Basic Microlocal Analysis

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I. Microlocal Analysis

- **Generalization of Fourier analysis to include nonconstant coefficient operators**

- **Analysis of singularities of functions/distributions and how operators transform them**

- **Locations of singularities described in terms of phase space: spatial position and frequency direction**

Applications

- **Approximate Green's functions (parametrices) for elliptic PDE**

- **Approximate inverses and estimates for Radon transforms**

PDE

Notation: On $\mathbb{R}^n$, $\partial_j u = \frac{\partial u}{\partial x_j}$, $D_j u = \frac{1}{i}\partial_j u$.

Multi-index of degree m: $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, $|\alpha| = \alpha_1 + \cdots + \alpha_n = m$

Differential monomials:

$$\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}}$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} = i^{-|\alpha|}\partial^\alpha$$

Constant coefficient partial differential operators

$$P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha \in \mathbb{C}[\xi_1, \ldots, \xi_n] \longrightarrow$$

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

Def. $P(\xi)$ **is the full symbol of** $P(D)$

Exs. 1. $P(D) =$ **Laplacian**

$$\Delta = \sum_{j=1}^{n} \partial_j^2 = -\sum D_j^2 \longrightarrow P(\xi) = -|\xi|^2$$

2. On $\mathbb{R}^{n+1}_{x,t}$**, the heat operator**

$$P(D) = \frac{\partial}{\partial t} - \Delta \longrightarrow P(\xi, \tau) = |\xi|^2 + i\tau$$

3. d'Alembertian, $\square = \frac{\partial^2}{\partial t^2} - \Delta = \rightarrow P(\xi, \tau) = |\xi|^2 - \tau^2$

4. Cauchy-Riemann operator on $\mathbb{R}^2 \sim \mathbb{C}$

$$\overline{\partial} = \partial_x + i\partial_y \rightarrow P(\xi, \eta) = i(\xi + i\eta)$$

Basic Questions About PDE

1. Given a function/distribution f, does the equation $P(D)u = f$ have a solution? Is it unique? What side conditions ensure uniqueness?

2. Is there an explicit representation for u?

3. How are qualitative properties of u related to those of f?

4. Do the singularities of f determine the singularities of u?

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1. Malgrange-Ehrenpreis Theorem. Any $P(D) \neq 0$ is locally solvable: for any distribution f of compact support, there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ s.t. $P(D)u = f$.

2. Taking $f = \delta =$ Dirac delta 'function' at $0 \in \mathbb{R}^n \implies$ $P(D)$ admits a fundamental solution (Green's function):

$$K \in \mathcal{D}'(\mathbb{R}^n) \text{ s.t. } P(D)K = \delta.$$

• **Convolution:**

$$g * h(x) = \int_{\mathbb{R}^n} g(x-y)h(y)dy.$$

Properties: (i) $(h * g) * f = h * (g * f)$**, (ii)** $\delta * f = f$

(iii) $D^\alpha(g * f) = (D^\alpha f) * g = f * (D^\alpha g)$

• **If K is a fundamental solution for $P(D)$,**

$$P(D)(K * f) = (P(D)K) * f = \delta * f = f$$

A solution to $P(D)u = f$ is

$$u(x) := K * f(x) = \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

Ex. $P(D) = \Delta$**. Newtonian potential on** $\mathbb{R}^n$**,**

$$N(x) = c_n |x|^{2-n},\ n \geq 3; \quad = (2\pi)^{-1} \log |x|,\ n = 2$$

Ex. $P(D) = \frac{\partial}{\partial t} - \Delta$**. Heat kernel on** $\mathbb{R}^{n+1}$**,**

$$W(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t},\ t > 0; \quad = 0,\ t \leq 0$$

3. Regularity (smoothness) of solutions.

Q. When we solve $P(D)u = f$, can we predict where u is smooth from where f is smooth?
Or: determine where f is singular from where u is singular?

Def. $P(D)$ **is (C^∞)-hypoelliptic if, for any open set $\mathcal{O} \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(\mathbb{R}^n)$,**

$$P(D)u \in C^\infty(\mathcal{O}) \implies u \in C^\infty(\mathcal{O})$$

K **a fundamental solution for** $\implies PK = \delta \in C^\infty(\mathbb{R}^n \setminus 0)$**. Thus,**

$$P(D) \text{ hypoelliptic } \implies K \in C^\infty(\mathbb{R}^n \setminus 0)$$

Thm. The converse is also true.

Laplacian, heat operator are hypoelliptic; d'Alembertian is not.

Def. $P(D)$ **is elliptic if its principal symbol,**

$$\sigma_{prin}(P)(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha,$$

satisfies $|\sigma_{prin}(P)(\xi)| \geq c|\xi|^m$, $c > 0$**.**

- $\sigma_{prin}(\Delta) = -|\xi|^2$ **elliptic**
- $\sigma_{prin}(\frac{\partial}{\partial t} - \Delta) = |\xi|^2$**, which is** $= 0$ **on** τ**-axis, so not elliptic.**

Thm. $P(D)$ **elliptic** $\implies$ **hypoelliptic.**

The converse is not true, e.g., heat operator.

4. Thus: for an elliptic PDE, $P(D)u = f$ **, the singularities of** f **determine the singularities of** u**, and** *vice versa***.**

Support and Singular Support. For $u \in \mathcal{D}'(\mathbb{R}^n)$,

$supp(u) =$ **the complement of largest** $\mathcal{O}$ **on which** $u = 0$.

$sing\, supp(u) =$ **the complement of largest** $\mathcal{O}$ **on which** $u \in C^\infty$.

Ex. $H(x) =$ **Heaviside function:** $supp(H) = [0, \infty)$, $sing\, supp(H) = \{0\}$

$N(x) =$ **Newtonian potential:** $supp(N) = \mathbb{R}^n$, $sing\, supp(N) = \{0\}$

$\delta =$ **Dirac delta:** $supp(\delta) = sing\, supp(\delta) = \{0\}$

Differential operators are local: $supp\,(P\,(D)\,u) \subseteq supp(u)$,

and also pseudolocal: $sing\, supp\,(P\,(D)\,u) \subseteq sing\, supp(u)$

Def. $P(D)$ **is hypoelliptic iff the reverse containment holds, i.e.,**

$$sing\, supp\,(P\,(D)\,u) = sing\, supp(u)$$

Ex. Poisson equation. $\Delta u = f \implies u$ **is smooth outside** $sing\, supp(f)$.

Operators with C^∞ coefficients: $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$, $a_\alpha \in C^\infty$

Such a $P(x,D)$ is local and pseudolocal.

Def. The full symbol of $P(x,D)$ is

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in C^\infty(T^*\mathbb{R}^n)$$

and the principal symbol of $P(x,D)$ is

$$\sigma(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in C^\infty(T^*\mathbb{R}^n).$$

Cotangent space: $T^*\mathbb{R}^n = \big\{(x,\xi) : x \in \mathbb{R}^n,\ \xi \in (T_x\mathbb{R}^n)^*\big\} \simeq \mathbb{R}^n \times \mathbb{R}^n$

Ex. Conductivity equation: $\nabla\cdot(\gamma(x)\nabla u) = \gamma(x)\nabla\cdot\nabla u + \nabla\gamma\cdot\nabla u$
$\implies \sigma(P)(x,\xi) = -\gamma(x)|\xi|^2$.

Def. $P(x,D)$ **is uniformly elliptic on $\mathcal{O}$ if** $\exists\, C_0 > 0$ **s.t.** $\big|\sigma(P)\big| \ge C_0|\xi|^m$

Calculus of pseudodifferential operators $\implies$
analogues of #1,2,3 and 4 (modulo C^∞ errors)

Constant coefficient $P(D) \longrightarrow$ **analyze via Fourier transform**

C^∞ **coefficient** $P(D) \longrightarrow$ **use pseudodifferential operators**

$$f(x) \longrightarrow \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

- $f \in L^1 \implies \widehat{f} \in C_0(\mathbb{R}^n)$**:** $\widehat{f}$ **continuous and** $\to 0$ **as** $|\xi| \longrightarrow \infty$
- **If** $f \in L^1 \cap L^2$**, then** $\widehat{f} \in L^2$**,**

$$||\widehat{f}||_{L^2}^2 := \int |\widehat{f}(\xi)|^2\, d\xi = (2\pi)^n \int ||f||_{L^2}^2 \text{ and}$$

$\widehat{\ }$ **extends to a unitary isometry** $L^2(\mathbb{R}^n_x) \to L^2(\mathbb{R}^n_\xi)$ **(Parseval-Plancherel)**

- $\widehat{D_j f}(\xi) = \xi_j \widehat{f}(\xi) \implies \widehat{D^\alpha f}(\xi) = \xi^\alpha \widehat{f}(\xi) \implies \widehat{P(D) f}(\xi) = P(\xi) \cdot \widehat{f}(\xi)$
- $\widehat{(}f * g(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi)$

Fourier inversion formula

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \widehat{f}(\xi)\, d\xi = (2\pi)^{-n} \widehat{(\widehat{f})}(-x) \implies$$

$$\begin{aligned} P(D)u(x) &= (2\pi)^{-n} \int e^{ix\cdot\xi}\, P(\xi)\, \widehat{f}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi}\, P(\xi)\, f(y)\, dy\, d\xi \end{aligned}$$

Smoothness of $f(x) \leftrightarrow$ decay of $\widehat{f}(\xi)$ and *vice versa*

• $D^\alpha f \in L^1 \implies \widehat{D^\alpha f}(\xi) = \xi^\alpha \widehat{f}(\xi) \in C_0(\mathbb{R}^n) \implies |\xi^\alpha \cdot \widehat{f}(\xi)| \leq B_\alpha$
If true for all $|\alpha| \leq M$, then $|\widehat{f}(\xi)| \leq B(1+|\xi|)^{-M} \quad (*)$

• **Conversely, if (*) holds for some $M > n$, then F.I.F. $\implies f \in C_0$**

If (*) holds for $M > n+k$, then for $|\alpha| \leq k$, $D^\alpha f = \widehat{\xi^\alpha \widehat{f}}(-x) \in C_0$
$\implies f \in C_0^k(\mathbb{R}^n)$

• **Thus, $\widehat{f}$ rapidly decreasing $\implies f \in C^\infty$.**

Wavefront set $WF(u)$

Describe singularities of $u \in \mathcal{D}'(\mathbb{R}^n)$ in terms of both

spatial location in x space,

frequency direction in ξ space.

$WF(u)$ **is a closed subset of** $T^*\mathbb{R}^n \setminus 0 = \{(x,\xi) \in T^*\mathbb{R}^n : \xi \neq 0\}$**, invariant under** $(x,\xi) \longrightarrow (x, t\xi)$, $0 < t < \infty$ **(conic).**

Define $WF(u)$ in terms of its complement:

Def. $(x_0,\xi_0) \notin WF(u)$ **if** $\exists \phi \in \mathcal{D}(\mathbb{R}^n)$ **s.t.** $\phi(x_0) \neq 0$ **and** $\widehat{\phi u}(\xi)$ **is rapidly decreasing on some conic neighborhood of** ξ_0**.**

Ex. $\phi \cdot \delta = \phi(0)\delta$ **and** $\widehat{\delta}(\xi) \equiv 1$, $WF(\delta) = \{(x,\xi) : x = 0, \xi \neq 0\} = T_0^*\mathbb{R}^n$

Similarly, $WF(H) = T_0^*\mathbb{R}$**.**

However, $WF((x+i0)^{-1}) = \{(x,\xi) \in T^*\mathbb{R} : x = 0, \xi > 0\}$**.**

Ex. $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ **smooth** $\Longrightarrow$

$$WF(\chi_\Omega) = \{(x,\xi) : x \in \partial\Omega, \xi \perp T_x\partial\Omega\} := N^*\partial\Omega$$

In the examples above, the x's that occur in $WF(u)$ are exactly those that comprise $sing\,supp(u)$.

Prop. If $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ denotes the projection onto the spatial variable, $\pi(x,\xi) = x$, then

$$\pi\left(WF\left(u\right)\right) = sing\,supp(u).$$

Using the calculus of pseudodifferential operators (ΨDOs), for an elliptic $P(x,D)$, we can:

- **Construct an approximate fundamental solution (parametrix), s.t. $Q(x,D)P(x,D) = I + E_L$ and $P(x,D)Q(x,D) = I + E_R$, with E_L, E_R infinitely smoothing.**

- **Show that $P(x,D)u = f$ has a solution modulo C^∞.**

- **Prove that $P(x,D)$ is hypoelliptic: $sing\,supp(P(x,D)u) = sing\,supp(u)$, and in fact $WF(P(x,D)u) = WF(u)$.**

- **Obtain estimates for $||u||$ in terms of $||f||$ for various function space norms.**

ΨDOs can be represented either as integral operators:

$$A(x,D)f(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy$$

$K \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$
with estimates on K and its derivatives,

or as oscillatory integral operators:

$$Af(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, f(y)\, d\xi\, dy,$$

with the amplitude $a(x,y,\xi)$ belonging to a symbol class.

Both points of view are valuable, but we will focus on the latter.

II. Pseudodifferential Operators

• **Convolution operators are Fourier multiplier operators:**

$$K * f(x) = c \int e^{ix\cdot\xi}\, \widehat{K * f}(\xi)\, d\xi = c \int e^{ix\cdot\xi} a(\xi)\, \widehat{f}(\xi)\, d\xi, \quad a = \widehat{K},$$
$$= c \int\int e^{i(x-y)\cdot\xi}\, a(\xi)\, f(y)\, d\xi\, dy, \quad c = (2\pi)^{-n}.$$

Compositions of operators correspond to product of multipliers:

$$K_1 * (K_2 * f)(x) = c \int\int e^{i(x-y)\cdot\xi}\, a_1(\xi)\cdot a_2(\xi)\, f(y) d\xi\, dy, \quad a_j = \widehat{K_j}, j = 1, 2$$

Idea of ΨDOs: Generalize $a(\xi)$ to $a(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ or $a(x,y,\xi)$.

• **Allows for operators with variable coefficients**

• **Need to allow error terms, smoothing in various senses**

$$Af(x) = c \int \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, f(y)\, d\xi\, dy$$

Phase function: $\phi(x,y,\xi) = (x-y)\cdot\xi$**,** $\nabla\phi = (\xi, -\xi, x-y) \neq (0,0,0)$

Amplitude: $a(x,y,\xi)$ **belongs to a symbol class.**

Def. (i) For $m \in \mathbb{R}$**, the symbol class** $S^m = S^m_{1,0}$ = **those** $a \in C^\infty$ **s.t.**

$$|\partial_x^\gamma \partial_y^\beta \partial_\xi^\alpha a(x,y,\xi)| \leq C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\alpha|},\ \forall \alpha,\beta,\gamma \in \mathbb{Z}_+^n$$

(ii) S^m_{cl} = **classical symbols of order** m = **those** $a \in S^m$ **s.t.**

$a \sim \sum_{j=0}^{\infty} a_{m-j}$ **with** $a_{m-j}(x,y,\xi)$ **homogeneous of degree** $m-j$ **in** ξ **for** $|\xi| \geq C$

where $\sim$ **means**

$$a - \sum_{j=0}^{N} a_{m-j} \in S^{m-N-1}, \quad \forall N \in \mathbb{Z}_+.$$

• $P(x,D)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ **PDO order** $m\in\mathbb{Z}_+ \implies$ **both** $p(x,\xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ **and** $\sigma(P)(x,\xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in S^m_{cl}$

If $P(x,D)$ **is elliptic, then** $|\sigma(P)(x,\xi)|\ge c|\xi|^m \ne 0$ **for** $\xi\ne 0$**, and** $|p(x,\xi)|\ge c|\xi|^m$ **for** $|\xi|$ **large.**

Let $\chi\in C^\infty(\mathbb{R}^n), \chi\equiv 0$ **near 0,** $\chi\equiv 1$ **near** ∞**. Then**

$$\chi(\xi)[\sigma(P)(x,\xi)]^{-1} \text{ and } \chi(\xi)[p(x,\xi)]^{-1}\in S^{-m}_{cl}$$

• **Still true if extend the notion of ellipticity to amplitudes in** S^m**:**

$$a\in S^m \text{ is elliptic if } |a(x,y,\xi)|\ge C|\xi|^m \text{ for } |\xi| \text{ large.}$$

• **If** $a\in S^m$ **and** $F\in C^\infty(\mathbb{C})$ **and all** $\partial^\alpha F$ **are bounded, then** $F(a)\in S^m$**.**

• $a\in S^m$ **takes values in** $\mathbb{C}\setminus(-\infty,0]$ **and** $r\in\mathbb{R}\setminus 0 \implies \chi\cdot a^{1/r}\in S^{m/r}$**.**

• $S^m\times S^{m'}\hookrightarrow S^{m+m'}$**.**

Error classes. $m' < m \implies S^{m'} \subsetneq S^m$.

Def. **(i)** $S^\infty = \bigcup_{m\in\mathbb{R}} S^m$ **and** $S^\infty_{cl} = \bigcup_{m\in\mathbb{R}} S^m_{cl}$

(ii) $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m =$ **space of rapidly decreasing amplitudes.**

- $S^{-\infty}$ **corresponds to infinitely smoothing** Ψ**DOs,** $A : \mathcal{E}' \longrightarrow C^\infty$**.**

- $S^{m-1} \leftrightarrow$ **operators one order lower down in the calculus.**

- **What "one order lower down" means varies from calculus to calculus.**

Simple vs. compound amplitudes

$a(x, y, \xi)$ **is simple if only depends on** x, ξ**; otherwise, is compound.**

Pseudodifferential operators (ΨDOs). **For $a \in S^m$ or S^m_{cl}, let**

$$Af(x) = c \int \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, f(y)\, d\xi\, dy$$

- $A : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ **with Schwartz kernel**

$$K_A(x,y) = c \int e^{i(x-y)\cdot\xi}\, a(x,y,\xi)\, d\xi$$

- **Absolutely convergent if $m < -n$, not otherwise.**

- **Interpret oscillatory integrals as distributions and manipulate them consistently via Hörmander's theory.**

- $\Psi^m =$ **all ΨDOs with amplitudes from S^m, similar for $\Psi^m_{cl} \subset \Psi^m$**

- **Identity operator I, with $K_I(x,y) = \delta(x-y)$, is represented by $a \equiv 1 \in S^0 \implies I \in \Psi^0_{cl}$.**

- $a \in S^{-\infty} \implies K_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \implies A : \mathcal{E}'(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$**.**

Say that A is infinitely smoothing.

Thm. ΨDOs are pseudolocal: $sing\,supp(Af) \subseteq sing\,supp(f)$.

Pf. **Follows from** $K_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$.

For any $\alpha \in \mathbb{Z}_+^n$, $D_\xi^\alpha(e^{i(x-y)\cdot\xi}) = (x-y)^\alpha \cdot e^{i(x-y)\cdot\xi}$, **so**

$$\begin{aligned} (i(x-y))^\alpha \cdot K_A(x,y) &= c \int D_\xi^\alpha(e^{i(x-y)\cdot\xi})\, a(x,y,\xi)\, d\xi \\ &= c \int e^{i(x-y)\cdot\xi}\, (D_\xi^t)^\alpha(a(x,y,\xi))\, d\xi \\ &= c \int e^{i(x-y)\cdot\xi}\, b(x,y,\xi)\, d\xi, \quad b \in S^{m-|\alpha|}. \end{aligned}$$

Taking $|\alpha| > m + n$, **the last integral converges absolutely and is thus a continuous function of the parameters,** x, y.

Taking $|\alpha| > m + n + q$, **the integral converges well enough so that we can differentiate** q **times in** x, y, **so that** $(x-y)^\alpha \cdot K_A \in C^q(\mathbb{R}^n \times \mathbb{R}^n)$. **Since we can do this for arbitrary** α, **and the common zero set of all the** $(x-y)^\alpha$ **is** $\{x = y\}$, **we get**

$$K_A(x,y) \in C^q(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\})$$

for every q, **and** K_A **is infinitely smooth there.**

Compound symbols are unnecessary

Let $a(x,y,\xi) \in S^m$**. Expand** a **about the diagonal** $\{x=y\}$**:**

$$a(x,y,\xi) = \sum_{|\alpha|\le N} a_\alpha(x,\xi)(y-x)^\alpha + \mathcal{O}(|y-x|^{N+1})$$

$$\textbf{where} \quad a_\alpha(x,\xi) = \frac{1}{\alpha!}\partial_y^\alpha a(x,y,\xi)|_{y=x} \in S^m \ \textbf{(simple)}$$

$$\begin{aligned}
\Longrightarrow \ K_A(x,y) &= \sum_{|\alpha|\le N} c\int e^{i(x-y)\cdot\xi}\,(y-x)^\alpha\,a_\alpha(x,\xi)\,d\xi + \dots \\
&= \sum_{|\alpha|\le N} c\int (iD_\xi)^\alpha(e^{i(x-y)\cdot\xi})a_\alpha(x,\xi)\,d\xi + \dots \\
&= \sum_{|\alpha|\le N} c\int e^{i(x-y)\cdot\xi}\,(-1)^{|\alpha|}\,\partial_\xi^\alpha a_\alpha(x,\xi)\,d\xi + \dots.
\end{aligned}$$

$\partial_\xi^\rho a_\alpha \in S^{m-|\alpha|}$, $|\alpha| \le N$, **and** $\dots \in S^{m-N-1} \Longrightarrow$

Prop. **An mth order ΨDO A can be represented, modulo an infinitely smoothing operator, by a simple amplitude, called the symbol of A,**

$$\sigma_A(x,\xi) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)|_{y=x} = a(x,x,\xi) + \dots$$

Def. **If A is an mth order ΨDO defined by $a(x,y,\xi)$, then the principal symbol of A, denoted $\sigma_{prin}(A)(x,\xi)$ is**

(i) $a_m(x,x,\xi)$ **if** $a \in S^m_{cl}$, $a \sim \sum_{j=0}^{\infty} a_{m-j}$**, and**

(ii) the equivalence class $[a(x,x,\xi)] \in S^m/S^{m-1}$ **if** $a \in S^m$**.**

Note: **If B is of order $\leq m-1$, then** $\sigma_{prin}(A+B) = \sigma_{prin}(A)$

Symbol calculus: σ_{prin} **is an isomorphism** $\Psi^m/\Psi^{m-1} \longrightarrow S^m/S^{m-1}$**.**

Application: Adjoints

- **For any operator T with integral kernel $K_T(x,y)$, its L^2 adjoint T^* has kernel**

$$K_{T^*}(x,y) = \overline{K_T(y,x)}$$

- **If A is a ΨDO with amplitude $a(x,y,\xi)$, then**

$$K_{A^*}(x,y) = \overline{K_A(y,x)} = c\int e^{i(x-y)\cdot\xi}\,\overline{a}(y,x,\xi)\,d\xi.$$

$\overline{a}(y,x,\xi)$ **belongs to the same symbol class as** $a \implies$
A^* **is also a ΨDO, with symbol**

$$\sigma_{A^*}(x,\xi) \sim \overline{a}(x,x,\xi) + \dots \textbf{ and } \sigma_{prin}(A^*) = \overline{\sigma_{prin}(A)}$$

Application: Composition

For properly supported Ψ**DOs** A, B **be of orders** m, m'**,**

$$\begin{aligned} K_{AB}(x,y) &= \int K_A(x,z)\, K_B(z,y)\, dz \\ &= c^2 \int\int\int e^{i[(x-z)\cdot\xi + (z-y)\cdot\eta]}\, a(x,z,\xi)\, b(z,y,\eta)\, d\xi\, d\eta\, dz. \end{aligned}$$

Now apply stationary phase: **with** x, y, ξ **as parameters, consider integral** $dz\, d\eta$**. Phase**

$$\Phi(z,\eta) = (x-z)\cdot\xi + (z-y)\cdot\eta$$

has a unique critical point $(z_0, \eta_0) := (y, \xi)$**:**

$$\nabla_z \Phi = -\xi + \eta = 0,\ \nabla_\eta \Phi = z - y = 0 \leftrightarrow z = y,\ \eta = \xi.$$

Furthermore, $det(\nabla^2 \Phi) \neq 0$ **and** $\Phi(z_0, \eta_0) = (x-y)\cdot\xi \implies$

$$K_{AB}(x,y) = c \int e^{i(x-y)\cdot\xi}\, (a \circ b)(x,y,\xi)\, d\xi, \qquad a \circ b \in S^{m+m'}$$

$$\sigma_{prin}(AB) = \sigma_{prin}(A) \cdot \sigma_{prin}(B)$$

Boundedness properties of ΨDOs.

Thm. **(i) If $A \in \Psi^0$ compactly supported, then A is a bounded operator on $L^2(\mathbb{R}^n)$; on the L^2-based Sobolev spaces $H^s(\mathbb{R}^n)$, $\forall s \in \mathbb{R}$; on the Hölder-Zygmund spaces $C_*^{k,\alpha}$, $k \in \mathbb{Z}_+$, $0 \le \alpha \le 1$; and on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.**

(ii) If $A \in \Psi^m$, then

$$A : H^s \longrightarrow H^{s-m}, \forall s \in \mathbb{R}$$

and

$$C_*^{k,\alpha} \longrightarrow C_*^{k-[m],\alpha-(m-[m])}.$$

(iii) Thus, if $m < 0$, $A \in \Psi^m$ is a compact operator on L^2 and any H^s.

Application: Parametrices

Let $A(x,D) \in \Psi^m$ **or** Ψ^m_{cl} **be elliptic:** $|\sigma_{prin}(A)(x,\xi)| \geq c|\xi|^m$, $|\xi| \longrightarrow \infty$.

For a cutoff χ **as before,** $\chi \equiv 1$ **near** ∞.

$$b_0(x,\xi) := \chi(\xi) \cdot [\sigma_{prin}(x,\xi)]^{-1} \in S^{-m}.$$

By symbol calculus, $b_0 \longrightarrow B_0 \in \Psi^{-m}$**, and**

$$\sigma_{prin}(B_0 A) = [\chi] = 1 \implies B_0 A = I \textbf{ mod } \Psi^{-1}.$$

Let

$$b_1 = -\chi \cdot [\sigma_{prin}(x,\xi)]^{-1} \cdot \sigma_{prin}(B_0 A - I) \in S^{-m-1},$$

and $B_1 \in \Psi^{-m-1}$ **the corresponding** Ψ**DO. As an element of** Ψ^{-1}**,** $(B_0 + B_1)A - I$ **has principal symbol 0, and hence it** $\in \Psi^{-2}$**.**

Continuing iteratively, construct a sequence $B_0, B_1, B_2, \ldots$**, with** $B_j \in \Psi^{-m-j}$ **and**

$$(B_0 + \cdots + B_N)A - I \in \Psi^{-N-1}, \, \forall N$$

ΨDO calculus allows one to asymptotically sum: $\exists B \in \Psi^{-m}$ s.t. $B \sim \sum_{j=0}^{\infty} B_j$, and $BA - I \sim 0$. I.e., $BA = I +$ an infinitely smoothing operator. We say that B is a left parametrix for A.

Similarly, can construct a right parametrix, B'.

$$BAB' = B(AB') \sim B \cdot I \textbf{ mod } \Psi^{-\infty}$$

and similarly $\sim B' \cdot I$ mod $\Psi^{-\infty}$, so B and B' differ by a smoothing operator. Thus, either one is a two-sided parametrix, i.e., a Green's function modulo a smoothing error.

If A is only elliptic on some open cone $\Gamma \subset T^*\mathbb{R}^n$, can construct microlocal parametrices.

Thus, if $P(x, D)$ is an elliptic PDO or ΨDO, and we are interested in the inhomogeneous equation $Pu = f$, let $Q(x, D) \in \Psi^{-m}$ be a two-sided parametrix for P.

(i) **Can solve $Pu = f$ mod C^∞. Letting $u = Qf$, get**

$$Pu = PQf = f \textbf{ mod } C^\infty, \forall f \in \mathcal{E}'(\mathbb{R}^n).$$

(ii) **$P(x, D)$ is hypoelliptic:**

$$sing\,supp(u) = sing\,supp(f)$$

(iii) **Elliptic estimates.** $Pu = f \implies \forall s \in \mathbb{R}$,

$$||u||_{H^{s+m}} \leq C_s\big(||f||_{H^s} + ||u||_{H^s}\big)$$

Didn't cover: Fourier integral operators (FIOs). More general phase functions than ΨDOs, useful for analyzing generalized Radon transforms, $\mathcal{R}$.

In particular, under the Bolker condition, $\mathcal{R}^*\mathcal{R}$ is a ΨDO, elliptic on the microlocal illuminated region.

$\Longrightarrow$ Singularities of f are determined by singularities of $\mathcal{R}f$, estimates (stability), ...

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