\gamma^m,x)\cdot\nabla z_h + \tau^N(\gamma^m,x)z_h\,dx\Big)
\end{aligned} \tag{16}$$

The discrete optimality system is of the form

$$(v^{N,h,M},\zeta^{N,h,M}) + T^{N,h,M}G(\lambda,(v^{N,h,M},\zeta^{N,h,M})) = 0. \tag{17}$$

We now proceed with the verification of (8), (9). For clarity of exposition, we split the error into the errors due to Karhunen-Loève truncation, finite-element approximation and the Lagrange interpolation error:

$$\begin{aligned}\|(v-v^{N,h,M},\zeta-\zeta^{N,h,M})\|_X \le \|(v-v^N,\zeta-\zeta^N)\|_X + \|(v^N-v^{N,h},\zeta^N-\zeta^{N,h})\|_X \\ + \|(v^{N,h}-v^{N,h,M},\zeta^{N,h}-\zeta^{N,h,M})\|_X.\end{aligned}$$

From (14) we have

$$\|(v-v^N,\zeta-\zeta^N)\|_X \le C\|(\theta-\theta^N,\Theta-\Theta^N,f-f^N,\tau-\tau^N)\|_Y, \tag{18}$$

(where C is a constant independent of N) and by Mercer's result it follows that

$$\|(v-v^N,\zeta-\zeta^N)\|_X \to 0 \text{ as } N\to\infty. \tag{19}$$

Similarly, the error due to the finite element approximation is controlled by the approximation properties of the finite element space

$$\begin{aligned}\|(v^N-v^{N,h},\zeta^N-\zeta^{N,h})\|_X &\le Ch\|(v^N,\zeta^N)\|_{L^6_P(\Omega,W^{2,6}(D))} \\ &\le Ch\|(\theta^N,\Theta^N,f^N,\tau^N)\|_{L^6_P(\Omega,L^6(D))},\end{aligned} \tag{20}$$

for all $\theta^N,\Theta^N,f^N,\tau^N \in L^6_P(\Omega,L^6(D))$.

To extend (20) to the whole space Y, i.e., $\theta^N, \Theta^N, f^N, \tau^N \in L^6_P(\Omega, W^{-1,\frac{6}{5}}(D))$ we use a density argument.

Lemma

For $(\theta^N, \Theta^N, f^N, \tau^N) \in L^6_P(\Omega, W^{-1,\frac{6}{5}}(D))$

$$\|(v^N - v^{N,h}, \zeta^N - \zeta^{N,h})\|_X \to 0, \qquad \text{as } h \to 0. \tag{21}$$

Proof For any $\theta^N, \Theta^N, f^N, \tau^N \in L^6_P(\Omega, W^{-1,\frac{6}{5}}(D))$ there exist sequences of functions $\{\theta_i^N\}, \{\Theta_i^N\}, \{f_i^N\}, \{\tau_i^N\} \subset L^6_P(\Omega, L^6(D))$ such that

$$\|(\theta_i^N - \theta^N, \Theta_i^N - \Theta^N, f_i^N - f^N, \tau_i^N - \tau^N)\|_Y \to 0 \qquad \text{as } i \to \infty. \tag{22}$$

Denoting by (v_i^N, ζ_i^N) the solution to (14) corresponding to $(\theta_i^N, \Theta_i^N, f_i^N, \tau_i^N)$ and by $(v_i^{N,h}, \zeta_i^{N,h})$ the solution to (15) corresponding to $(\theta_i^N, \Theta_i^N, f_i^N, \tau_i^N)$, we have that

$$\begin{aligned}
&\|(v^N - v^{N,h}, \zeta^N - \zeta^{N,h})\|_X \qquad (23)\\
&\le \|(v^N - v_i^N, \zeta^N - \zeta_i^N)\|_X + \|(v_i^N - v_i^{N,h}, \zeta_i^N - \zeta_i^{N,h})\|_X + \|(v_i^{N,h} - v^{N,h}, \zeta_i^{N,h} - \zeta^{N,h})\|_X \\
&\le C\|(\theta^N - \theta_i^N, \Theta^N - \Theta_i^N, f^N - f_i^N, \tau^N - \tau_i^N)\|_Y + Ch\|(\theta_i^N, \Theta_i^N, f_i^N, \tau_i^N)\|_{L^6_P(\Omega, L^6(D))}.
\end{aligned}$$

For the first and last terms we used the same argument as in (18), for the second term (20). Putting together (22) and (23) completes the argument. ∎

The quadrature error, collocation, Smolyak, etc.

$$\|(u^{N,h} - u^{N,h,M}, \eta^{N,h} - \eta^{N,h,M})\|_X \leq \mathcal{F}(w, M) \to 0 \text{ as } M \to \infty. \tag{24}$$

Lemma

The following approximations hold

$$\forall(\theta,\Theta,f,\tau)\in Y \qquad \|(T-T^{N,h,M})(\theta,\Theta,f,\tau)\|_X \to 0, \tag{25}$$

and moreover,

$$\|T-T^{N,h,M}\|_{\mathcal{L}(Z,X)} \to 0 \tag{26}$$

as $h\to 0,$ *and* $N,M\to\infty.$

Proof. For any $(\theta,\Theta,f,\tau)\in Y$, using (19), (20) and (24), we have

$$\begin{aligned} &\|(T-T^{N,h,M})(\theta,\Theta,f,\tau)\|_X = \|(v-v^{N,h,M},\zeta-\zeta^{N,h,M})\|_X \\ &\le \|(v-v^N,\zeta-\zeta^N)\|_X + \|(v^N-v^{N,h},\zeta^N-\zeta^{N,h})\|_X \\ &\qquad + \|(v^{N,h}-v^{N,h,M},\zeta^{N,h}-\zeta^{N,h,M})\|_X \to 0, \end{aligned}$$

as $N\to\infty, h\to 0, M\to\infty$. This proves (25), while (26) follows from the continuous embedding of $Z\subset Y$. ∎

The Fréchet derivative of $G(\lambda,(u,\eta))$ is

$$D_{(u,\eta)}G(\lambda,(u,\eta))\cdot(\underline{u},\underline{\eta})$$

$$= \Big[-\arctan[\mathcal{R}(-\frac{1}{\beta}\nabla u\nabla\eta)]\nabla\underline{u} - \frac{\mathcal{R}'(-\frac{1}{\beta}\nabla u\cdot\nabla\eta)\big(-\frac{1}{\beta}(\nabla\underline{u}\cdot\nabla\eta+\nabla u\cdot\nabla\underline{\eta})\big)}{1+\mathcal{R}^2(-\frac{1}{\beta}\nabla u\nabla\eta)}\nabla u,$$

$$-\arctan[\mathcal{R}(-\frac{1}{\beta}\nabla u\nabla\eta)]\nabla\underline{\eta} - \frac{\mathcal{R}'(-\frac{1}{\beta}\nabla u\nabla\eta)\big(-\frac{1}{\beta}(\nabla\underline{u}\cdot\nabla\eta+\nabla u\cdot\nabla\underline{\eta})\big)}{1+\mathcal{R}^2(-\frac{1}{\beta}\nabla u\nabla\eta)}\nabla\eta, 0, \mathbb{E}(\underline{u})\Big]$$

for all $(\underline{u},\underline{\eta})\in X$.

Lemma

$D_{(u,\eta)}G(\lambda,(u,\eta))\in\mathcal{L}(X,Z)$ for all $(u,\eta)\in X$. Moreover, G is C^2 mapping, and D^2G is bounded on all bounded sets of X.

Proof. For any $(u,\eta),(\underline{u},\underline{\eta})\in X$, because $\arctan,\mathcal{R}'$ are bounded with bounded derivatives, we have

$$\|D_{(u,\eta)}G(\lambda,u,\eta)\cdot(\underline{u},\underline{\eta})\|_Z \le \|\nabla\underline{u}\|_{L^2_\rho(\Gamma;L^2(D))} + \|\nabla\underline{\eta}\|_{L^2_\rho(\Gamma;L^2(D))}$$

$$+\frac{1}{\beta}\big(\|\underline{u}\|_X\|\underline{\eta}\|_X\|u\|_X + \|\underline{u}\|_X^2\|\underline{\eta}\|_X + \|\underline{u}\|_X\|\underline{\eta}\|_X^2 + \|u\|_X\|\underline{\eta}\|_X\|\eta\|_X\big) \le C\|(\underline{u},\underline{\eta})\|_X.$$

The proof of the second part is a consequence of the boundedness of $\arctan,\mathcal{R}'$ and $\mathcal{R}''$. ∎

Theorem

Let $\{(\lambda = \beta, \varphi(\lambda) = (u(\lambda), \kappa(\lambda)); \lambda \in \Lambda\}$ *be a nonsingular branch of solutions. Then there exists a neighborhood* $\mathcal{O}$ *of the origin in* X *and, for* $h \leq h_0$ *small enough and* N, M *sufficiently large, a unique* C^2 *branch of solutions of* (17) *such that*

$$\varphi^{N,h,M}(\lambda) \in \varphi(\lambda) + \mathcal{O}.$$

Moreover, if

ASSUMPTIONS IN K-L EXPANSION, SMOLYAK

we have the estimate

$$\begin{aligned}
&\|u - u^{N,h,M}\|_{L^6_P(\Omega; W_0^{1,6}(D))} + \|\eta - \eta^{N,h,M}\|_{L^6_P(\Omega; W_0^{1,6}(D))} \\
&\leq C(h + \text{ERRORS IN N, M})\mathcal{E}(u, \eta, f, \overline{u}),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}(u, \eta, f, \overline{u}) = \Big(&\|\nabla u\|_{L^6_P(\Omega; W^{-1,\frac{6}{5}}(D))} + \|\nabla \eta\|_{L^6_P(\Omega; W^{-1,\frac{6}{5}}(D))} \\
&+ \|f\|_{L^6_P(\Omega; W^{-1,\frac{6}{5}}(D))} + \|\mathbb{E}(u - \overline{u})\|_{L^6_P(\Omega; W^{-1,\frac{6}{5}}(D))}\Big).
\end{aligned}$$

From the form of κ in (13), we have (using Taylor expansion and boundedness of $\mathcal{R}'$, Cauchy-Schwarz)

$$
\begin{aligned}
&\|\kappa - \kappa^{N,h,M}\|_{L^3_P(\Omega;L^3(D))} \\
&\leq C\big(\|(\nabla u - \nabla u^{N,h,M})\nabla\eta\|_{L^3_P(\Omega;L^3(D))} + \|\nabla u(\nabla\eta - \nabla\eta^{N,h,M})\|_{L^3_P(\Omega;L^3(D))}\big) \\
&\leq C\big(\|\nabla u - \nabla u^{N,h,M}\|_{L^6_P(\Omega;L^6(D))} + \|\nabla\eta - \nabla\eta^{N,h,M}\|_{L^6_P(\Omega;L^6(D))}\big).
\end{aligned}
$$

We remark here that this κ gives the sought coefficient through formula (3).

Numerical Example

Stochastic parameter identification

Couple an adjoint-based gradient algorithm with gSCFEM to compute the optimal pair $(\widehat{u}, \widehat{a})$ s.t. $J_{3,4}(\widehat{a}, \widehat{u}) = \inf_{(a,u)\in\mathcal{A}_{ad}} J_{3,4}(a,u)$

$$J_3(a,u) = \mathbb{E}\left[\frac{1}{2}\|u(\omega,x) - \overline{u}(\omega,x)\|^2_{L^2(D)} + \frac{\beta}{2}\|a(\omega,x)\|^2_{L^2(D)}\right] \qquad (P_3)$$

$$J_4(a,u) = \frac{1}{2}\left\|\mathbb{E}[u](x) - \mathbb{E}[\overline{u}](x)\right\|^2_{L^2(D)} + \frac{\beta}{2}\left\|\mathbb{E}[a](x)\right\|^2_{L^2(D)} \qquad (P_4)$$

Stochastic target: $\overline{u} = x(1-x^2) + \sum_{n=1}^{N} \sin\left(\frac{n\pi x}{L}\right) Y_n(\omega)$

Exact random coeff.: $\widehat{a} = (1+x^3) + \sum_{d=1}^{D} \cos\left(\frac{m\pi x}{L}\right) Y_d(\omega)$

Deterministic initial guess: $a = 1 + x$

Exact given RHS: $f = \nabla\cdot(\widehat{a}(\omega,x)\nabla\overline{u}(\omega,x))$

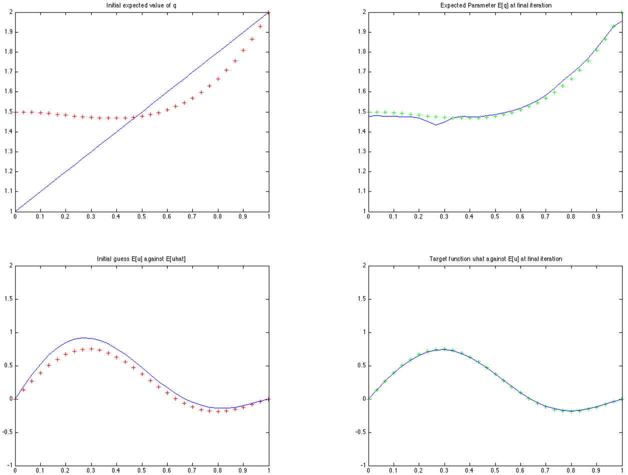
- $\mathbb{E}[Y_i] = 0$ and $\mathbb{E}[Y_iY_j] = \delta_{ij}$ for $i,j \in \mathbb{N}_+$, are taken uniform in the interval $[0,1]$, and $x \in \mathbb{R}^1$

GOAL: given the random f, identify the expectation of both the paramter $\mathbb{E}[a]$ and the state $\mathbb{E}[u]$ and compare with the exact statistical quantities.

$N = 1$ example: $\inf J_3(a, u)$

Tracking the expectation of the parameter $\mathbb{E}[a]$ and the state $\mathbb{E}[u]$

$$M = 5$$



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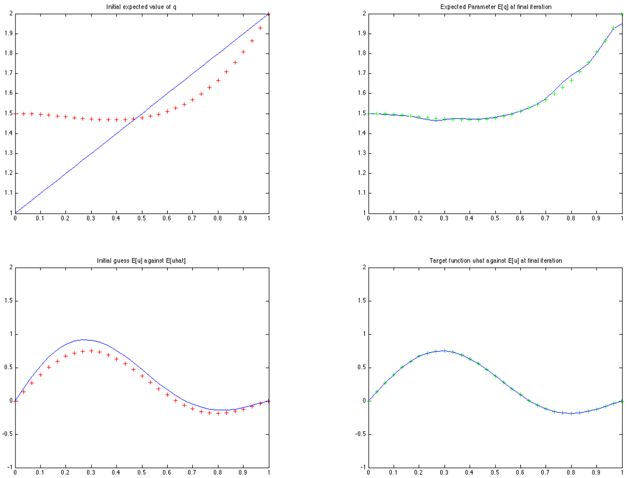
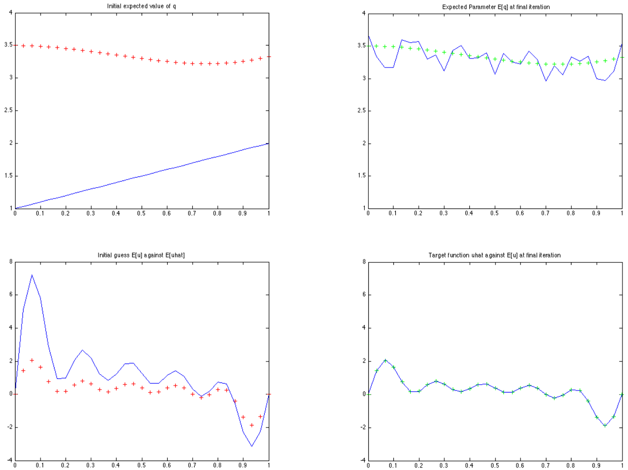


$N = 11$ example: $\inf J_3(a, u)$

Tracking the expectation of the parameter $\mathbb{E}[a]$ and the state $\mathbb{E}[u]$

$$M = 1581$$



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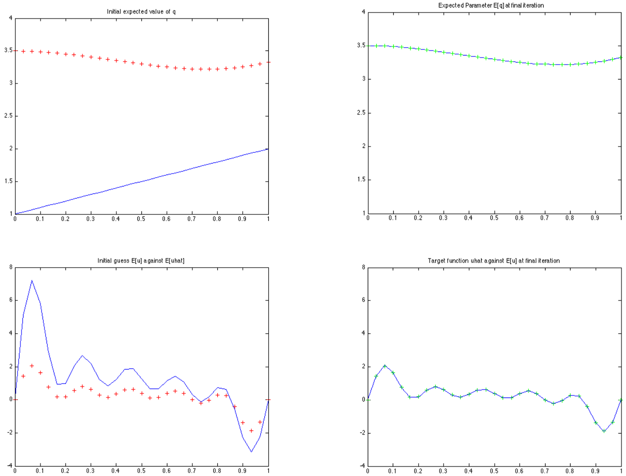
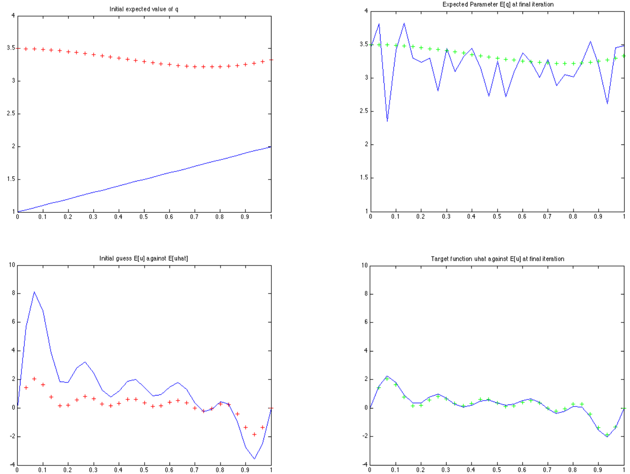


$N = 11$ example: $\inf J_4(a, u)$

MC vs. gSC

$$M \sim 10^9$$



$N = 5, 10$ and 20 examples: MC vs. SC

Tracking the expectation of the state $\mathbb{E}[u]$ and control $\mathbb{E}[\kappa]$

N	gSC	MC
11	360	7e+09
51	1581	9e+10
121	4801	8e+12
251	11561	1e+15

Table: For Γ^N, with $N = 11, 51, 121$ and 251, we compare the number of deterministic solutions required by the generalized Stochastic Collocation (gSC) using Chebyshev abscissas and the Monte Carlo (MC) method using random abscissas, to reduce the original error in both $\|\mathbb{E}[u] - \mathbb{E}[\overline{u}]\|_{L^2(D)}$ and $\|\mathbb{E}[a] - \mathbb{E}[\widehat{a}]\|_{L^2(D)}$ by a factor of 10^6.

Concluding remarks

- High-dimensional problems are a characteristic of modern forward and inverse UQ. Accurate Monte Carlo-type results take too long and tensor product methods suffer from the *curse of dimensionality*
- Extensions to *fully adaptive gSC*: Notice that the coefficient are defined by the following hierarchical description:

$$c(\mathbf{i}) = u_p - \mathscr{A}^{m,g}(p-1, N)(u) = u_p - u_{p-1}$$

and thus local adaptivity is achieved by replacing the basis with hierarchical piecewise interpolants (unstable multiscale L^2 splitting).

Optimal L^2 splitting and two sided estimates for the coefficient using semi-orthogonal hierarchical basis functions, e.g. prewavelets

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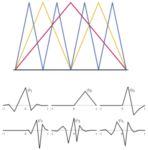

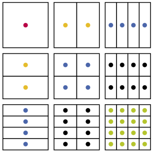

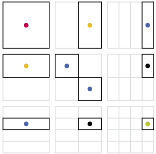

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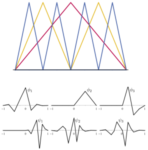
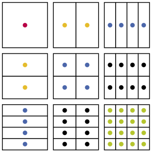
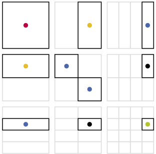
Optimal L^2 splitting and two sided estimates for the coefficient using semi-orthogonal hierarchical basis functions, e.g. prewavelets







- **TASMANIAN**: **T**oolkit for **A**daptive **S**tochastic **M**odeling **A**nd **N**on-**I**ntrusive **A**pproximatio**N**. ORNL multilevel parallel object-oriented framework for adaptive UQ in high-dimensional spaces supporting terascale laptop to petascale cluster to exascale supercomputer

References

[NTW08a] F. Nobile, R. Tempone and **C. Webster**
A sparse grid stochastic collocation method for PDEs with random input data, SIAM J. Numer. Anal., 2008.

[NTW08b] F. Nobile, R. Tempone and **C. Webster**
An anisotropic sparse grid stochastic collocation method for PDEs with random input data, SIAM J. Numer. Anal., 2008.

[GW11] M. Gunzburger and **C. Webster**
Numerical Methods for SPDEs and UQ, SIAM book, Expected 2012.

[GTW11] M. Gunzburger, C. Trenchea and **C. Webster**
A generalized methodology for the solution of stochastic identification problems, IOU, IJ4UQ, 2011.

[W11] **C. Webster**
A fully adaptive $h \times p$ *stochastic collocation method for high-dimensional discontinuous stochastic simulations.* ORNL Technical Report, 2011.