Optimal Allocations of deductibles and policy limits with generalized dependence structures

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Outline

- Introduce the motivation and background.
- Define new dependence structures.
- Study the properties of new dependence structures.
- Application in optimal allocation problems.

Motivations

- Continuation of optimal reinsurance problem.
    - Classical optimal reinsurance problem.
    $$X \wedge d \leq_{cx} I(X).$$
    - Multivariate case with dependent risks.
    Suppose the risks $X_1, X_2, ..., X_n$ are PDS. Then,
    $$\sum_{i=1}^{n} X_i \wedge d_i \leq_{cx} \sum_{i=1}^{n} I_i(X_i).$$
- Difficulties in identifying the parameters of the optimal form;
$$\inf_{\vec{d}} \mathbb{E}\left[u(\sum_{i=1}^{n} X_i \wedge d_i)\right].$$
- Alternative approach: consider optimal allocation problem.

Literature Review

- Kijima et al (1996): Optimal weights between different assets.

$$\sup_k \mathbb{E}\left[u(kX + (1-k)Y)\right].$$

- Cheung (2007): Optimal deductibles and policy limits.

$$\inf_{\vec{d}} \mathbb{E}\left[u(\textstyle\sum_{i=1}^n X_i \wedge d_i)\right], \text{ or } \inf_{\vec{d}} \mathbb{E}\left[u(\textstyle\sum_{i=1}^n (X_i - d_i)_+)\right].$$

- Zhuang et al (2008): incorporated the discounted factor.

$$\inf_{\vec{d}} \mathbb{E}\left[u(\textstyle\sum_{i=1}^n X_i \wedge d_i \times e^{-\delta T_i})\right], \text{ or}$$

$$\inf_{\vec{d}} \mathbb{E}\left[u(\textstyle\sum_{i=1}^n (X_i - d_i)_+ \times e^{-\delta T_i})\right].$$

- Limitation and Improvement.
  - Only comonotonicity and independence have been studied.
  - Generalize dependence structure to unify the existing models.

Stochastic Orders

Ordinary Stochastic Orders

- $X \leq_{st} Y$ if and only if $\bar{F}_X(x) \leq \bar{F}_Y(x)$ for all $x \in \mathbb{R}$;
- $X \leq_{hr} Y$ if and only if $\bar{F}_Y(x)\bar{F}_X(y) \leq \bar{F}_Y(y)\bar{F}_X(x), \forall x \leq y$;
- $X \leq_{lr} Y$ if and only if $f_Y(x)f_X(y) \leq f_Y(y)f_X(x), \forall x \leq y$.

Bivariate Stochastic Orders

- $X \leq_{hr:j} Y$ if and only if $\frac{\partial}{\partial x}\bar{F}(x,y) \leq \frac{\partial}{\partial x}\bar{F}(y,x), \forall x \leq y$.
- $X \leq_{lr:j} Y$ if and only if $f_{X,Y}(y,x) \leq f_{X,Y}(x,y), \forall x \leq y$.

Remarks

- Ordinary orders and bivariate orders do not imply each other. For example,
  $X \le_{hr} Y \nRightarrow X \le_{hr:j} Y, \qquad X \le_{hr:j} Y \nRightarrow X \le_{hr} Y.$
- Bivariate orders involve dependence, ordinary orders don't. Define through joint d.f. vs through marginal d.f.
- Inspiration to generalize the bivariate stochastic orders. Integrate the inequality describing joint hazard rate order,

$$\int_x^y \frac{\partial}{\partial s}\bar{F}(s,y) \le \int_x^y \frac{\partial}{\partial s}\bar{F}(y,s) \Longrightarrow \bar{F}(y,x) \le \bar{F}(x,y), \forall x \le y.$$

Arrangement Increasing Function

Arrangement Order

Let $\vec{x}$ and $\vec{y}$ be two vectors. We say $\vec{x} \leq_a \vec{y}$, if $\vec{x}$ can be obtained from $\vec{y}$ through successive pairwise interchanges of its components, with each interchange resulting in a decreasing order of the two interchanged component. For example, we say $(3, 2, 1) \leq_a (1, 3, 2)$ since $(1, 3, 2) \rightarrow (3, 1, 2) \rightarrow (3, 2, 1)$.
Also, $(y, x) \leq_a (x, y)$ if $x \leq y$.

Arrangement Increasing Function

Definition: Multivariate function $f(\vec{x})$ is said to be arrangement increasing (AI), if $\vec{x} \leq_a \vec{y}$ implies $f(\vec{x}) \leq f(\vec{y})$.
Example: $f(x_1, x_2, x_2) = a_1x_1 + a_2x_2 + a_3x_3$, with $0 \leq a_1 \leq a_2 \leq a_3$.

Generalization of Bivariate Stochastic Orders

Definition - Upper Orthant Arrangement Increasing (UOAI)

Random vector $\vec{X}$ is said to be upper orthant arrangement increasing (UOAI), if the survival function $\bar{F}(x_1, \cdots, x_n)$ is arrangement increasing.
$\vec{X}$ is conditionally upper orthant arrangement increasing (CUOAI), if $(X_i, X_j)|\vec{X}_{K_{ij}} = \vec{x}_{K_{ij}}$ is UOAI for any $i < j$ and any fixed $\vec{x}_{K_{ij}} \in S(\vec{X}_{K_{ij}})$, with $K_{ij} = \{1, 2, \cdots, n\}/\{i, j\}$.

Definition - Stochastic Arrangement Increasing (SAI)

Assume $\vec{X}$ has joint density function $f(\vec{x})$. $\vec{X}$ is said to be stochastically arrangement increasing (SAI) if $f(\vec{x})$ is arrangement increasing.

Equivalent Characterization of SAI

Definition - Partially Arrangement Increasing

Function $f : \mathbb{R}^n \to \mathbb{R}$ is called partially arrangement increasing (PAI), if there exists $K \subseteq \{1, \cdots, n\}$ such that, for any fixed $\vec{x}_{\bar{K}} \in \mathbb{R}^{n-|K|}$, $g(\vec{x}_K) \equiv f(\vec{x}_K, \vec{x}_{\bar{K}}) : \mathbb{R}^{|K|} \to \mathbb{R}$ is arrangement increasing. In this case, f is also referred as K-PAI.

Obviously, if f is AI, then f is K-PAI for any index set $K \subseteq \{1, \cdots, n\}$.

Equivalent Definition of SAI

Random vector $\vec{X} = (X_1, \cdots, X_n)$ is called stochastically arrangement increasing (SAI), if for any $K-$PAI function g such that the following expectations exist, it always holds that $\mathbb{E}[g(\vec{X})] \geq \mathbb{E}[g(\pi_{ij}(\vec{X}))]$ for any $i < j \in K$.

Properties of UOAI/CUOAI

Preliminaries

- If $\vec{X}$ is CUOAI, then $\vec{X}$ is UOAI.
- If $\vec{X}$ are mutually independent, then $\vec{X}$ is UOAI if and only if $X_i \leq_{hr} X_{i+1}$ for all $1 \leq i \leq n-1$.
- If $\vec{X}$ are comonotonic, then $\vec{X}$ is UOAI if and only if $X_i \leq_{st} X_{i+1}$ for all $1 \leq i \leq n-1$.

Upper Orthant Comparison

Assume $\vec{X}$ is UOAI and $X_i \geq 0$. Then
$(X_j \wedge x_i, \vec{X}_{K_{ij}}) \times \mathbb{I}\{X_i > x_j\} \leq_{uo} (X_i \wedge x_i, \vec{X}_{K_{ij}}) \times \mathbb{I}\{X_j > x_j\}$

Construction of CUOAI Random Vector

Proposition - Copula and CUOAI

Assume $\vec{X} = (X_1, \cdots, X_n)$ with positive joint density is linked by Archimedean survival copula: $C(u_1, \cdots, u_n) = \Psi^{-1}(\sum_{k=1}^{n} \Psi(u_k))$, with $X_i \leq_{hr} X_{i+1}$ for $i = 1, \cdots, n-1$. If $\Psi(e^t)$ is convex in $t \in (-\infty, 0]$, then $\vec{X}$ is CUOAI.

Examples

- Gumbel copula: $\Psi(x) = (-\log x)^{\alpha}$ with $\alpha \geq 1$. Independent and comonotonic copulas are special Gumbel copulas with $\alpha = 1$ and $\alpha = \infty$ respectively.
- Clayton copula: $\Psi(x) = x^{-\theta} - 1$ with $\theta > 0$.

Properties of SAI

Preliminaries

- If $\vec{X}$ is SAI, then $\vec{X}$ is CUOAI and thus UOAI.
- Assume $\vec{X} = (X_1, \cdots, X_n)$ is mutually independent. Then $\vec{X}$ is SAI if and only if $X_i \leq_{lr} X_{i+1}$ for all $i = 1, \cdots, n-1$.
- Assume $\vec{X} = (X_1, \cdots, X_n)$ is comonotonic. Then $\vec{X}$ is SAI if and only if $X_i \leq_{st} X_{i+1}$ for all $i = 1, \cdots, n-1$.

Example - Bivariate Normal Distribution

Suppose $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ with $\rho \geq 0$. Then (X, Y) is SAI if

- $\mu_1, \mu_2 \geq 0$ and $\mu_2/\mu_1 \geq \sigma_1^2/\sigma_2^2 \geq 1$; or
- $\mu_2 \geq \mu_1$ and $\sigma_1^2 = \sigma_2^2$.

Model Formulation

- Original Risks are modeled as $X_1, X_2, ..., X_n$.
- Strategy: policy limit or deductible. Retained risk is

$$I(\vec{d}) = \sum_{i=1}^{n} X_i \wedge d_i, \text{ or } R(\vec{d}) = \sum_{i=1}^{n} (X_i - d_i)_+.$$

- Target: $\inf_{\vec{d}} \mathbb{E}\left[u(I(\vec{d}))\right]$ with $u \in \mathcal{U}_1($ or $\mathcal{U}_2,$ or $\mathcal{U}_3)$ where

$$\mathcal{U}_1 = \{u : u(x) \text{ is increasing}\}, \mathcal{U}_2 = \{u : u(x) \text{ is increasing convex}\},$$
$$\mathcal{U}_3 = \{u : u(x) = x^n, n \in \mathbb{N} \text{ or } e^{\gamma x}, \gamma > 0\}.$$

- Assumptions: comonotonicity, independence vs UOAI/CUOAI.

Statement of the optimization problem

$$\inf_{\sum \vec{d}=l} \mathbb{E}\left[u\left(\sum \vec{X} \wedge \vec{d}\right)\right], \text{ for any } u \in \mathcal{U}_3; \tag{1}$$

$$\inf_{\sum \vec{d}=l} \mathbb{E}\left[u\left(\sum \vec{X} \wedge \vec{d}\right)\right], \text{ for any } \in \mathcal{U}_2; \tag{2}$$

$$\inf_{\sum \vec{d}=l} \mathbb{E}\left[u\left(\sum (\vec{X} - \vec{d})_+\right)\right], \text{ for any } u \in \mathcal{U}_2. \tag{3}$$

Denote the optimal solution as $\vec{d}^* = (d_1^*, \ldots, d_n^*)$.

Models with Discounted Factor

Let δ be the discounted factor and $\vec{S} = (S_1, ..., S_2)$ be the occurrence times of claims. Suppose $\vec{S}$ is independent of $\vec{X}$.

- Total discounted risk under different policy:
  With limit: $I_{\vec{X},\vec{S}}(\vec{d}) = \sum \left((\vec{X} \wedge \vec{d}) \star \exp\{-\delta \vec{S}\} \right)$
  With deductible: $R_{\vec{X},\vec{S}}(\vec{d}) = \sum \left((\vec{X} - \vec{d})_+ \star \exp\{-\delta \vec{S}\} \right)$
  where $\vec{x} \star \vec{y} = (x_1 y_1, \cdots, x_n y_n)$.
- Optimization Problems:

$$\inf_{\sum \vec{d} = l} \mathbb{E}\left[u(I_{\vec{X},\vec{S}}(\vec{d}))\right] \quad , \quad \text{for any } u \in \mathcal{U}_2, \qquad (4)$$

$$\inf_{\sum \vec{d} = l} \mathbb{E}\left[u(R_{\vec{X},\vec{S}}(\vec{d}))\right] \quad , \quad \text{for any } u \in \mathcal{U}_2. \qquad (5)$$

Main Results (1)

Theorem - Comparison of m.g.f. and Moments under UOAI

Assume $\vec{X}$ is UOAI and nonnegative. Then for any $i \leq j$ such that $d_i \geq d_j$, it holds that $\mathbb{E}\left[u\left(\sum \vec{X} \wedge \vec{d}\right)\right] \leq \mathbb{E}\left[u\left(\sum \vec{X} \wedge \pi_{ij}(\vec{d})\right)\right]$, for any $u \in \mathcal{U}_3$.

Theorem - Comparison of Convex Utility under CUOAI

Assume $\vec{X}$ is CUOAI. Then for any $i \leq j$ such that $d_i \geq d_j$, we have $\mathbb{E}\left[u\left(\sum \vec{X} \wedge \vec{d}\right)\right] \leq \mathbb{E}\left[u\left(\sum \vec{X} \wedge \pi_{ij}(\vec{d})\right)\right]$, for any $u \in \mathcal{U}_2$.

Interpretations

The above two Theorems means that the solutions to problems (1) and (2), denoted as $(d_1^*, \ldots, d_n^*)$, should satisfy $d_1^* \geq \ldots \geq d_n^*$.

Main Results (2)

Theorem - Solution to (4)

Assume $\vec{X}$ is CUOAI and nonnegative, $-\vec{S}$ is SAI, then the solution to (4) satisfies: $d_1^* \geq \cdots \geq d_n^*$.

Theorem - Solution to (5)

Assume $\vec{X}$ is SAI and nonnegative, $-\vec{S}$ is SAI, then the solution to (5) satisfies: $d_1^* \leq \cdots \leq d_n^*$.

Summary

Under limit policy, the limits should be arranged in a descending order; while under deductible policy, deductibles should be arranged in an ascending order.

Conclusion and Future Work

Conclusion

- Generalize the dependence structures.
- Restudy the optimal allocation problems.

Future Work

- Construct more general examples of SAI.
- Examine the relation between UOAI and PDS, to make the study consistent.
- Systematically develop the properties of UOAI/CUOAI and SAI, explore more applications.

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Thank You!