Geometric Approaches in Image Diffusion

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Introduction

1. Image Diffusion: Basic Problem and Applications
2. Basic History:
   - Witkin
   - Perona-Malik
3. Geometric Approaches
   - Beltrami Diffusion
   - Hypoelliptic Diffusion
   - Sobolev Diffusion
4. Conclusion: Microlocal Diffusions ?

Basic Problem:

- Define a suitable **function space** H and a suitable **one-parameter semigroup** $(T_t)_{t\geq 0}$ on H such that for any $u_0 \in H$, the family $(T_t u_0)_{t\geq 0}$ corresponds to a one-parameter family of deformations of u_0 in a "desired way", such as:
  - (a) yields "smoother" (i.e. less noisy) versions of u_0 as $t \to \infty$, while preserving the "important" structure in u_0 (e.g. object boundaries, ...), or
  - (b) "sharpens" u_0 more and more as $t \to \infty$, or
  - (c) "missing parts" of u_0 are "filled in", or ...

  Or (often equivalently),
- Define a suitable **function space** H and a suitable **operator** A on H such that the one-parameter family $(u(t))_{t\geq 0}$ solution to the evolution equation

$$\begin{aligned} \frac{du}{dt} &= Au, \quad t > 0, \\ u(0) &= u_0, \end{aligned}$$

  (with $u_0 \in H$) deforms u_0 in a "desired way" as above.

Image Diffusion: Basic Applications

Figure: **Image denoising**

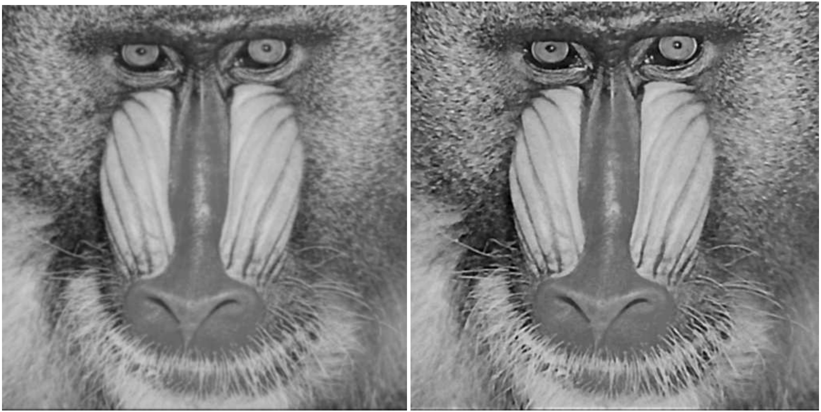


Figure: **Image sharpening**

Image Diffusion: Basic Applications

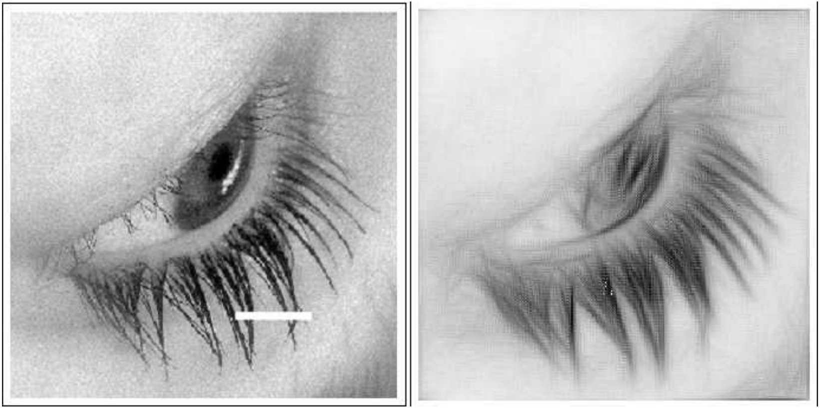


Figure: **Image reconstruction (inpainting)** (images courtesy of Ugo Boscain)

Basic Diffusion PDE: The Heat Equation (Witkin'83)

- Given an image $u_0 : \Omega \to \mathbb{R}$, consider the partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\Delta u, \quad t > 0, \\ u(\cdot, 0) &= u_0, \\ u(\cdot, t) &= 0 \text{ on } \partial\Omega, \ \forall t \geq 0, \end{aligned}$$

  where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, and $c > 0$ is the diffusion constant.
- Basic regularity result: $\forall t > 0$, $\forall u_0 \in L^2(\Omega)$, $u(\cdot, t)$ is C^∞.
- Basic consequence of the regularity result:
  - $u(\cdot, t)$ is "infinitely smoother" than $u(\cdot, s)$ for $t > s$.

As a result:

- Image denoising is performed, **but** ...
- Substantial image structure is lost in the process! Key reason: **Isotropicity**.
- **Δ rotationally invariant** $\Rightarrow$
  - Diffusion is performed equally in all directions,
  - Diffusion is performed independently of any underlying image structure.

The Perona-Malik Equation ('89)

Basic idea:

(a) Re-express the heat equation as

$$\frac{\partial u}{\partial t} = \nabla \cdot (c \nabla u), \quad t > 0,$$

(b) Replace the **constant** diffusion coefficient c with a **variable** diffusion coefficient $s \mapsto c(s)$, yielding:

$$\frac{\partial u}{\partial t} = \nabla \cdot (c(\|\nabla u\|) \nabla u), \quad t > 0,$$

with $c : \mathbb{R}^+ \to \mathbb{R}^+$ monotonically decreasing, $c(s) \to 0$ as $s \to \infty$, and $c(s) \to 1$ as $s \to 0$ (e.g. $c(s) = \frac{1}{1+s^2/K^2}$ or $c(s) = e^{-s^2/K^2}$).

(c) The diffusion obtained is **anisotropic**: More diffusion **along** image gradient directions than **across** image gradient directions.

(d) Key (theoretical) problem: **Highly ill-posed!**

Anisotropic Diffusion

Example: The 1-D Perona-Malik Equation (with $c(s) = \frac{1}{1+s^2}$):

$$u_t = \partial_x\left(\frac{1}{1+u_x^2}u_x\right) = \frac{1-u_x^2}{(1+u_x^2)^2}u_{xx}.$$

Hence: We have heat flow for $|u_x| < 1$ and **reverse heat flow** for $|u_x| > 1$.

- But: **Provably stable numerical schemes do exist!** ("Perona-Malik paradox")

Theoretical Fix: **Mollification!**

$$\frac{\partial u}{\partial t} = \nabla \cdot (c(\|\nabla u_\sigma\|)\nabla u), \quad t > 0,$$

where u_σ is the convolution of u with a Gaussian of variance σ^2.

Theorem (Catté et al. '92)

Let $\sigma > 0$. For any $u_0 \in L^2(\Omega)$, there is a unique **weak** *solution $u \in C^0([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^1(\Omega))$ to*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (c(\|\nabla u_\sigma\|)\nabla u) \text{ on }]0,T] \times \Omega, \\
\partial_\nu u &= 0 \text{ on }]0,T] \times \partial\Omega, \\
u(0) &= u_0;
\end{aligned}$$

Moreover, this unique solution is in $C^\infty(]0,T] \times \bar{\Omega})$, hence is a **strong** *solution.*

Isotropic Diffusion: Denoising Results

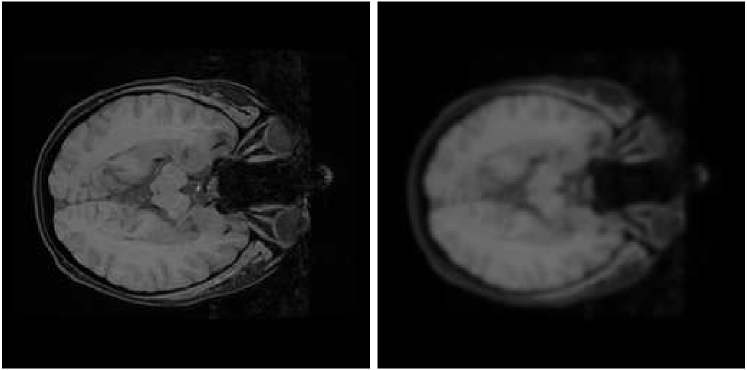


Figure: **Isotropic diffusion**

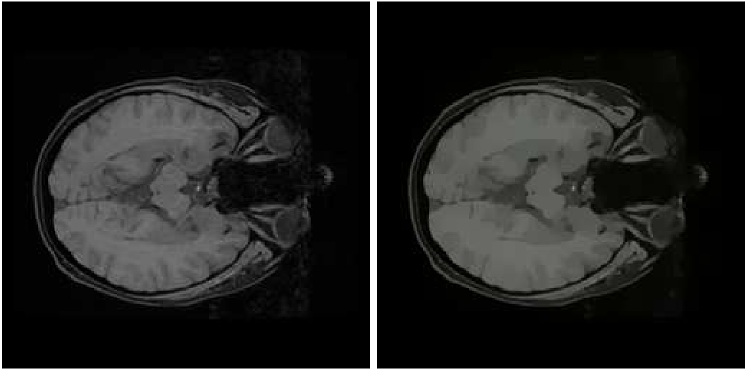


Figure: **Perona-Malik Anisotropic diffusion**

Basic Idea:

- Formulate image diffusion as a **geometric problem**.
- For example:
  - Consider the **graph** of the image function as a **Riemannian manifold** (Beltrami diffusion), or
  - **Lift** the image function to the **projective cotangent bundle** of $\mathbb{R}^2$ (Hypoelliptic diffusion), or
  - Consider the **space of images** as a **Riemannian manifold** (Sobolev diffusion), or
  - ...

Beltrami Diffusion (Kimmel et al.)

- Consider the image function $I : \Omega \rightarrow \mathbb{R}^K$ (assumed smooth) as an **embedding** $X : \Omega \rightarrow \mathbb{R}^N = R^{K+2}$, $(u_1, u_2) \mapsto (x(u_1, u_2), y(u_1, u_2), I(u_1, u_2))$;
- For the map $(u_1, u_2) \mapsto (u_1, u_2, I(u_1, u_2))$, the image of X becomes the **graph** Γ_I of the image function I;
- Let $h = h_{ij} dy^i dy^j$ be a Riemannian metric on $\mathbb{R}^N$;
- Let $g = X^\star h$ be the pullback metric on Ω under the embedding $X : \Omega \rightarrow \mathbb{R}^N$;
- (Ω, g) is a Riemannian manifold;
- Example: With $h = \sum_i dx^i dx^i + \sum_k \beta^2 dI^k dI^k$, we obtain

$$(g_{ij}) = \begin{pmatrix} \sum_i (x_1^i)^2 + \beta^2 \sum_j (I_1^j)^2 & \sum_i x_1^i x_2^i + \beta^2 \sum_j I_1^j I_2^j \\ \sum_i x_1^i x_2^i + \beta^2 \sum_j I_1^j I_2^j & \sum_i (x_2^i)^2 + \beta^2 \sum_j (I_2^j)^2 \end{pmatrix}$$

Beltrami Diffusion (Kimmel et al.)

- Basic idea: **Minimizing the area functional**

$$S = \int \sqrt{g}\, du_1\, du_2$$

with respect to the image function I leads to the **Beltrami flow**

$$\frac{\partial I}{\partial t} = \Delta_g I,$$

where

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu)$$

is the **Laplace-Beltrami operator** of the Riemannian manifold (Ω, g).

- Minimizing

$$S(X, g, h) = \int d\sigma \sqrt{g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}(X)$$

selectively with respect to X and g yields known flows (e.g. heat flow, mean curvature flow, etc.).

- Making the induced metric non-definite yields inverse diffusion across the edges.

Beltrami Diffusion (Kimmel et al.)

Figure: **Beltrami diffusion** (http://www.cs.technion.ac.il/∼ron/)

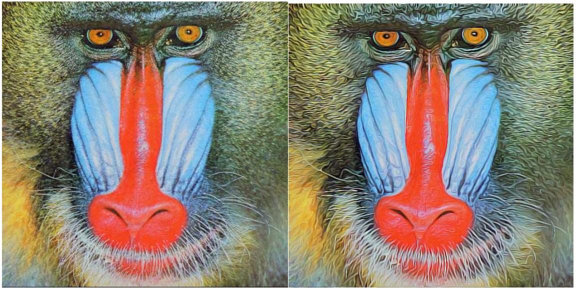
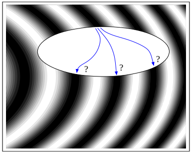


Figure: **Inverse Beltrami** (http://www.cs.technion.ac.il/∼ron/)

Basic problem: Reconstructing level sets of images (image courtesy of Ugo Boscain)



Basic idea: For one level curve, favor short and straight completions by Minimizing the functional

$$\gamma \mapsto J(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \sqrt{1 + K_\gamma^2(t)}\, dt$$

over a suitable space of curves.

Theorem (Boscain et al. 2012)

For every $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ *with* $(x_0, y_0) \neq (x_1, y_1)$ *and every* $v_0, v_1 \in \mathbb{R}^2 \setminus \{0\}$, *the functional* J *has a minimizer over the set*

$$\mathcal{D} = \{\gamma \in C^2([0,1];\mathbb{R}^2) \mid t \mapsto \|\dot{\gamma}(t)\| \sqrt{1 + K_\gamma^2(t)} \in L^1([0,1];\mathbb{R}),$$
$$\gamma(0) = (x_0, y_0),\ \gamma(1) = (x_1, y_1),\ \dot{\gamma}(0) \sim v_0,\ \dot{\gamma}(1) \sim v_1\}$$

Key point: This problem can be reformulated as an optimal control problem on a **sub-Riemannian manifold** by **lifting** γ to the projective cotangent bundle $PT^\star\mathbb{R}^2$ of $\mathbb{R}^2$

Hypoelliptic Diffusion (Boscain et al.)

Sub-Riemannian problem: Find $q = (x, y, \theta)$ that minimizes

$$l(q) = \int_0^1 (u_1^2(t) + u_2^2(t))^{1/2} dt$$

subject to

$$\begin{aligned} \dot{q} &= u_1 X_1(q) + u_2 X_2(q), \\ X_1(q) &= (\cos(\theta), \sin(\theta), 0),\ \ X_2(q) = (0, 0, 1), \\ q(b) &= (x_b, y_b, \theta_b),\ \ q(c) = (x_c, y_c, \theta_c), u_1, u_2 \in L^1([0, 1]). \end{aligned}$$

Considering the Stratonovitch stochastic differential equation

$$dq = dw_1 X_1(q) + dw_2 X_2(q)$$

leads to the Fokker-Planck diffusion equation

$$\frac{\partial \phi}{\partial t} = \Delta_H \phi$$

where $\Delta_H = X_1^2 + X_2^2$ (**hypoelliptic** Laplacian).

Hypoelliptic Diffusion (Boscain et al.)

Basic steps:

- **Lifting**: Lift image function $I : \mathbb{R}^2 \to \mathbb{R}$ to $\bar{I} : PT^\star\mathbb{R}^2 \to \mathbb{R}$;
- **Hypoelliptic diffusion**: For fixed $T > 0$, solve the solution at time T of the hypoelliptic heat equation

$$\frac{\partial \phi}{\partial t} = \Delta_H \phi, \quad \phi(0) = \bar{I}.$$

- **Projection**: Compute the reconstructed image by choosing the maximum fiber value: $I_T(x, y) = \max_{\theta \in P^1} \phi(x, y, \theta, T)$.

Note:

- $SE(2)$ is a double cover of $PT^\star\mathbb{R}^2$, hence the hypoelliptic heat kernel for Δ_H can be obtained from that on $SE(2)$.

Hypoelliptic Diffusion (Boscain et al.)

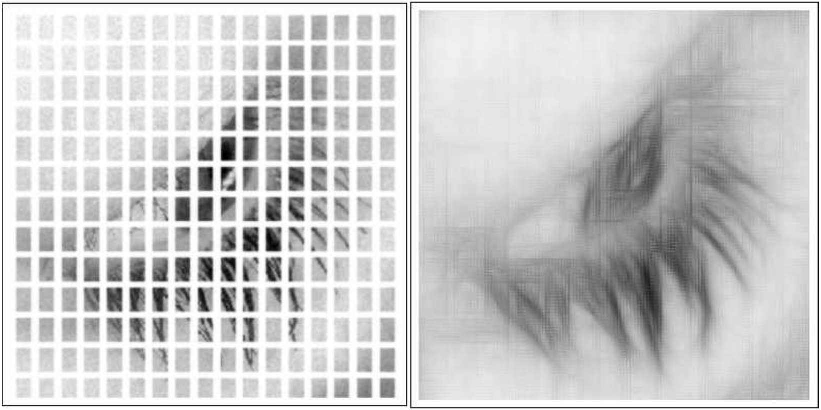


Figure: **Hypoelliptic Diffusion** (images courtesy of Ugo Boscain)

L^2 Diffusion: Basic Idea

- Let $E : H_0^k(\Omega) \to \mathbb{R}$ be a functional.
- Assume E is Gateaux differentiable at every $u \in H_0^k(\Omega)$, i.e. there exists a continuous linear mapping $dE|_u : H_0^k(\Omega) \to \mathbb{R}$ such that:

$$\lim_{t \to 0} \frac{E(u + tv) - E(u)}{t} = dE|_u(v), \quad \forall v \in H_0^k(\Omega).$$

- $dE|_u$ is the Gateaux differential of E at u (we shall assume $u \mapsto dE|_u$ is continuous on $H_0^k(\Omega)$).
- Let $u \in H_0^k(\Omega)$ and assume there exists $\nabla E|_u \in L^2(\Omega)$ such that

$$dE|_u(v) = \langle \nabla E|_u, v \rangle_{L^2}, \quad \forall v \in H_0^k(\Omega).$$

- $\nabla E|_u$ is the L^2-gradient of E at u.
- Consider the differential equation in H_0^k given by

$$\frac{\partial u}{\partial t}(t) = -\nabla E|_{u(t)}, \quad t > 0.$$

- We have:

$$\frac{d(E \circ u)}{dt}(t) = -\langle \nabla E|_{u(t)}, \nabla E|_{u(t)} \rangle_{L^2} \leq 0, \quad \forall t > 0.$$

- Hence the solution to the PDE evolves so as to minimize E (gradient descent).

L^2 Diffusion: Heat Flow

- Consider the functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by:

$$u \mapsto E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2.$$

- E is Gateaux differentiable on $H_0^1(\Omega)$ with Gateaux differential at $u \in H_0^1(\Omega)$ given by

$$dE|_u(v) = \int_\Omega \nabla u \nabla v, \quad \forall v \in H_0^1(\Omega)$$

- For each $u \in H^2(\Omega) \cap H_0^1(\Omega)$, the L^2 gradient of E at u is defined and is given by

$$\nabla E|_u = -\Delta u$$

- The gradient descent PDE for the functional E is then:

$$\frac{\partial u}{\partial t} = -\nabla E|_u = \Delta u$$

  ($\Rightarrow$ heat equation)

- We can therefore interpret the **isotropic diffusion** equation as a gradient descent equation for the functional

$$u \mapsto E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2.$$

L^2 Diffusion: Perona-Malik

What about **anisotropic** diffusions ?

- Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ of class C^2 and consider the functional

$$u \mapsto E(u) = \frac{1}{2} \int_\Omega g(\|\nabla u\|^2).$$

- $\forall u \in H^2(\Omega) \cap H_0^1(\Omega)$, the L^2 gradient of E at u is given by:

$$\nabla E|_u = -\nabla \cdot (g'(\|\nabla u\|^2)\nabla u)$$

- The gradient descent PDE for the functional E is then:

$$\frac{\partial u}{\partial t} = -\nabla E|_u = \nabla \cdot (g'(\|\nabla u\|^2)\nabla u)$$

- Different choices of g yield different anisotropic diffusion PDEs. Example: $g(s) = \log(1+s)$ yields

$$\frac{\partial u}{\partial t} = \nabla \cdot (\frac{1}{1+\|\nabla u\|^2}\nabla u)$$

- We can therefore also interpret anisotropic diffusion as gradient descent on a suitable functional.

A Geometric Picture of L^2 Diffusion PDES (Isotropic and Anisotropic)

Basic Observation:

- The gradient descent PDE

$$\frac{\partial u}{\partial t} = -\nabla E|_u$$

  can be recast in terms of **Riemannian geometry**.

With M a smooth manifold, let

- T_pM the tangent space to M at p,
- $T_p^\star M$ the cotangent space to M at p,
- g a Riemannian metric on M, $g_p : T_pM \times T_pM \to \mathbb{R}$, $(p \in M)$, yielding an isomorphism $T_pM \to T_p^\star M$ through $v \mapsto g_p(v, \cdot)$.

Application to our setting:

- $H_0^k(\Omega)$ is a Hilbert space and hence a (infinite-dimensional Hilbert) manifold,
- For each $u \in H_0^k(\Omega)$, the tangent space $T_u H_0^k(\Omega)$ is canonically identified with $H_0^k(\Omega)$ itself (since $H_0^k(\Omega)$ is a vector space),
- For each $u \in H_0^k(\Omega)$, the cotangent space $T_u^\star H_0^k(\Omega)$ is canonically identified with the dual space of $H_0^k(\Omega)$,
- The L^2 metric $g_0 : (v, w) \mapsto g_0(v, w) = \langle v, w \rangle_{L^2}$ on each tangent space $T_u H_0^k(\Omega)$ gives $H_0^k(\Omega)$ the structure of a (infinite-dimensional) Riemannian manifold,
- g_0 defines a **distance** on $H_0^k(\Omega)$ (our **space of images** ...)

A Geometric Picture of Diffusion PDES (Isotropic and Anisotropic)

- **Basic Observation:** $g_0(\nabla E, \cdot) = dE$; hence, starting from a functional E on $H_0^k(\Omega)$, g_0 is what allows us to go from the Gateaux differential dE to the L^2 gradient ∇E, and hence to the gradient descent equation $\frac{\partial u}{\partial t} = -\nabla E|_u$.
- Gradient descent on E with respect to g_0 yields a **path on the space of images** $H_0^k(\Omega)$.
- **Basic Question**: Is g_0 an appropriate metric on the space of images ?
- **Changing the metric** g_0 to some other metric changes the gradient ∇E and hence yields a **different path** on the space of images $H_0^1(\Omega)$, i.e. **a new class of diffusion equations**.

H^1 Sobolev Diffusion: Dirichlet Functional

- For each $\lambda > 0$, define the metric $g_\lambda : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by $(v, w) \mapsto g_\lambda(v, w) = (1-\lambda)\langle v, w\rangle_{L^2} + \lambda\langle v, w\rangle_{H^1}$,
- Consider the functional $E : H_0^1(\Omega) \to \mathbb{R}$, $u \mapsto E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2$,
- The gradient of E at $u \in H_0^1(\Omega)$ with respect to g_λ is defined by

$$g_\lambda(\nabla_{g_\lambda} E|_u, v) = dE|_u(v), \quad \forall v \in H_0^1(\Omega).$$

- With $\xi = 1$ (resp. $\xi = -1$), the equation

$$\frac{du}{dt} = -\xi \nabla_{g_\lambda} E|_{u(t)}, \; t > 0,$$

  is a gradient descent (resp. ascent) equation on E.
- Recall: Gradient ascent on E is ill-posed under the L^2 metric (reverse heat equation).

H^k $(k \geq 1)$ Sobolev Diffusion: Dirichlet Functional

With the H^1 Sobolev metric, we have:

Theorem (Calder, M., Yezzi 2010)

Let $\lambda > 0$, let $\Omega \subset \mathbb{R}^n$ open with smooth boundary, let $\xi \in \{+1, -1\}$, $k \in \mathbb{N}$, $0 < \gamma < 1$. Then, $\forall u_0 \in H_0^1(\Omega)$ (resp. $L^2(\Omega)$, $C^{k,\gamma}(\bar{\Omega})$), there exists a unique $u \in C^1([0,\infty[; H_0^1(\Omega))$ (resp. $C^1([0,\infty[; L^2(\Omega))$, $C^{k,\gamma}(\bar{\Omega})$) satisfying

$$\begin{aligned} \frac{du}{dt} &= -\xi \nabla_{g_\lambda} E|_{u(t)}, \ t > 0, \\ u(0) &= u_0. \end{aligned}$$

We can similarly define H^k Sobolev gradients, with $k > 1$; we obtain:

Theorem (Calder, M., Yezzi 2010)

Let $\lambda > 0$, let $\Omega \subset \mathbb{R}^n$ open with smooth boundary, let $\xi \in \{+1, -1\}$, let $1 \leq m \leq k$. Then, $\forall u_0 \in H_0^m(\Omega)$ (resp. $L^2(\Omega)$, $C^{k,\gamma}(\bar{\Omega})$) there exists a unique $u \in C^1([0,\infty[; H_0^\infty(\Omega))$ (resp. $C^1([0,\infty[; L^2(\Omega))$, $C^1([0,\infty[; C^{k,\gamma}(\bar{\Omega}))$) satisfying

$$\begin{aligned} \frac{du}{dt} &= -\xi \nabla_{k,\lambda} E|_{u(t)}, \ t > 0, \\ u(0) &= u_0. \end{aligned}$$

Theorem (Calder, M., Yezzi 2010)

Let E denote the Dirichlet functional on $H_0^1(\Omega)$, let $0 < \gamma < 1$, let $u_0 \in C^{0,\gamma}(\bar{\Omega}) \cap H_0^1(\Omega)$, and let $u \in C^1([0,\infty[; C^{0,\gamma}(\bar{\Omega})) \cap C^1([0,\infty[; H_0^1(\Omega))$ be the unique solution to the H^1-gradient descent equation on E:

$$\begin{aligned} \frac{du}{dt} &= -\nabla_{g_\lambda} E|_{u(t)}, \;\; t > 0, \\ u(0) &= u_0; \end{aligned}$$

then, $\forall (x,t) \in \Omega \times [0,\infty[$:

$$\min_{y\in\Omega} u_0(y) \leq u(x,t) \leq \max_{y\in\Omega} u_0(y).$$

- Maximum principles can also be derived for other function spaces (but with H^1 metric);
- Maximum principle does not hold for H^k metric with $k > 1$;
- Existence of a maximum principle $\Rightarrow$ no "extra edges" created by the semigroup (Hummel '89).

Sobolev Diffusion: Sharpening PDEs

- Under the Sobolev metric, the functional

$$E : u \mapsto E(u; u_0) = \frac{1}{4} \left(\int_\Omega \|\nabla u_0\|^2 \right) \left(\frac{\int_\Omega \|\nabla u\|^2}{\int_\Omega \|\nabla u_0\|^2} - \alpha \right)^2,$$

  leads to the gradient descent equation

$$\frac{du}{dt} = \left(\frac{\int_\Omega \|\nabla u\|^2}{\int_\Omega \|\nabla u_0\|^2} - \alpha \right) \Delta (I - \Delta)^{-1} u, \ \ t > 0,$$

  where $\alpha \in \mathbb{R}$ ($\alpha < 1 \Rightarrow$ **blurring**, $\alpha > 1 \Rightarrow$ **sharpening**).
- For the functional

$$E : u \mapsto E(u; u_0) = -\frac{1}{2} \int_\Omega \|\nabla u\|^2 + \frac{\mu}{2} \|u - u_0\|^2_{H^1}$$

  the gradient descent equation under the Sobolev metric is

$$\frac{du}{dt} = -\Delta (I - \lambda \Delta)^{-1} u + \mu (u_0 - u), \ \ t > 0.$$

Sobolev Diffusion: Generalization of Perona-Malik/You-Kaveh

- Let $\mathcal{G}$ be a compatible inner product on $H_0^k(\Omega)$,
- Let the linear operator $\mathcal{L} : H^{-k}(\Omega) \to H_0^k(\Omega)$ be defined by

$$\mathcal{G}(\mathcal{L}u, v) = \langle u, v \rangle_{H^{-k}, H_0^k}, \; \forall u \in H^{-k}(\Omega), v \in H_0^k(\Omega)$$

Theorem (Calder, M. 2011)

Let $k \in \mathbb{N}$, $g : [0, \infty[\to \mathbb{R}$ bounded and C^1 such that $|sg'(s)| \leq C \; \forall s \geq 0$ (for some $C > 0$). Then, $\forall u_0 \in H^k(\Omega)$, there exists a unique $u \in C^1([0, T]; H^k(\Omega))$ such that:

$$\begin{aligned} \frac{du}{dt} &= \mathcal{L} \begin{cases} -\Delta^{k/2}(g(|\Delta^{k/2}u|^2)\Delta^{k/2}u), & k \text{ even} \\ \Delta^{(k-1)/2}\mathrm{div}(g(\|\nabla\Delta^{(k-1)/2}u\|^2)\nabla\Delta^{(k-1)/2}u) & k \text{ odd} \end{cases} \\ u(0) &= u_0. \end{aligned}$$

Furthermore, if $u_0 \in H_0^k(\Omega)$ then $u(t) \in H_0^k(\Omega)$ for all $t \in]0, T]$

Note:

- $k = 1 \Rightarrow$ Perona-Malik.
- Sobolev regularizes Perona-Malik while remaining a descent equation on the Perona-Malik functional (unlike Catté et al.'s regularization scheme).

- Let $E : H_0^1(\Omega) \to \mathbb{R}$ be given by $E(u) = \int_\Omega \phi(\|\nabla u\|^2)$, with $\phi(s) = s - \beta(1 - e^{-s})$ $(\beta > 1)$,
- Let $g : s \mapsto g(s) = 1 - \beta e^{-s/K^2}$;

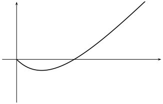
The evolution equation

$$\begin{aligned} \frac{du}{dt} &= -(I - \lambda\Delta)^{-1}\nabla \cdot (g(\|\nabla u\|^2)\nabla u), \ \ t > 0 \\ u(0) &= u_0 \end{aligned}$$

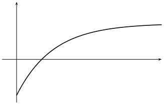
can be rewritten as

$$\begin{aligned} \frac{du}{dt} &= -(I - \lambda\Delta)^{-1}\Delta u + \beta(I - \lambda\Delta)^{-1}\nabla \cdot (e^{-(\|\nabla u\|^2)/K^2}\nabla u), \\ u(0) &= u_0 \end{aligned}$$

$\Rightarrow$ smoothing of weak edges + sharpening of strong edges:



(a) $\phi(s)$



(b) $g(s) = \phi'(s)$

Figure 1: The qualitative properties of (a) the combined smoothing/sharpening potential $\phi(s)$ and (b) the diffusion coefficient, $g(s) = \phi'(s)$.

Sobolev Diffusion: Experimental Results

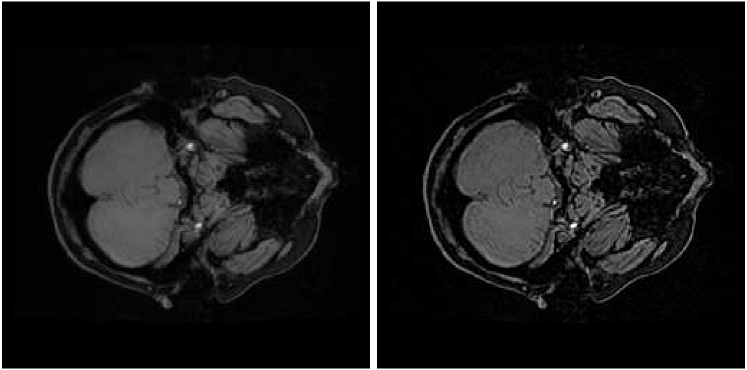


Figure: **Isotropic Sobolev sharpening**

Sobolev Diffusion: Experimental Results

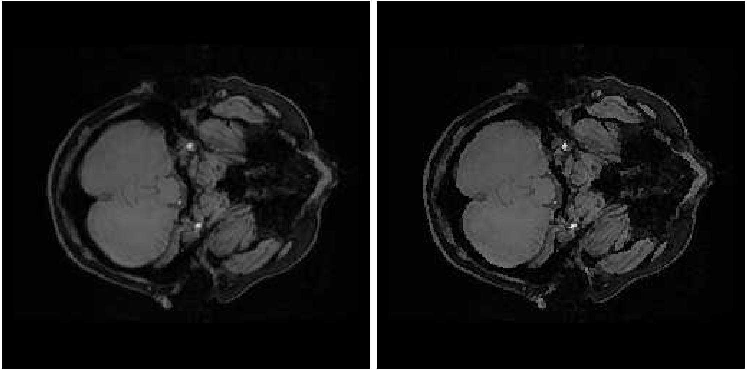


Figure: **Shock Filter**

Sobolev Diffusion: Experimental Results

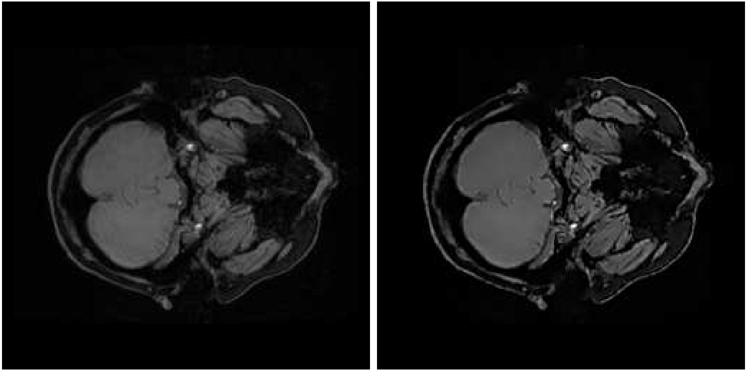


Figure: **Combined Sobolev sharpening/smoothing**

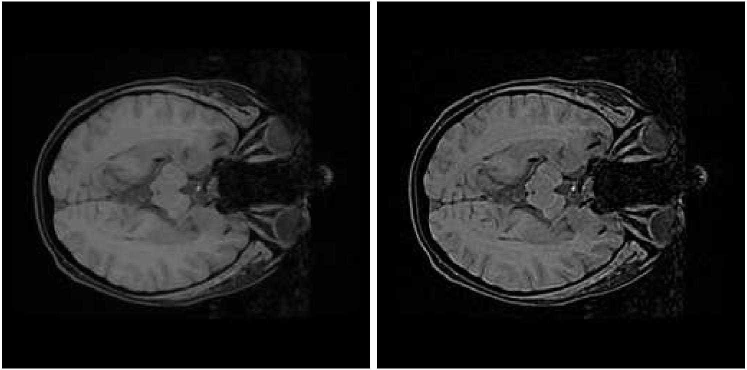
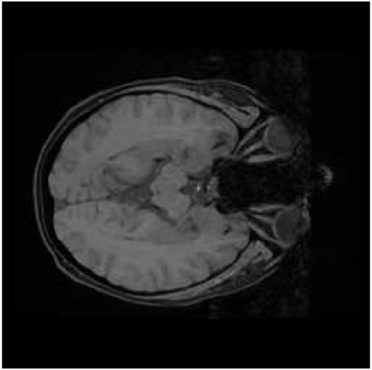


Figure: **Isotropic Sobolev sharpening**



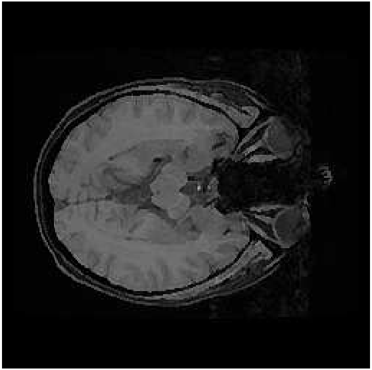


Figure: **Shock Filter**

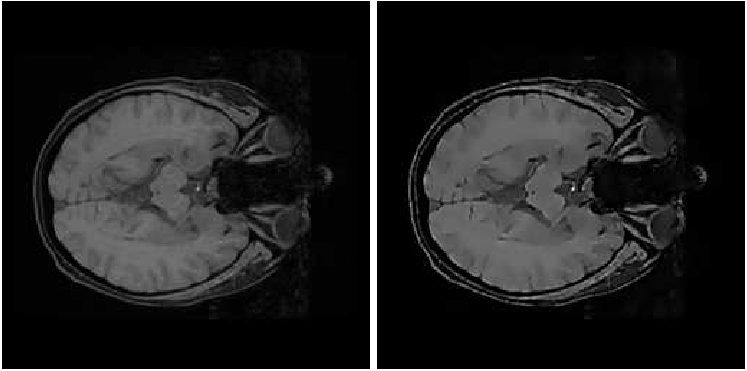


Figure: **Combined Sobolev sharpening/smoothing**

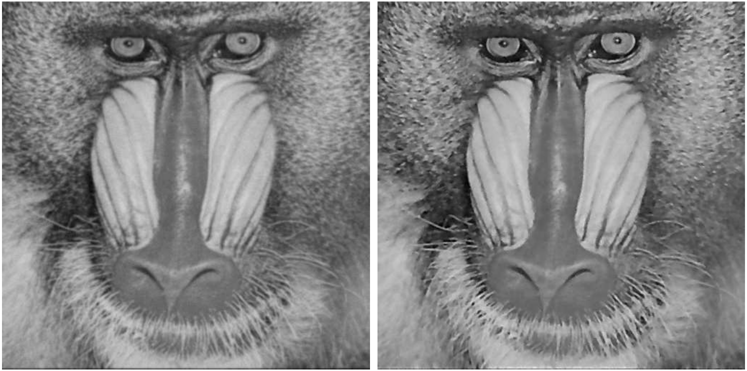


Figure: **Combined Sobolev sharpening/smoothing on noisy blurred image**

Figure: **Combined Sobolev sharpening/smoothing on noisy blurred image**

Key Points:

- Changing the geometry on the space of images from L^2 to Sobolev leads to new families of diffusion equations which overcome instabilities associated with L^2 diffusions.
- Immediate extension: Non-constant Riemannian metrics on the space of images.

But Sobolev diffusion is still a diffusion in $\mathbb{R}^2$ ...

Let us revisit the main objective of image diffusion:

- To remove certain types of image singularities ...
- while preserving other types of singularities.

But:

- The wavefront set of a singularity is a subset of $T^\star R^2$ ...

This suggests the following attempt at defining a microlocal diffusion:

Definition (Microlocal Diffusion)

A one-parameter semigroup $(T_t)_{t\geq 0}$ defined on $\mathcal{D}'(T^\star\mathbb{R}^2)$ such that, with $u \in \mathcal{D}'(\mathbb{R}^2)$ given by

$$u = u_S + u_N,$$

where $u_S \in \mathcal{D}'_S(\mathbb{R}^2)$ ("signal") and $u_N \in \mathcal{D}'_N(\mathbb{R}^2)$ ("noise"), we have

$$WF(\pi(T_t(l(u)))) \to WF(u_S)$$

as $t \to \infty$, in some appropriate topology + other conditions ($l(u)$ denoting the lift of u to $\mathcal{D}'(T^\star\mathbb{R}^2)$ and π the projection back to $\mathcal{D}'(\mathbb{R}^2)$).

THANK YOU

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