Recent advances in tomographic image reconstruction from x-ray interior data

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Support

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Introduction

Starting from a breakthrough paper by Gelfand and Graev (1991), inversion of the Hilbert transform became a very important tool for image reconstruction in tomography.

1. M. Defrise, F. Noo, R. Clackdoyle, H. Kudo, *Truncated Hilbert transform and image reconstruction from limited tomographic data*, Inverse Problems, 2006.
2. Y. B. Ye, H. Y. Yu, G. Wang, *Local reconstruction using the truncated Hilbert transform via singular value decomposition*, Journal of X-Ray Science and Technology, 2008.
3. M. Courdurier, F. Noo, M. Defrise, H. Kudo, *Solving the interior problem of computed tomography using a priori knowledge*, Inverse Problems, 2008.

Introduction

Let $D_\phi(y, \beta)$ be the cone beam transform of ϕ:

$$D_\phi(y, \beta) = \int_0^\infty \phi(y + t\beta) dt,$$

Let $y(s)$ be a piecewise smooth curve. We assume that $y(s)$ is continuous, does not self-intersect, and is traversed in one direction as s varies over some interval I. Pick any two $s_1, s_2 \in I$, $s_1 \neq s_2$. Let α be a unit vector along the chord $y(s_1), y(s_2)$.

Then

$$\frac{1}{2}\int_{s_1}^{s_2} \frac{1}{|x-y(s)|} \, \frac{\partial}{\partial\lambda} D_\phi \left(y(\lambda), \frac{x-y(s)}{|x-y(s)|} \right) \Big|_{\lambda=s} ds$$
$$= \int \frac{\phi(x+t\alpha)}{t} dt.$$

Knowing the cone beam transform of ϕ one can compute the Hilbert transform of ϕ on chords of the source trajectory.

The problem

What to do if the data are truncated and not sufficient for computing the Hilbert transform of ϕ along the entire line, or even on the intersection of $\operatorname{supp}\phi$ with the line? In this case finding ϕ involves analytic continuation. We study a number of problems related to this situation.

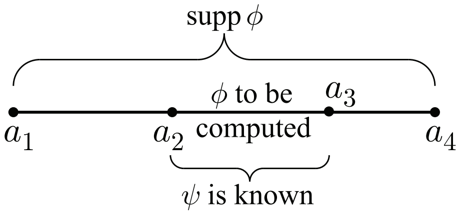
We may consider each chord $y(s_1), y(s_2)$ (and the associated Hilbert line) separately from all others, so we assume that the function $\phi(x)$ depends on a one-dimensional argument $x \in \mathbb{R}$.

Example: Interior Problem

Suppose supp$\phi \subset [a_1, a_4]$ and the data are ψ:

$$(\mathcal{H}\phi)(y) := \frac{1}{\pi} \int \frac{\phi(x)}{y - x} dx = \psi(y)$$

on the measured segment $y \in [a_2, a_3]$. The problem is to find $\phi(x)$ on $[a_2, a_3]$.



Truncated Hilbert Transform

Consider the operator

$$\mathcal{H}: \ L^2([a_1, a_4]) \to L^2([a_2, a_3]).$$

Our main results are the following:

- SVD for $\mathcal{H}$;
- Complete description of the null space of $\mathcal{H}$,
- Uniform asymptotics of the singular functions of $\mathcal{H}$ as $n \to \infty$ on $[a_2, a_3]$,
- Asymptotics of singular values of $\mathcal{H}$ as $n \to \infty$.
- A study of related transforms.

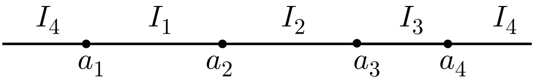
Fix any four real numbers $a_i \in \mathbb{R}$, $a_1 < a_2 < a_3 < a_4$, and define the operator

$$L(x, d_x)f := (P(x)f'(x))' + 2(x-\sigma)^2 f(x),$$

where f is piecewise smooth with isolated singularities, and

$$P(x) := (x-a_1)(x-a_2)(x-a_3)(x-a_4),$$
$$\sigma := (a_1 + a_2 + a_3 + a_4)/4.$$

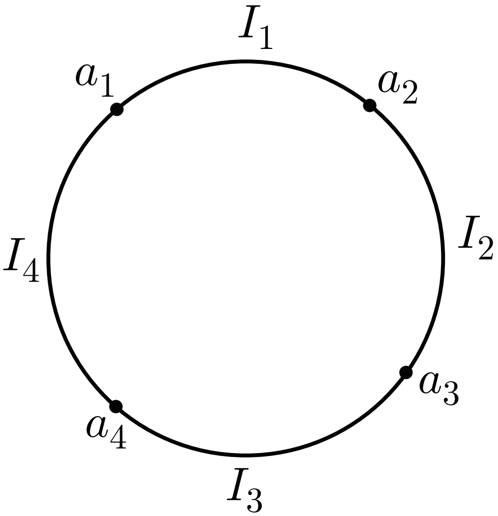
We will frequently write L instead of $L(x, d_x)$.



Denote $I_k = [a_k, a_{k+1}]$, $k = 1, 2, 3$, and $I_4 = (-\infty, a_1] \cup [a_4, \infty)$. Let $\mathcal{H}_k$, $1 \leq k \leq 4$, be the FHT that integrates over I_k:

$$(\mathcal{H}_k \psi)(x) := \frac{1}{\pi} \int_{I_k} \frac{\psi(y)}{y - x} dy, \ \ \psi \in C^2(I_k)$$

If $k = 4$, then ψ is supposed to decay sufficiently fast at infinity, so the integral is absolutely convergent.


I_1
a_1
a_2
I_4
I_2
a_3
a_4
I_3

Proposition. [AK 2010, 2011] One has

$$(\mathcal{H}_k L(y, d_y)\psi)(x) = L(x, d_x)(\mathcal{H}_k\psi)(x), \ x \neq a_k, a_{k+1},$$

for any $\psi \in C^\infty(I_k)$ and $k = 1, 2, 3$.

The Cauchy operator approach

Let γ be a simple smooth bounded oriented contour in $\mathbb{C}$, and $\phi(t)$ be a Hòlder continuous complex-valued function on γ. By definition, the Cauchy transform $\Phi(z) = (C_\gamma \phi)(z)$ of $\phi(t)$ is

$$\Phi(z) = (C_\gamma \phi)(z) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(t)dt}{t - z}.$$

It is clear that $\Phi(z)$ is analytic in $\mathbb{C} \setminus \gamma$ and

$$\Phi(z) = \frac{\Phi_1}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty,$$

where $\Phi_1 = -\frac{1}{2\pi i} \int_\gamma \phi(t)dt$.

The Cauchy operator approach

Let a_j, $j = 1, 2, 3, 4$, denote distinct finite points in $\mathbb{C}$, $P(z) = \prod_{j=1}^{4}(z - a_j)$, and let L_Q denote the differential operator $L_Q f = (P(z)f'(z))' + Q(z)f(z)$, where $Q(z)$ is a polynomial. Let γ be a simple smooth bounded oriented contour with endpoints a_j, a_k, $j \neq k$, that does not contain any of the remaining two points, and let f be a restriction of an analytic function to γ. We want to find Q, such that L_Q commutes with C_γ.

The Cauchy operator approach

Assume that $L_Q C_\gamma f = C_\gamma L_Q f$. Then $L_Q C_\gamma f = O(z^{-1})$ as $z \to \infty$. This equation could be written in the form

$$P'(z) \int_\gamma \frac{f(t)dt}{(t-z)^2} + 2P(z) \int_\gamma \frac{f(t)dt}{(t-z)^3} + Q(z) \int_\gamma \frac{f(t)dt}{t-z} = O(z^{-1}),$$

which, after expanding at $z = \infty$, yields

$$2zM_0 + 2M_1 - 4\sigma M_0 - Q(z)\left[\frac{M_0}{z} + \frac{M_1}{z^2} + \cdots\right] = O(z^{-1}),$$

where $\sigma = (a_1 + a_2 + a_3 + a_4)/4$ and $M_j = \int_\gamma t^j f(t)dt$. In general, $M_0 \neq 0$, so $Q(z)$ is a quadratic polynomial with the leading coefficient 2, the linear coefficient -4σ and a free constant term. Thus, $Q(z) = 2z^2 - 4\sigma z + \mathrm{const}$ is a necessary condition for $L_Q C_\gamma f = C_\gamma L_Q f$. For convenience, we choose $Q(z) = 2(z-\sigma)^2$ and $L_Q = L$.

The Cauchy operator approach

To show that $L = L_Q$ indeed commutes with C_γ, consider the following Riemann-Hilbert problem (RHP): find a function $F(z)$, such that:

$$\begin{cases} F(z) \ \text{ is analytic in } \mathbb{C} \setminus \gamma \\ F_+(z) - F_-(z) = f(z) \quad z \in \gamma, \\ F(z) \to 0 \ \text{ as } \ z \to \infty. \end{cases}$$

This problem has a unique solution $F(z) = C_\gamma[f]$ if we require $F_\pm(z) \in L^2(\gamma)$.

The Cauchy operator approach

Consider now $\Phi_1(z) = LF(z) = LC_\gamma f(z)$. It satisfies the RHP

$$\begin{cases} \Phi(z) \ \text{ is analytic in } \mathbb{C}\setminus\gamma \\ \Phi_+(z) - \Phi_-(z) = Lf(z) \quad z \in \gamma, \\ \Phi(z) \to 0 \ \text{ as } \ z \to \infty, \end{cases}$$

where the last condition follows from the construction of L, described above. On the other hand, $\Phi_2(z) = C_\gamma Lf(z)$ also satisfies the RHP. Thus, the fact that L commutes with C_γ will follow from the uniqueness of solution of the RHP, provided that $C_\gamma Lf \in L^2(\gamma)$ and $LC_\gamma f \in L^2(\gamma)$. The former inclusion follows from the smoothness of f. So, to prove $C_\gamma Lf = LC_\gamma f$, it is sufficient to show that $LC_\gamma f \in L^2(\gamma)$. Generically, $C_\gamma f$ has the behavior $\phi_1(z-a) + \phi_2(z-a)\ln(z-a)$, where $\phi_{1,2}$ are analytic functions, near an endpoint a of γ. It is trivial to check that the limiting values of $(P(C_\gamma f)')'$ are square integrable on γ near a and, thus, $LC_\gamma f \in L^2(\gamma)$.

The Cauchy operator approach

Since the FHT of a function is a linear combination of the limiting values of its Cauchy transform, we immediately prove that L commutes with $\mathcal{H}$.

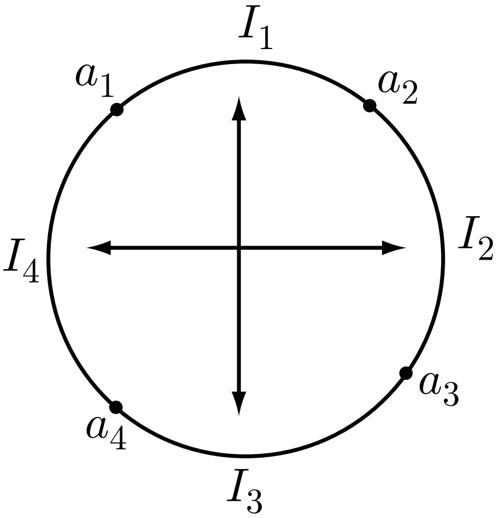
Next we consider the following two pairs of Sturm-Liouville boundary value problems (SLPs):

$$Lg_1 = \lambda g_1 \text{ on } I_1, g_1 \text{ is bounded on } I_1,$$
$$Lg_3 = \lambda g_3 \text{ on } I_3, g_3 \text{ is bounded on } I_3,$$

and

$$-Lg_2 = \lambda g_2 \text{ on } I_2, g_2 \text{ is bounded on } I_2,$$
$$-Lg_4 = \lambda g_4 \text{ on } I_4, g_4 \text{ is bounded on } I_4.$$

To the second problem in the last pair we add the requirement that the function $g_4(1/z)/z$ have a removable singularity at $z = 0$. All four are singular SLPs with limit-circle non-oscillatory end-points.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a_1
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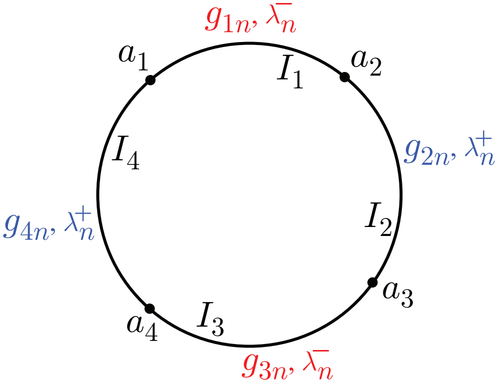
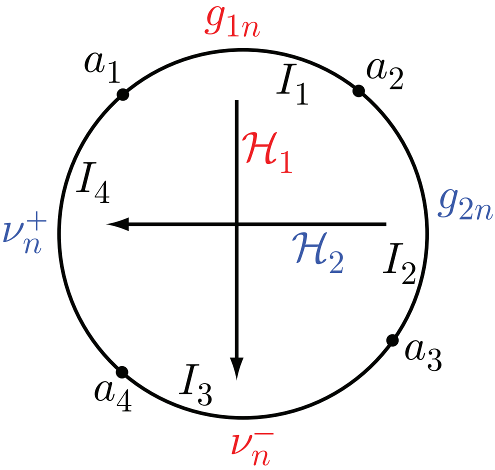
Proposition. [AK 2010, 2011] All four SLPs have simple and discrete spectra, the associated eigenfunctions $\{g_{k,n}\}$, $n \geq 0$, are orthogonal and complete in $L^2(I_k)$, $k = 1, 2, 3, 4$. Moreover, the eigenvalues of the two problems in the first pair coincide, and the eigenvalues of the two problems in the second pair coincide as well.

In what follows we assume that the functions $g_{k,n}$ are normalized so that $\|g_{k,n}\|_k = 1$, $n \geq 0$, where $\|\cdot\|_k$ is the norm in $L^2(I_k)$, $k = 1, 2, 3, 4$. The eigenvalues on I_1, I_3 and I_2, I_4 are denoted λ_n^- and λ_n^+, respectively.



Denote $\nu_n^- := \|\mathcal{H}_1 g_{1,n}\|_3$ and $\nu_n^+ := \|\mathcal{H}_2 g_{2,n}\|_4$, $n \geq 0$.

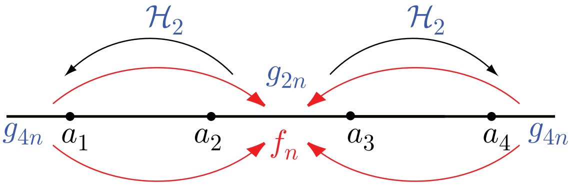


Proposition [AK 2010, 2011] One has $\mathcal{H}_1 g_{1,n} = \nu_n^- g_{3,n}$ on I_3 and $\mathcal{H}_2 g_{2,n} = \nu_n^+ g_{4,n}$ on I_4. In other words, the sets of triples $(\nu_n^-, g_{1,n}, g_{3,n})$ and $(\nu_n^+, g_{2,n}, g_{4,n})$, $n \geq 0$, form the SVD for the operators $\mathcal{H}_1 : L_2(I_1) \to L_2(I_3)$ and $\mathcal{H}_2 : L_2(I_2) \to L_2(I_4)$, respectively.

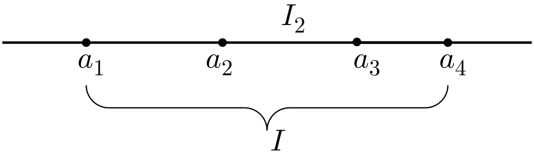
We will use additional notations:

$I := [a_1, a_4]$;
$\mathcal{H}$ is the FHT that integrates over I;
$w(x) := \sqrt{(a_4 - x)(x - a_1)}$;
$\langle \cdot, \cdot \rangle_k$ and $\langle \cdot, \cdot \rangle$ are the inner products in $L^2(I_k)$, $k = 1, \dots, 4$, and $L^2(I)$, respectively;
$\| \cdot \|_I$ denotes the norm in $L^2(I)$.

Consider the eigenfunctions $g_{4,n}$ analytically extended to all of $\mathbb{C} \setminus I_2$. By an earlier proposition, $g_{4,n}$ and $\mathcal{H}_2 g_{2,n}$ are proportional on I_4 and, hence, on $\mathbb{C} \setminus I_2$. Define $g_{4,n}(x) = \frac{1}{2}(g_{4,n}(x+i0) + g_{4,n}(x-i0))$ for $x \in (a_2.a_3)$. By the Plemelj formulae, $g_{4,n}(x)$ and $\mathcal{H}_2 g_{2,n}(x)$ are proportional for all $x \in \mathbb{R}$, $x \neq a_2, a_3$. Denote $f_n = c_n g_{4,n}$, where c_n's are chosen so that $\|f_n\|_I = 1$. By a proposition of [AK2011], f_n's are orthonormal in $L^2(I)$.



Thus $\mathcal{H} f_n = \nu_n g_{2,n}$ on I_2 and **the triples** $(\nu_n, f_n, g_{2,n})$**,** $n \geq 0$**, form an SVD for the operator** $\mathcal{H}: L^2(I) \to L^2(I_2)$.



The null-space of $\mathcal{H}$

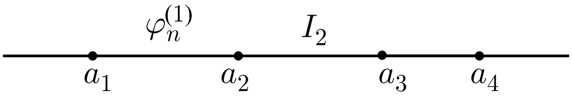
In particular, $\mathrm{Ker}(\mathcal{H}) = (\mathrm{span}(\{f_n\}_{n\geq 0}))^{\perp}$. Indeed, pick any function $\mu \in \mathrm{Ker}(\mathcal{H})$. By definition, $(\mathcal{H}\mu)(x) \equiv 0$ on I_2. Hence

$$0 = \langle \mathcal{H}\mu, g_{2,n}\rangle_2 = -\langle \mu, \mathcal{H}_2 g_{2,n}\rangle = -\nu_n \langle \mu, f_n\rangle.$$

Since $\nu_n \neq 0$, we get that $\langle \mu, f_n\rangle = 0$ for all $n \geq 0$. On the other hand, since $g_{2,n}$'s are complete in $L^2(I_2)$, we see that if a function is orthogonal to all f_n on I, then this function belongs to $\mathrm{Ker}(\mathcal{H})$. The problem is to find a basis in $\mathrm{Ker}(\mathcal{H})$.

The null-space of $\mathcal{H}$

Fix any pair (p, q) such that $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 2$. Let $\tilde{\varphi}_n^{(1)}$, $n \geq -1$, be a collection of linearly independent C^∞ functions, whose span is dense in $L^q(I_1)$. They are extended by zero to all of I. Functions in the range of $\mathcal{H}$ satisfy $\langle \varphi, 1/w \rangle = 0$. By adjusting $\tilde{\varphi}_n^{(1)}$, $n \geq -1$, we get another linearly independent collection $\varphi_n^{(1)}$, $n \geq 0$, which is dense in the subspace $\{\varphi \in L^q(I_1) : \langle \varphi, 1/w \rangle_1 = 0\}$. By construction, $\varphi_n^{(1)}/w \in L^1(I)$.



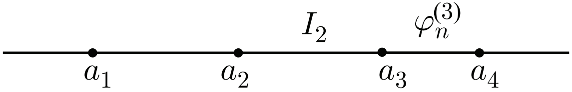
The null-space of $\mathcal{H}$

Consider $\mathcal{H}(\varphi_n^{(1)}/w)$. At worst, $\varphi_n^{(1)}/w$ behaves like $(x-a_1)^{-1/2}$ as $x \to a_1^+$ and is discontinuous at $x = a_2$. Using the results of Gakhov (1966), we get that $\mathcal{H}(\varphi_n^{(1)}/w)$, at worst, behaves like $|x-a_1|^{-1/2}$ as $x \to a_1^+$ and $\ln|x-a_2|$ as $x \to a_2$. Thus $w\mathcal{H}(\varphi_n^{(1)}/w)$ is bounded near a_1 and $w\mathcal{H}(\varphi_n^{(1)}/w) \in L^2(I)$. Using an FHT inversion formula, define

$$h_n^{(1)} = \mathcal{H}^{-1}(\varphi_n^{(1)}) = -w\mathcal{H}(\varphi_n^{(1)}/w),\ \ n \geq 0,$$

i.e. $\mathcal{H}(h_n^{(1)}) = \varphi_n^{(1)}$ on I. As we have just established, $h_n^{(1)} \in L^2(I)$. By construction, $\varphi_n^{(1)} \equiv 0$ on I_2 (and I_3), hence $h_n^{(1)} \in \mathrm{Ker}(\mathcal{H})$.

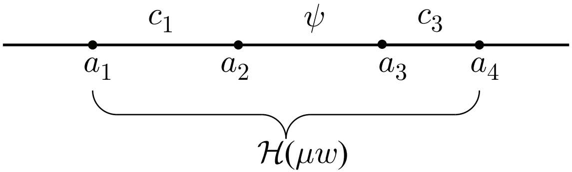
The null-space of $\mathcal{H}$



In a similar fashion we can add more functions to the basis by starting with a dense set in $C^\infty(I_3)$: $h_n^{(3)} = \mathcal{H}^{-1}(\varphi_n^{(3)}) \in \mathrm{Ker}(\mathcal{H})$. Since $\mathcal{H}(h_n^{(1)}) = \varphi_n^{(1)}$ and $\mathcal{H}(h_n^{(3)}) = \varphi_n^{(3)}$, it is clear that the functions $h_n^{(1)}, h_n^{(3)}$, $n \geq 0$, are linearly independent. It turns out that $h_n^{(1)}$, $h_n^{(3)}$, $n \geq 0$, are not dense in $\mathrm{Ker}(\mathcal{H})$. To find what is missing, pick $\mu \in \mathrm{Ker}(\mathcal{H})$ which is orthogonal to all $h_n^{(1)}$ and $h_n^{(3)}$.

The null-space of $\mathcal{H}$

We can show that $\mathcal{H}(\mu w) = c_1$ on I_1 and $\mathcal{H}(\mu w) = c_3$ on I_3. Denote $\psi = \mathcal{H}(\mu w)|_{I_2}$.



The null-space of $\mathcal{H}$

After some calculations and using, in particular, the Poincare-Bertrand transposition formula, we get

$$\psi_1(z) - \frac{1}{\pi^2}\int_{I_2} K(x,z)\psi_1(x)dx = \frac{c_1}{\pi}\phi_1(z) + \frac{c_3}{\pi}\phi_3(z),\ z \in I_2.$$

where $\psi_1 = \psi/w$,

$$K(x,z) := \frac{1}{x-z}\left[\ln\left|\frac{z-a_4}{z-a_1}\right| - \ln\left|\frac{x-a_4}{x-a_1}\right|\right],$$

and $\phi_1(z)$, $\phi_3(z)$, are the Hilbert transforms of $\int_{I_1}\frac{dx}{w(x)(x-y)}$, $\int_{I_3}\frac{dx}{w(x)(x-y)}$, respectively.

The null-space of $\mathcal{H}$

Consider the homogeneous equation, which is obtained by setting $c_1 = c_3 = 0$. It can be shown that the homogeneous equation has only the trivial solution. By the Fredholm alternative, equations

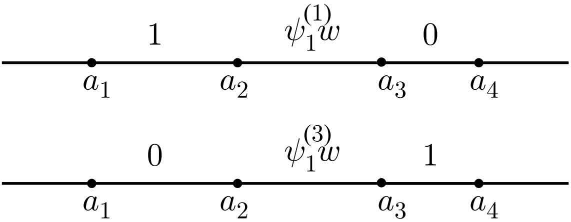
$$\psi_1^{(j)}(z) - \frac{1}{\pi^2}\int_{I_2} K(x,z)\psi_1^{(j)}(x)dx = \frac{1}{\pi}\phi_j(z),\ j = 1, 3,$$

have unique solutions $\psi_1^{(j)}$.

The null-space of $\mathcal{H}$

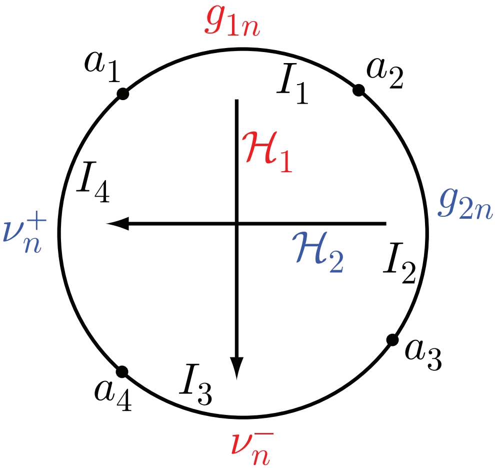
Hence we need to add only the following two functions to the families $h_n^{(1)}, h_n^{(3)}$:

$$\tilde{h}^{(1)} := (1/w)\mathcal{H}^{-1}([1, \psi_1^{(1)} w, 0]), \ \tilde{h}^{(3)} := (1/w)\mathcal{H}^{-1}([0, \psi_1^{(3)} w, 1]).$$



Proposition. Consider the operator $\mathcal{H}: L^2(I) \to L^2(I_2)$. The collection of functions consisting of $\{h_n^{(1)}, h_n^{(3)}\}$, $n \geq 0$, and $\tilde{h}^{(1)}$, $\tilde{h}^{(3)}$, forms a basis of $\mathrm{Ker}(\mathcal{H})$.

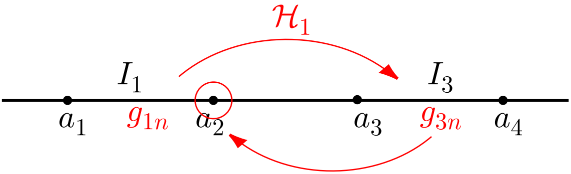
Asymptotics of singular values and eigenfunctions



$\mathcal{H}_1 g_{1,n} = \nu_n^- g_{3,n}$ and $\mathcal{H}_2 g_{2,n} = \nu_n^+ g_{4,n}$

Asymptotics of ν_n^- as $n \to \infty$

By construction, $\mathcal{H}_1 g_{1,n} = \nu_n^- g_{3,n}$ on I_3. After analytic continuation the equality holds on $\mathbb{C} \setminus I_1$. By defining $g_{3,n}(x) := \mathrm{Re}(g_{3,n}(x + i0))$, $x \in (a_1, a_2)$, the equality $\mathcal{H}_1 g_{1,n} = \nu_n^- g_{3,n}$ holds for all $x \in (a_1, a_2)$ as well. The two functions have logarithmic singularities at a_2. Comparing the leading coefficients in front of the logarithmic singularities of both functions as $x \to a_2^-$ allows us to estimate ν_n^-.

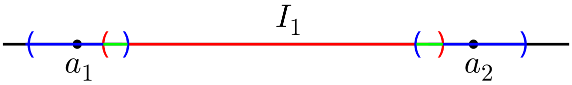


Asymptotics of ν_n^- as $n \to \infty$

We need:

1) The asymptotics of $g_{1,n}(a_2)$ as $n \to \infty$, and
2) The asymptotics of the coefficient in front of the logarithmic singularity of $g_{3,n}(x)$ for x close to a_2 as $n \to \infty$.

To find the asymptotics of $g_{1,n}(a_2)$ as $n \to \infty$ we match the WKB solution in the interior of (a_1, a_2) with the inner solutions (Bessel functions) in neighborhoods of a_1 and a_2. There exist two overlap regions that allow the matching.



The logarithmic singularity of $g_{3,n}(x)$ at a_2 can be estimated as follows:

1) Similarly to $g_{1,n}$, obtain the uniform asymptotics of $g_{3,n}(x)$ on all of I_3;
2) Analytically extend the WKB approximation of $g_{3,n}(x)$ to a neighborhood of a_2; and
3) Match the extended WKB approximation with the inner solution near a_2.

Analysis of the WKB approximation

Consider the matrix form

$$Y'(z) = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda+Q(z)}{P(z)} & -\frac{P'(z)}{P(z)} \end{pmatrix} Y(z) =: B(z)Y(z)$$

of the differential equation $(L+\lambda)g = 0$. If a vector $(g(z), g'(z))^T$ is a column of $Y(z)$, then $g(z)$ solves $(L+\lambda)g = 0$.

WKB approximation

Using a series of manipulations, the original differential equation can be transformed to a matrix Volterra integral equation

$$T(z) = I + \lambda^{-1/2} \int^{z} e^{i\sigma_3 \sqrt{\lambda} \int_{\zeta}^{z} \frac{d\xi}{\sqrt{P(\xi)}}} B(\zeta) T(\zeta) e^{-i\sigma_3 \sqrt{\lambda} \int_{\zeta}^{z} \frac{d\xi}{\sqrt{P(\xi)}}} d\zeta,$$

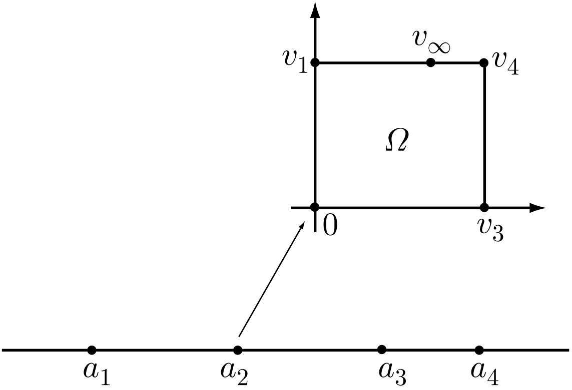
where different contours of integration with the same endpoint z will be selected for each entry of the matrix integrand. Here I is the identity matrix and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a Pauli matrix.

WKB approximation

Consider the Christoffel-Schwarz transformation given by $v(z) = \int_{a_2}^{z} \frac{d\zeta}{\sqrt{P(\zeta)}}$. It maps the upper halfplane of the complex z-plane onto the rectangle Ω with vertices $0v_3v_4v_1$ of the complex v-plane, where $v_j = v(a_j)$, $j = 1, 2, 3, 4$.



WKB approximation

The change of variables $\eta = v(\zeta)$ reduces the Volterra equation to

$$T = I + \lambda^{-1/2} \int_\gamma e^{i\sqrt{\lambda}(v-\eta)\sigma_3} P^{\frac{1}{2}} B T e^{-i\sqrt{\lambda}(v-\eta)\sigma_3} d\eta = I + \mathcal{I} T,$$

where the collection of contours of integration $\gamma_{ij} = \gamma_{ij}(v)$, $i, j = 1, 2$, will be specified later.

WKB approximation

The leading order term of the coefficient of the Volterra equation

$$P^{1/2}B \sim \left(-\frac{(P')^2}{32P} + \frac{Q}{2}\right) i\sigma_3 + \left(\frac{2PP'' - (P')^2}{16P} - \frac{Q}{2}\right)\sigma_1, \;\; \lambda \to \infty,$$

has second order poles at $v = v_\infty$ and at $v = v_j$, $j = 1, 2, 3, 4$. Hence v must be outside certain neighborhoods of these points. Here

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is another Pauli matrix.

Recall that

$$T = I + \lambda^{-1/2} \int_\gamma e^{i\sqrt{\lambda}(v-\eta)\sigma_3} P^{\frac{1}{2}} B T e^{-i\sqrt{\lambda}(v-\eta)\sigma_3} d\eta = I + \mathcal{I} T.$$

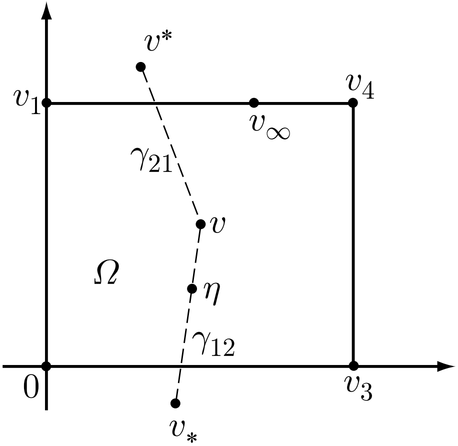
We are looking for an iterative solution in the form $T = \sum_{k=0}^{\infty} T_k$, where $T_0 = I$ and $T_{k+1} = \mathcal{I} T_k$. Convergence of this series for large $\lambda > 0$ requires

$$\mathrm{Im}(v-\eta) > 0 \quad \text{on} \quad \gamma_{12}(v) \quad \text{and} \quad \mathrm{Im}(v-\eta) < 0 \quad \text{on} \quad \gamma_{21}(v)$$

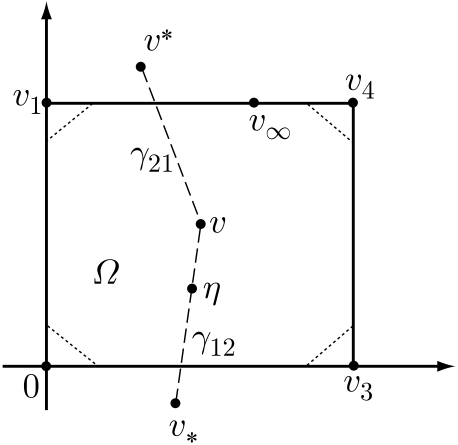
for all $v \in \Omega$ and for all $\eta \in \gamma_{12}(v)$, $\eta \in \gamma_{21}(v)$, respectively.

WKB approximation

Therefore, as the beginning of the contours γ_{12}, γ_{21} we chose points $v_* = \tilde{v}_* - iu_0$ and $v^* = \tilde{v}^* + iu_0$ respectively, where $u_0 > 0$ is some small fixed number, $\tilde{v}_* \in (0, v_3)$, $\tilde{v}^* \in (v_1, v_4)$ and $\tilde{v}^* \neq v_\infty$.

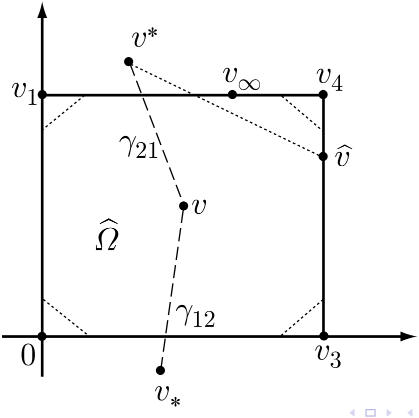


As mentioned above, the poles at $v = v_j$ do not allow us to construct a solution in all Ω. We start by cutting from Ω little triangles of some small λ-dependent size $\delta_0 > 0$ at the corners v_j, $j = 1, 2, 3, 4$.

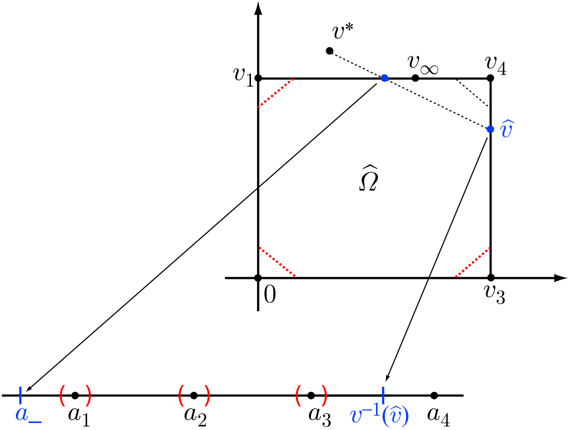


WKB approximation

Because of the pole at v_∞, the contour γ_{21} cannot pass through v_∞. Suppose, e.g., $\tilde{v}^* \in (v_1, v_\infty)$. We further reduce Ω by cutting off the triangle $v^*, v_4, \hat{v}$, where $\hat{v} \in (v_3, v_4)$ is some fixed point. We choose u_0 to be sufficiently small, so that v_∞ belongs to the triangle $v^*, v_4, \hat{v}$. The region, obtained by cutting all the triangles from Ω, is denoted by $\hat{\Omega}$.



Let Λ_- be the preimage of the region $\hat{\Omega}$ under the mapping $v = v(z)$. Then Λ_- contains the segment $[a_-, v^{-1}(\hat{v})]$, where a_- can be any large negative number and $v^{-1}(\hat{v}) \in (a_3, a_4)$, except for some neighborhoods of the points a_1, a_2, a_3.



WKB approximation

Theorem. Fix any $\mu_1 \in (0,1)$. The accuracy $O(\lambda^{-\mu_1/2})$ of the WKB approximation

$$\hat{Y}(z) = P^{-\frac{1}{4}} \mathrm{diag}\left(1, \sqrt{\frac{\lambda}{P}}\right) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{i\sqrt{\lambda}\int^z \frac{d\zeta}{\sqrt{P(\zeta)}}\sigma_3}$$

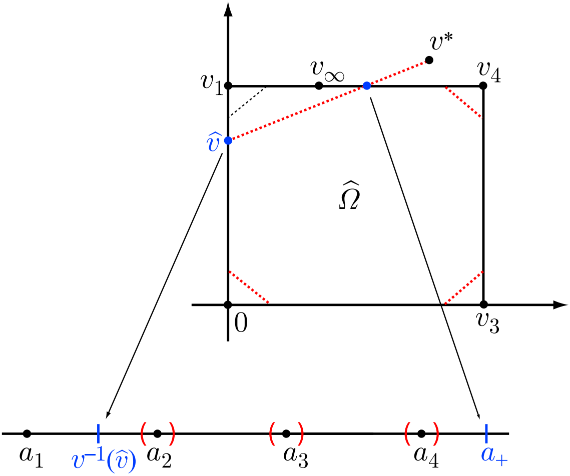
of the matrix solution

$$Y(z) = P^{-\frac{1}{4}} \mathrm{diag}\left(1, \sqrt{\frac{\lambda}{P}}\right) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I - \frac{\lambda^{-1/2} P'}{8\sqrt{P}}\sigma_1\right) \times T(z) e^{i\sqrt{\lambda}\int^z \frac{d\zeta}{\sqrt{P(\zeta)}}\sigma_3}$$

is valid uniformly in the region Λ_-, which does not contain $O(\lambda^{\mu_1 - 1})$ size neighborhoods of the points a_1, a_2, a_3.

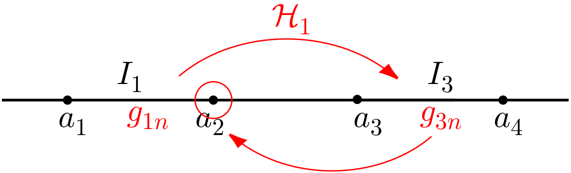
WKB approximation

By chosing the point $\tilde{v}^* \in (v_\infty, v_4)$ in the construction of $\hat{\Omega}$, we can prove the theorem in the corresponding region Λ_+ that contains points a_2, a_3, a_4.



Reminder

By construction, $\mathcal{H}_1 g_{1,n} = \nu_n^- g_{3,n}$ near a_2. Comparing the leading coefficients in front of the logarithmic singularities of both functions as $x \to a_2^-$ allows us to estimate ν_n^-.



Results

$$\ln(\nu_n^-) = -\pi n \frac{K_+}{K_-} + O(n^{\frac{1}{2}-\delta}), \;\; n \to \infty,$$

$$\ln(\nu_n^+) = -\pi n \frac{K_-}{K_+} + O(n^{\frac{1}{2}-\delta}), \;\; n \to \infty,$$

for any $\delta > 0$ sufficiently small, where

$$K_- := \int_{I_1} \frac{dx}{\sqrt{-P(x)}} = \int_{I_3} \frac{dx}{\sqrt{-P(x)}}, \;\; K_+ := \int_{I_2} \frac{dx}{\sqrt{P(x)}} = \int_{I_2} \frac{dx}{\sqrt{P(x)}}.$$