Risk Management of Portfolios by CVaR Optimization

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Joint work with Yuying Li and Lei Zhu, Univ of Waterloo:

L. Zhu, T. F. Coleman, and Y. Li, *Minmax robust and CVaR robust mean variance portfolios*, *Journal of Risk*, Vol 11, pp 55-85, 2009.

Mean-Variance Optimization: Harry Markowitz, 1950

Assume asset returns are jointly normal:

$\mu \in \Re^n$: expected rate of returns

Q : n-by-n covariance matrix

$x \in \Re^n$: percentage holdings

$\lambda \geq 0$: the risk aversion parameter

$$\min_{x \in \Re^n} -\mu^T x + \lambda \cdot x^T Q x$$

s.t.

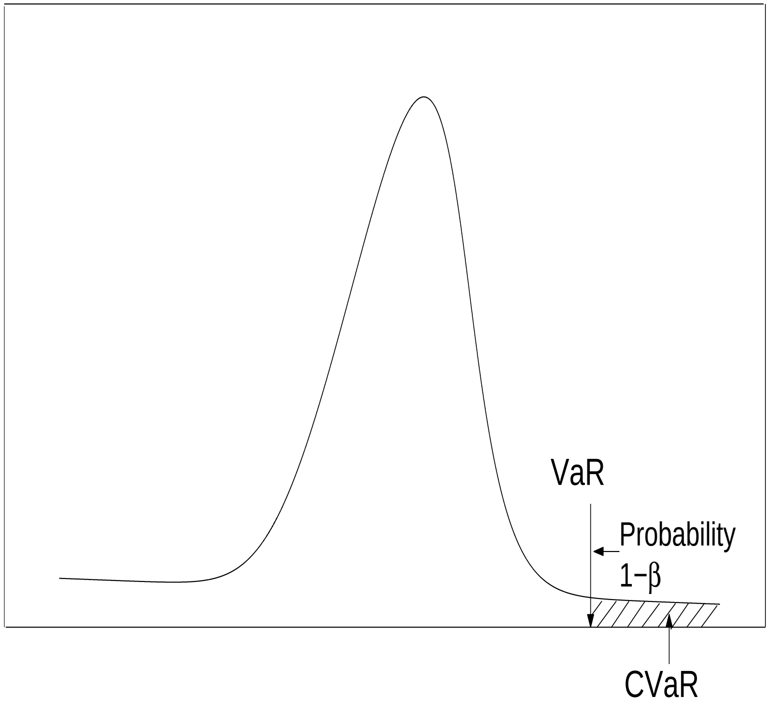
$$e^T x = 1$$

$$x \geq 0$$

where $e^T = [1, 1, \cdots, 1]$

Measure Tail Risk: CVaR$_\beta$ (β: a confidence level)

If the return distribution is not normal, tail risk becomes crucial.



Loss Distribution

OUTLINE

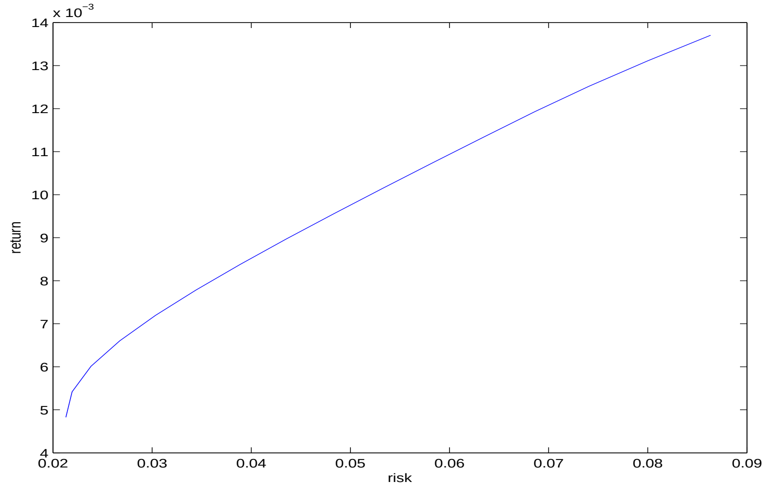
- Sensitivity to estimation error in MV portfolio optimization
- Min-max robust MV portfolio optimization
- Performance of min-max robust optimal portfolios
  - sensitivity to initial data
  - asset diversification
- CVaR robust MV portfolio optimization
- Efficient CVaR optimal portfolio computation

Assume that μ and Q are known.

The optimal portfolio x^* is efficient: it has the minimum risk for the given expected rate of return.

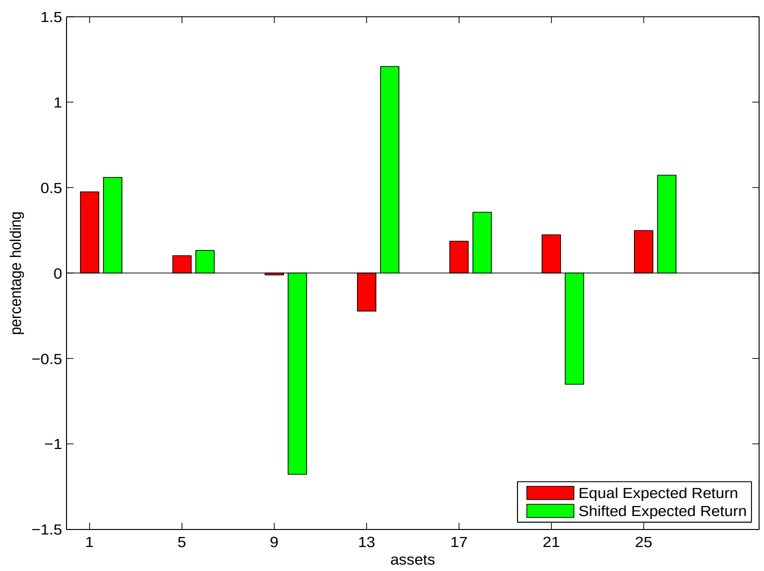
Let $x^*(\lambda)$ denote the optimal MV portfolio for λ.

The curve $(\sqrt{x^*(\lambda)^T Q x^*(\lambda)}, \mu^T x^*(\lambda)),\ \lambda \geq 0$, forms an efficient frontier.



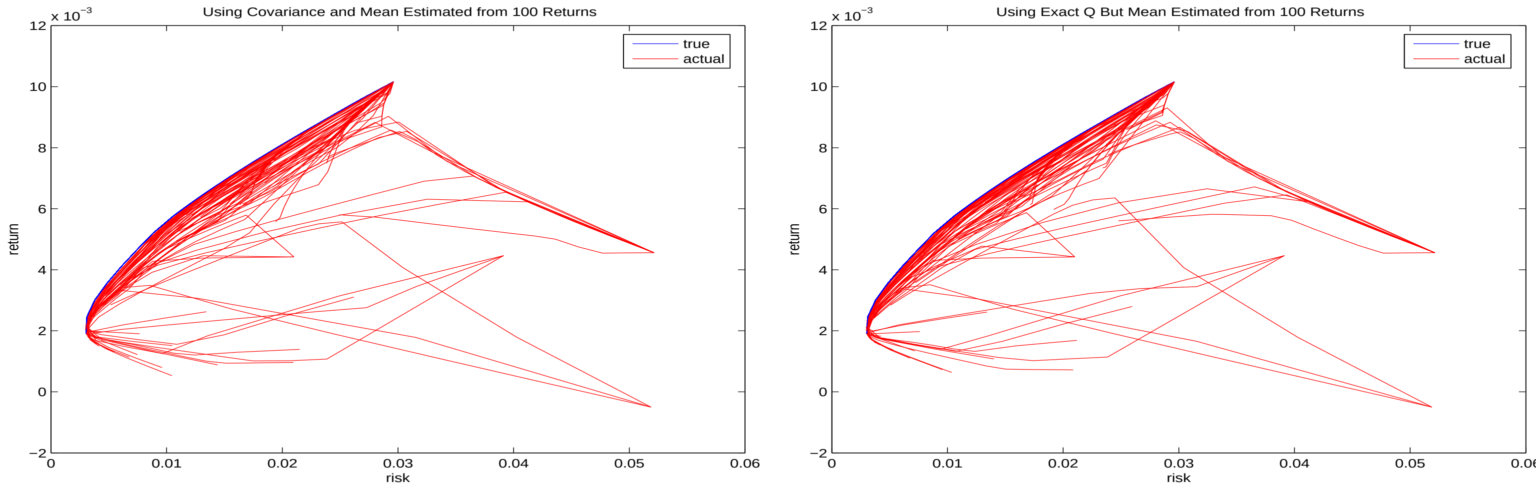
- In practice, only estimates $\tilde{\mu}, \tilde{Q}$ from a finite set of return samples are available.
- The MV optimization problem based on estimates $\tilde{\mu}, \tilde{Q}$ is called a nominal problem.
- Sensitivity of the optimal portfolio to mean returns: $\mu = 7\%$

$$\bar{\mu}_4 = \mu_4 + 2.5\%; \bar{\mu}_3 = \mu_3 - 2.5\%; \bar{\mu}_6 = \mu_6 - 2.5\%$$



- Optimal portfolio $\tilde{x}(\lambda)$ from estimates $\tilde{\mu}, \tilde{Q}$ may not perform well in reality.

- Actual frontier (Broadie, 1993): the curve $(\sqrt{\tilde{x}(\lambda)^T Q \tilde{x}(\lambda)}, \mu^T \tilde{x}(\lambda))$, $\lambda \geq 0$, describes the actual performance of optimal portfolios $\tilde{x}(\lambda)$ from nominal estimates.



A ten-asset example:

- (blue) true efficient frontier: computed using μ and Q
- (red) actual frontier: computed based estimates $\tilde{\mu}$ and $\tilde{Q}$ using 100 return samples

- Actual performance of the MV optimal portfolio from estimates can be very poor.
- Smaller variation for the minimum risk portfolio (left end).
- Larger variation for the maximum return portfolio (right end), which always concentrates on a single asset

- The optimal MV solution is particularly sensitive to estimation error in mean return.
- Mean return is notoriously difficult to estimate accurately.
- For a small number of assets, estimation error in covariance matrix is relatively small.

In this talk, we focus on uncertainty in mean returns.

Examples of research addressing estimation error in MV optimization:

- Incorporating additional views: Black-Litterman, 1992
- Robust optimization: Goldfarb and Iyengar (2003), Tütüncü and Koenig (2003), Garlappi, Uppal and Wang (2007)

Robust Optimization

- The notion of minmax robust has existed for a long time.
- Robust optimization offers a solution which has the best performance for all possible realizations in some uncertainty sets of the uncertain parameters.
- Minmax robust problems are typically semi-infinite programming problems.
- Recent advancement in efficient computation of solutions to robust (convex) optimization problems (semidefinite programming and conic programming) has attracted attention to robust portfolio selections.

What about Min-Max Robust Solutions?

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu \in \mathcal{S}_\mu, Q \in \mathcal{S}_Q} -\mu^T x + \lambda x^T Q x \\ \text{s.t.} & e^T x = 1 \end{array}$$

$\mathcal{S}_\mu, \mathcal{S}_Q$: uncertainty sets for μ and Q

Typical uncertainty sets:

- ellipsoidal uncertainty set: $(\bar{\mu} - \mu)^T A(\bar{\mu} - \mu) \leq \chi$
- interval uncertainty set: $\mu_L \leq \mu \leq \mu_R$

Specification of uncertainty sets plays crucial role in robust solutions.

A statistical result:

Assume that asset returns have a joint normal distribution and mean estimate $\bar{\mu}$ is computed from T samples of n assets. If the covariance matrix Q is known, then the quantity

$$\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$$

has a χ^2_n distribution with n degrees of freedom.

Garlappi, Uppal, Wang (2007) derive an explicit formula for the min-max robust solution using the ellipsoidal uncertainty set for μ, assuming Q is known and short selling is allowed, i.e., they consider

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu} -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & (\bar{\mu} - \mu)^T Q^{-1} (\bar{\mu} - \mu) \leq \chi \\ & e^T x = 1 \end{array}$$

With the no short selling constraint, the robust portfolio problem becomes:

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu} -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & (\bar{\mu} - \mu)^T Q^{-1} (\bar{\mu} - \mu) \leq \chi \\ & e^T x = 1, \quad x \geq 0 \end{array}$$

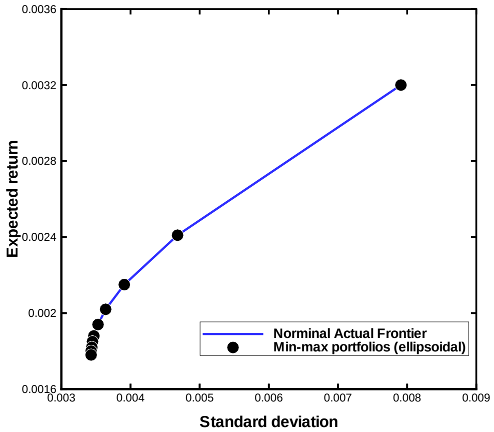
We show that [a] this min-max robust portfolio problem is equivalent to: is a solution to the nominal problem:

$$\begin{array}{ll} \min\limits_{x} & -\bar{\mu}^T x + \hat{\lambda} \cdot x^T Q x \\ \text{subject to} & e^T x = 1, \quad x \geq 0, \end{array}$$

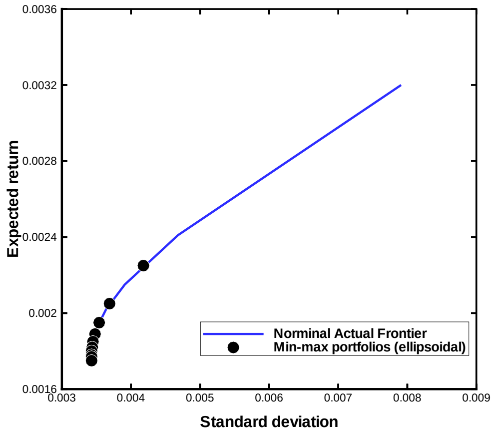
with $\hat{\lambda} \geq \lambda$.

[a] L. Zhu, T. F. Coleman, and Y. Li, *Minmax robust and CVaR robust mean variance portfolios*, *Journal of Risk*, Vol 11, pp 55-85, 2009.

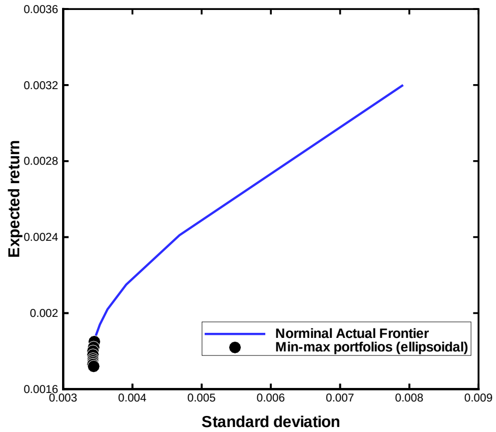
Min-max Robust Frontier vs Mean Variance Frontier



(a) $\chi = 0$



(b) $\chi = 5$



(c) $\chi = 50$

Minmax robust frontier: a squeezed segment of the frontier of the nominal problem.

Interval uncertainty set: $\mu_L \leq \mu \leq \mu_R$

$$\begin{array}{ll} \min\limits_{x} & \max\limits_{\mu_L \leq \mu \leq \mu_R} -\mu^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \ . \end{array}$$

The robust solution solves

$$\begin{array}{ll} \min\limits_{x} & -\ {\mu_L}^T x + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \ . \end{array}$$

$\Longrightarrow$ Minmax robust portfolios are now sensitive to specification of μ_L!

Uncertainty in parameter μ is an estimation risk.

The statistical result for the mean estimation that

$$\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$$

has a χ_n^2 distribution with n degrees of freedom can be used to yield a measure for the estimation risk.

CVaR-Robust Mean Variance Portfolio

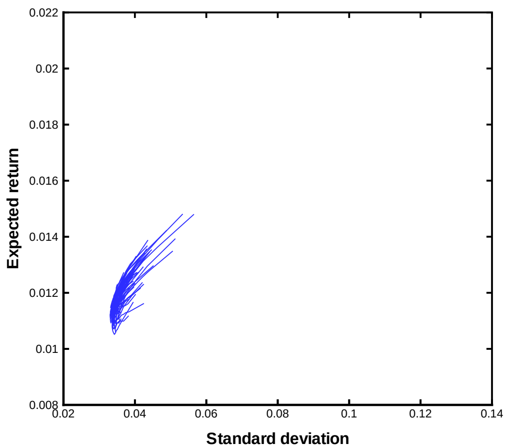
$$\begin{array}{ll} \min\limits_{x} & \mathrm{CVaR}_{\beta}^{\mu}(-\mu^T x) + \lambda \cdot x^T Q x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \end{array} \tag{1}$$

Assumption:

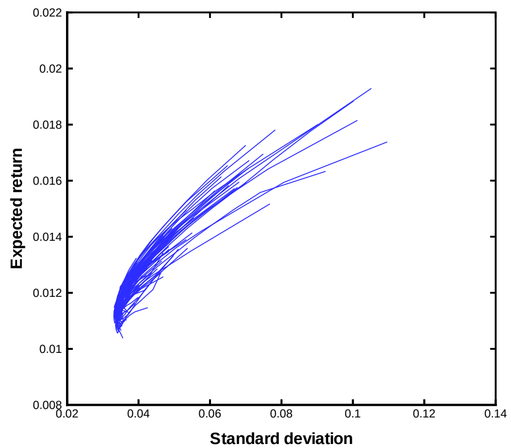
$$\frac{T(T-n)}{(T-1)n}(\bar{\mu}-\mu)^T Q^{-1}(\bar{\mu}-\mu)$$

has a χ_n^2 distribution with n degrees of freedom.

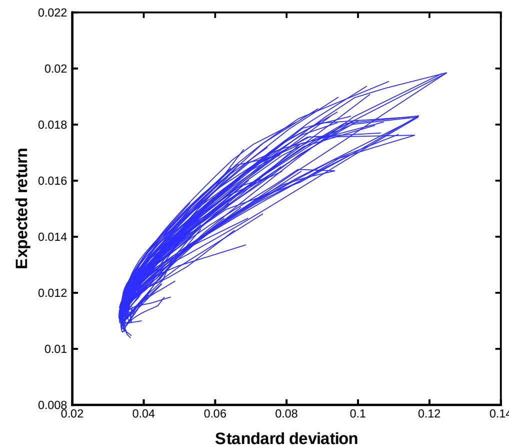
CVaR Robust Actual Frontiers: the curve $(\sqrt{\tilde{x}(\lambda)^T Q \tilde{x}(\lambda)}, \mu^T \tilde{x}(\lambda))$, $\lambda \geq 0$, describes the actual performance of the CVaR optimal portfolios $\tilde{x}(\lambda)$ from CVaR robust formulation (1).



(d) 90% confidence



(e) 60% confidence

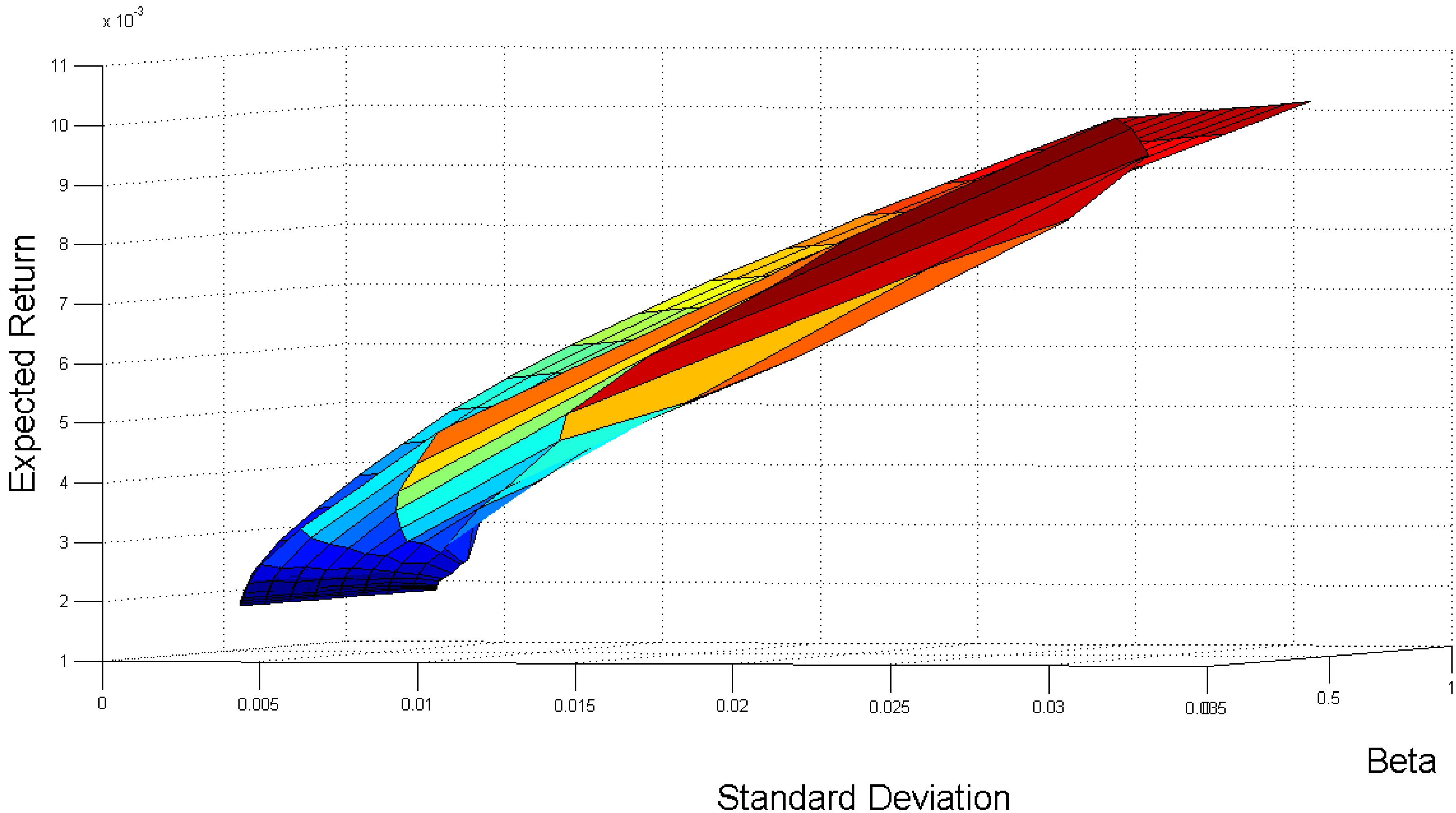


(f) 30% confidence

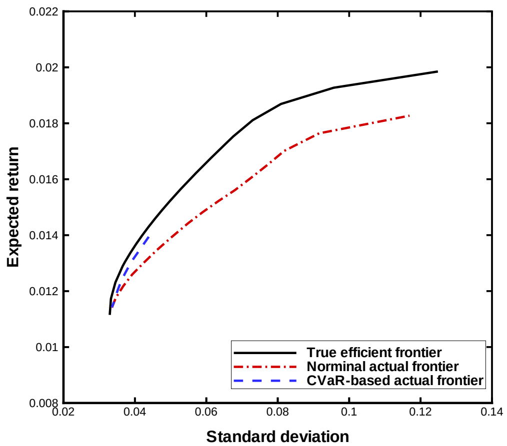
$\bar{\mu}$ is estimated from 100 return samples

10,000 Monte Carlo samples for μ

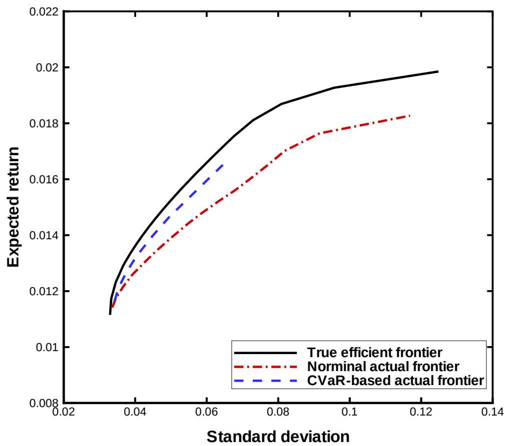
Surface of Efficient Frontier



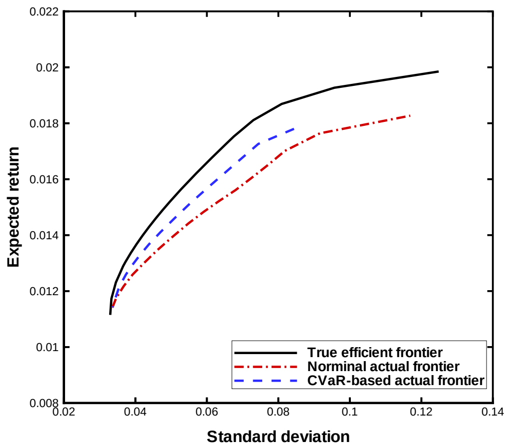
Note that the CVaR robust actual frontiers are different from actual frontiers from nominal estimates.



(g) $\beta = 90\%$



(h) $\beta = 60\%$

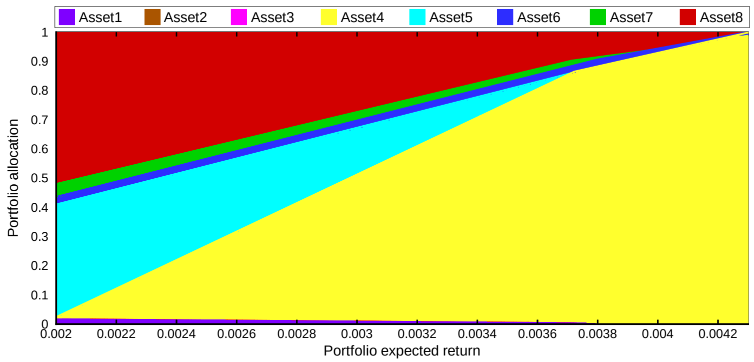


(i) $\beta = 30\%$

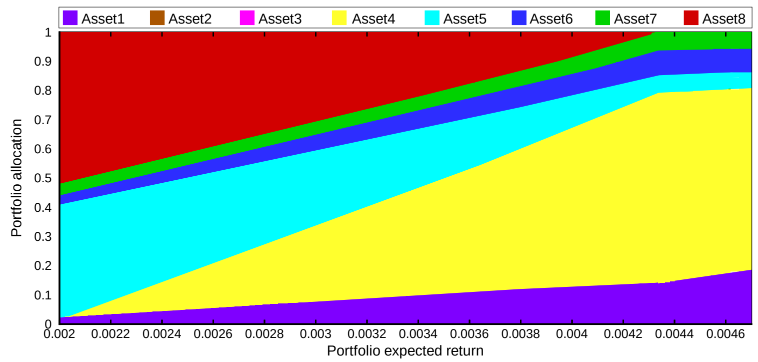
The confidence level can be interpreted as an estimation risk aversion parameter:

- As $\beta \Longrightarrow 1$, extreme loss due to uncertainty in μ is emphasized. This corresponds to increasingly strong aversion to estimation risk.
- As $\beta \Longrightarrow 0$, average loss due to uncertainty in μ is considered. This corresponds to increasing tolerance to estimation risk.

Composition Comparison: Min-max Robust and CVaR Robust



(j) Min-max robust portfolio ($\mu_L \leq \mu \leq \mu_R$)



(k) CVaR robust (90%) portfolios

For CVaR robust formulation, The maximum return portfolio are often diversified.

Computing CVaR Robust Portfolios

By definition,

$$\mathrm{CVaR}_\beta^\mu(-\mu^T x) = \min_\alpha \left(\alpha + (1-\beta)^{-1}\mathbf{E}^\mu([-\mu^T x - \alpha]^+)\right)$$

$$[-\mu^T x - \alpha]^+ \stackrel{\text{def}}{=} \max(-\mu^T x - \alpha, 0)$$

CVaR robust portfolios: stochastic optimization

$$\begin{array}{ll} \min\limits_{x,\alpha} & (\alpha + (1-\beta)^{-1}\mathbf{E}^\mu([-\mu^T x - \alpha]^+)) + \lambda \cdot x^T \bar{Q} x \\ \text{s.t.} & e^T x = 1, \quad x \geq 0 \end{array}$$

Min-max robust portfolios can be computed efficiently by solving a convex programming problem with n variables.

Computing CVaR Robust Portfolio by Solving a QP

Let $\{\mu_i,\ i = 1, \cdots, m\}$ be independent Monte Carlo samples from the specified distribution for μ.

CVaR robust portfolio can be computed by solving

$$
\begin{array}{ll}
\min\limits_{x,z,\alpha} & \alpha + \dfrac{1}{m(1-\beta)} \displaystyle\sum_{i=1}^{m} z_i + \lambda \cdot x^T \bar{Q} x \\
\text{s.t.} & e^T x = 1, \;\; x \geq 0, \;\; , \\
 & z_i \geq 0 \; , \\
 & z_i + \mu_i^T x + \alpha \geq 0, \; i = 1, \ldots, m \; .
\end{array}
$$

- $O(m+n)$ variables and $O(m+n)$ constraints, e.g., $n = 100$, $m = 10,000$
- Computational cost can become prohibitive as m and n become large.

Computing CVaR Robust Portfolio Can Be Expensive

	CPU sec		
# samples	8 assets	50 assets	148 assets
5000	0.39	1.75	7.06
10,000	0.77	4.25	10.38
25,000	2.56	10.83	34.97

CPU time for the QP approach when $\lambda = 0$: $\beta = 0.90$

To generate an efficient frontier, we need to solve QP for $\lambda \geq 0$.

Matlab 7.3 for Windows XP. Pentium 4 CPU 3.00GHz machine with 1GB RAM

A Simple Smoothing Technique

Let $\rho_\epsilon(z)$ be defined as:

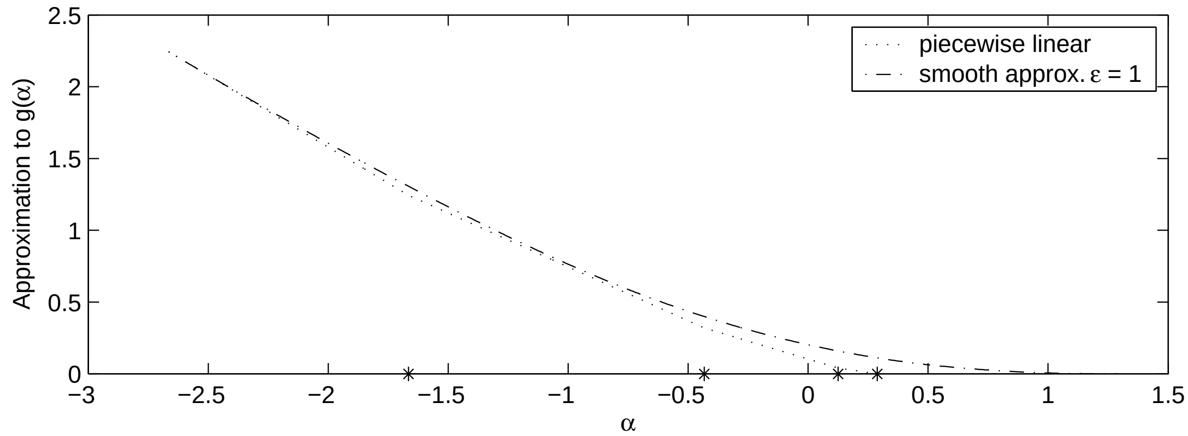
$$\begin{cases} z & \text{if } z \geq \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & \text{if } -\epsilon \leq z \leq \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

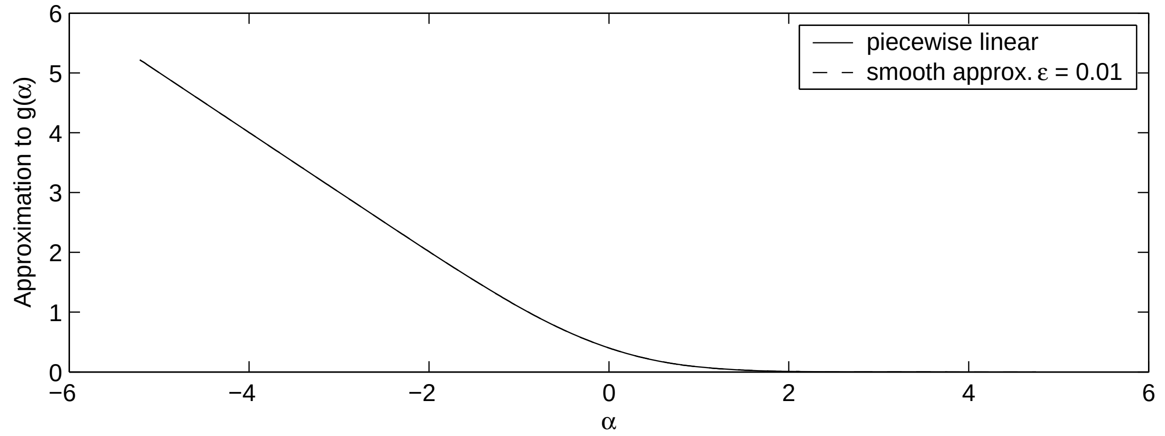
For a given resolution parameter $\epsilon > 0$,

- $\rho_\epsilon(z)$ is continuous differentiable, and approximates the piecewise linear function $[z]^+ = \max(z, 0)$

$$\rho_\epsilon(z) \approx [z]^+$$

Smooth Approximation: $g(\alpha) = \mathbf{E}^z\left((z-\alpha)^+\right)$





Computing CVaR Robust Portfolios Via Smoothing

$$
\begin{array}{ll}
\min\limits_{x,\alpha} & \alpha + \dfrac{1}{m(1-\beta)} \sum\limits_{i=1}^{m} \rho_\epsilon(-\mu_i^T x - \alpha) + \lambda \cdot x^T \bar{Q} x \\
\text{s.t.} & e^T x = 1, \quad x \geq 0 \ ,
\end{array}
$$

- $O(n)$ variables with $O(n)$ constraints

CPU Comparisons

	MOSEK (CPU sec)			Smoothing (CPU sec)		
# samples	8 assets	50 assets	148 assets	8 assets	50 assets	148 assets
5000	0.39	1.75	7.06	0.42	0.34	1.98
10,000	0.77	4.25	10.38	0.75	0.50	4.13
25,000	2.56	10.83	34.97	1.77	1.36	10.25

$$\lambda = 0, \ \beta = 90\%$$

Accuracy Comparisons (error in %): $\lambda = 0,\ \beta = 90\%$

# samples	50 assets	148 assets	200 assets
10000	-0.2974	-0.2236	-0.2234
25000	-0.0934	-0.0882	-0.0880
50000	-0.0504	-0.0454	-0.0466

$\epsilon = 0.001$

For convergence properties of the smoothing method for a class of stochastic optimization, see

Xu, H., D. Zhang. 2008. Smooth sample average approximation of stationary points in nonsmooth stochastic optimization and applications. *Math. Programming., Ser. A.* .

Concluding Remarks

When mean return is uncertain for mean variance portfolio selection,

- minmax robust with ellipsoidal uncertainty set: squeezed frontiers from MV based on nominal estimates
- minmax robust with interval uncertainty set: the maximum return portfolio is never diversified
- CVaR robust:
  - different frontiers from those based on nominal estimates
  - maximum return portfolios are typically diversified into multiple assets

- Broadie, M. 1993. Computing efficient frontiers using estimated parameters. *Annals of Operations Research* **45** 21–58.
- A. Ben-Tal and A. Nemirovski. 1998. Robust convex programming. *Mathematics of Operations Research* **23**(1) 769–805.
- Goldfarb, D., G. Iyengar. 2003. Robust portfolio selection problems. *Mathematics of Operations Research* **28**(1) 1–38.
- Tütüncü, R. H., M. Koenig. 2004. Robust asset allocation. *Annals of Operations Research* **132**(1) 157–187.
- Alexander, S., T. F. Coleman, Yuying Li. 2006. Minimizing var and cvar for a portfolio of derivatives. *Journal of Banking and Finance* **30**(2) 583–605.