Second Québec-Ontario Workshop on Insurance Mathematics
February 3, 2012

MATHEMATICAL AND COMPUTATIONAL ISSUES IN CALCULATING CAPITAL FOR CREDIT RISK

David Saunders
Department of Statistics and Actuarial Science
University of Waterloo

Joint work with: J.C. Garcia-Cespedes, J.A. de Juan Herrero, A. Memartoluie, D. Rosen, and T. Wirjanto

Preface…

Counterparty risk is

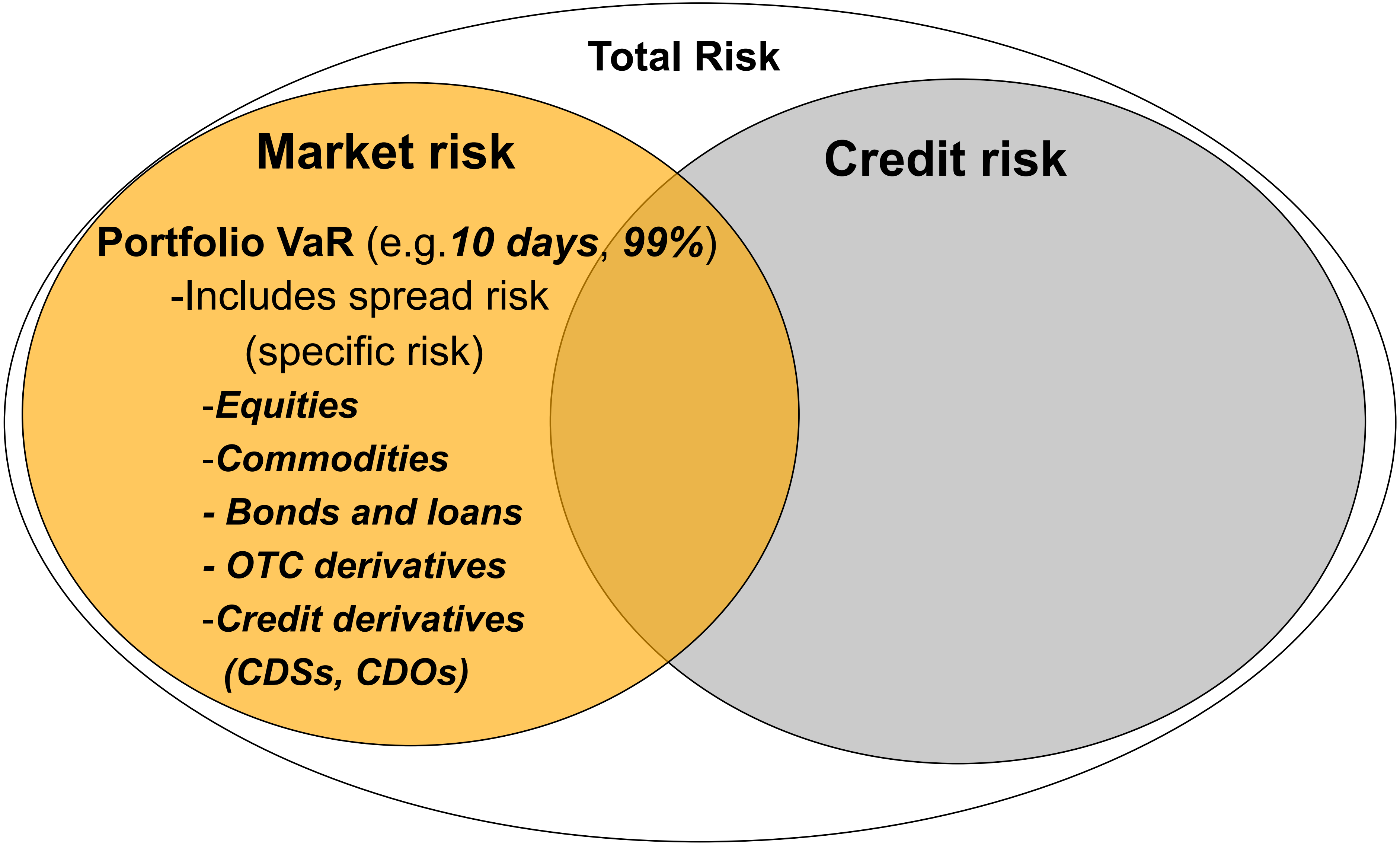
"… probably the single most important variable in determining whether and with what speed financial disturbances become financial shocks, with potential systemic traits"

Counterparty Risk Management Policy Group (CRMPG 2005)

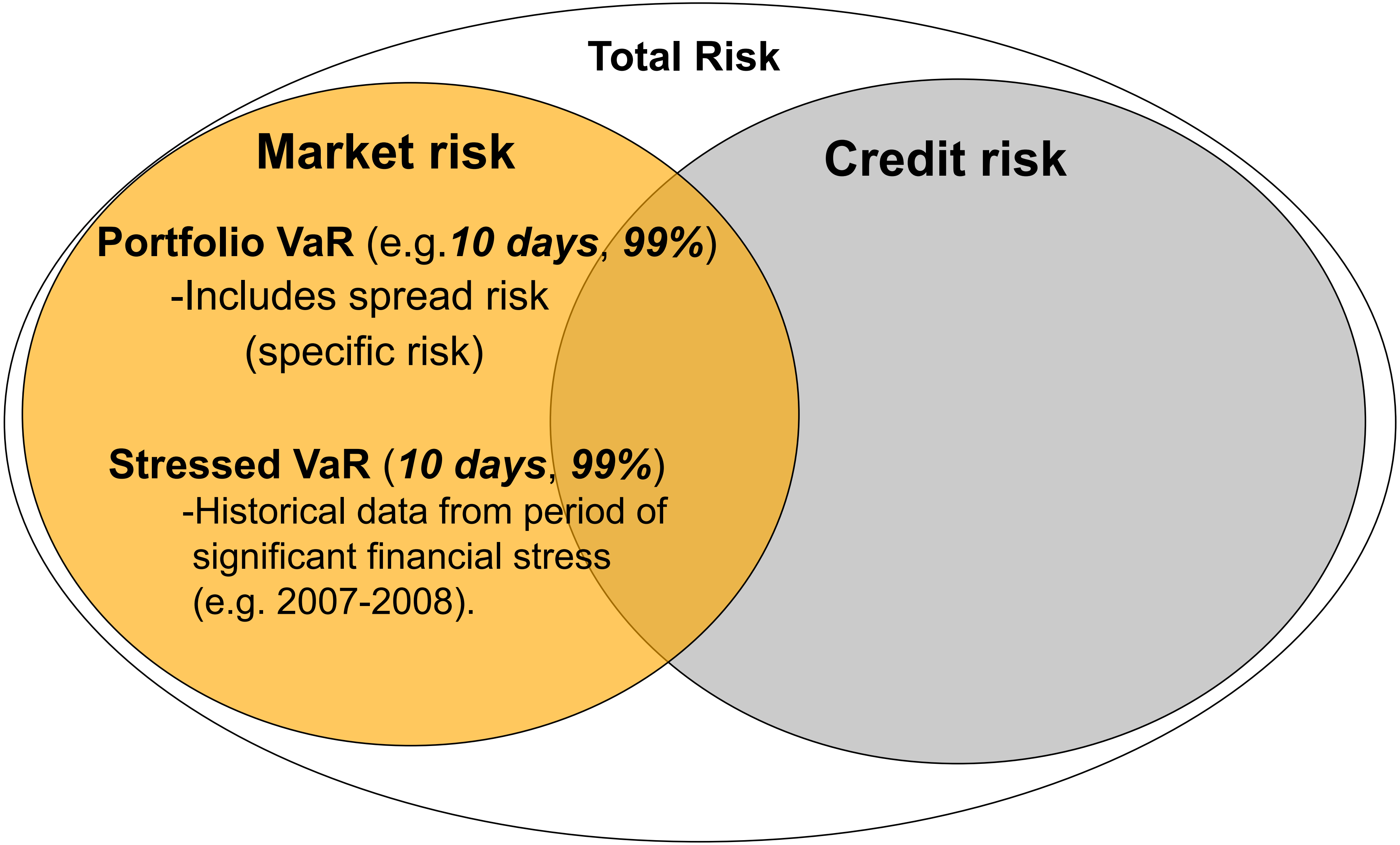
Outline

1. Credit Risk in the Trading Book
2. Stress Testing CCR in the Basel Accord
3. Worst Case Copulas: An Optimal Transportation Problem
4. Numerical Formulation
5. Results

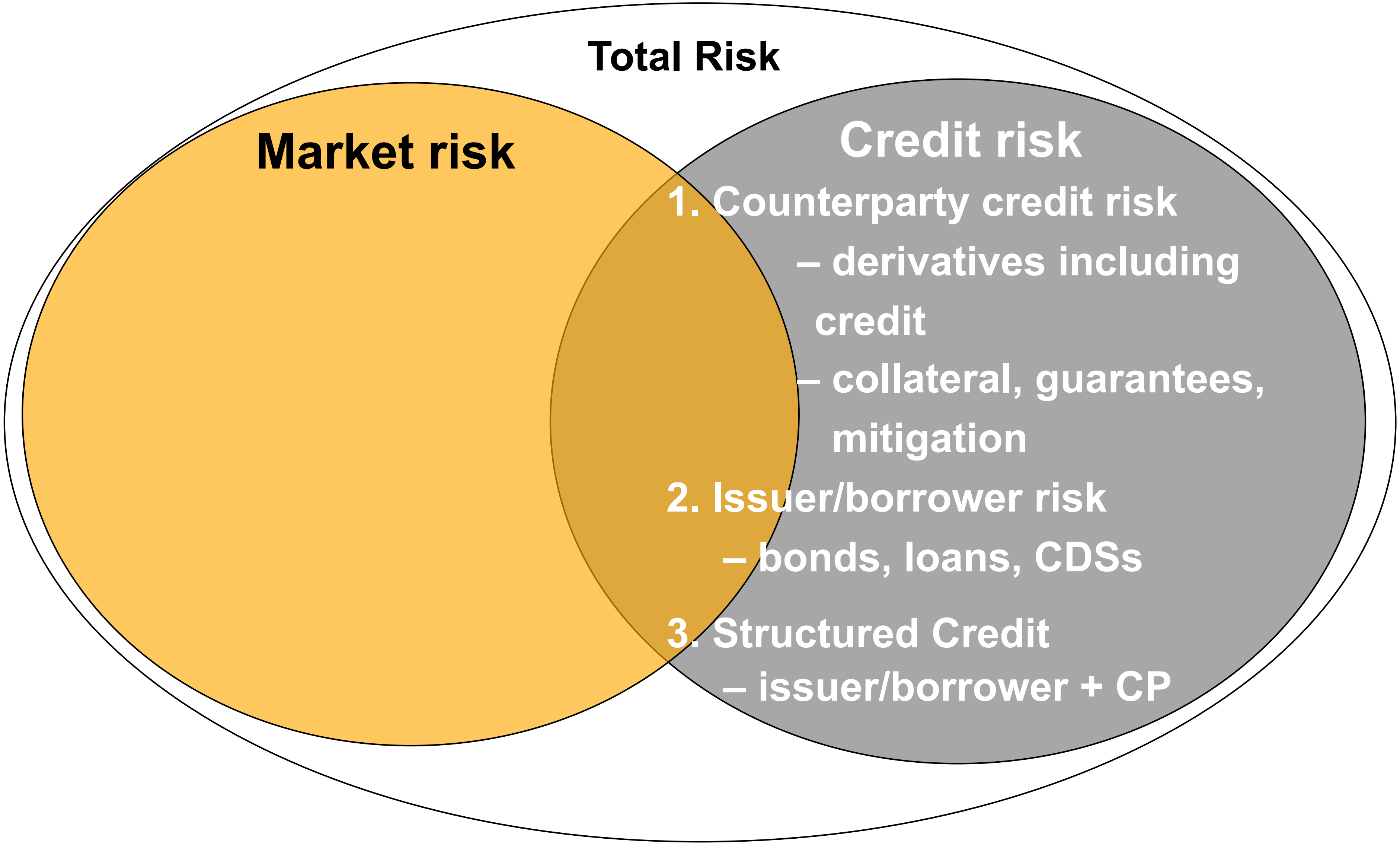
Risk in the Trading Book and Basel



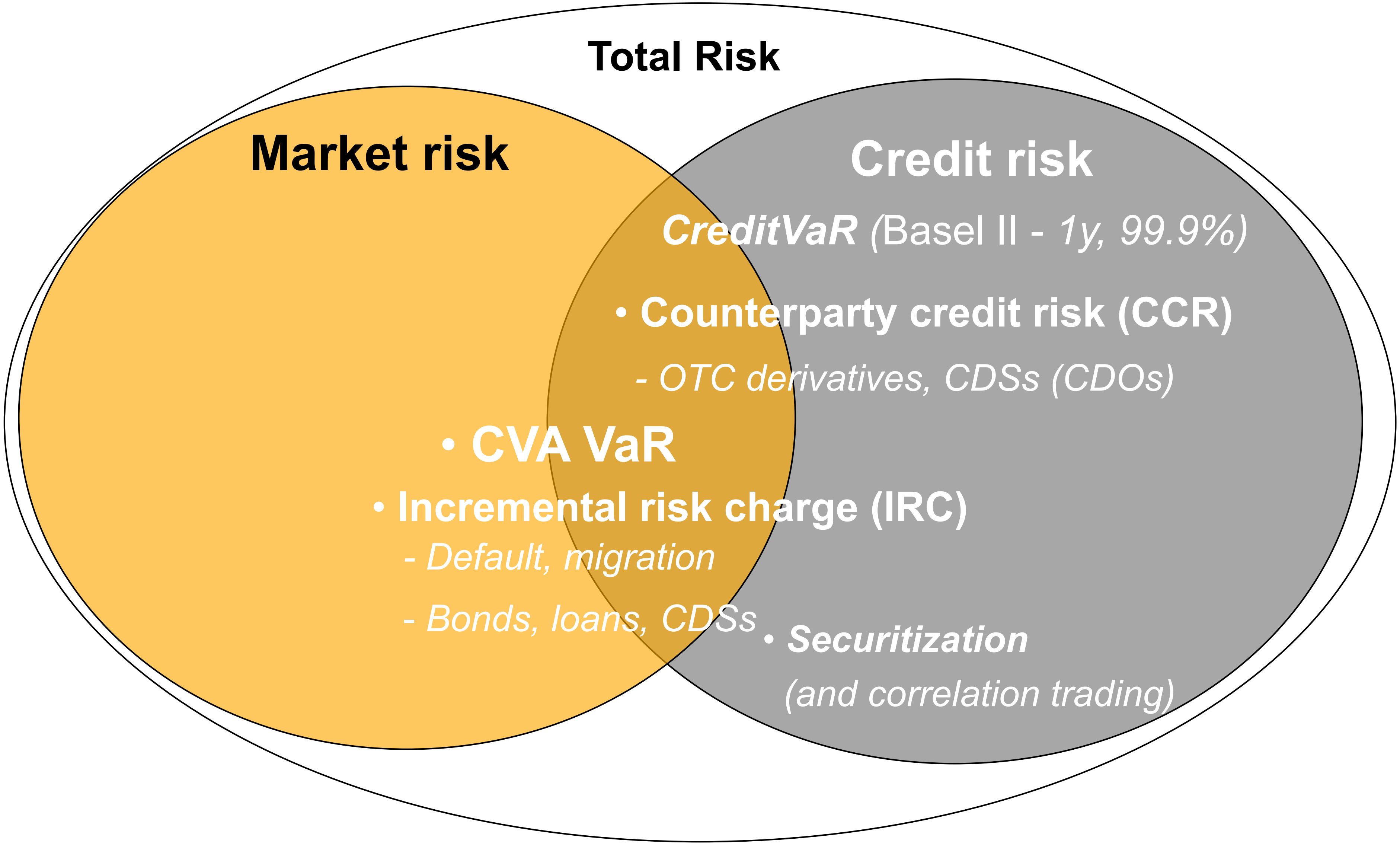
Risk in the Trading Book and Basel



Risk in the Trading Book and Basel



Risk in the Trading Book and Basel



Basel IRB Credit Capital Formula

- Basel II model: ASFM, heterogeneous portfolio, *default and migration risk*

$$Basel\ Capital = \sum_{j=1}^{N} LGD_j \cdot EAD_j \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD_j) - \sqrt{\rho_j}\ \Phi^{-1}(0.001)}{\sqrt{1-\rho_j}} \right) - PD_j \right] \cdot MF(M_j, PD_j)$$

RWAs calculation relies on four quantitative inputs (risk components):

1. **Probability of default** (PD): likelihood of borrower default over one year
2. **Exposure at default** (EAD): amount that could be lost upon default
3. **Loss given default** (LGD): proportion of exposure lost if default occurs
4. **Maturity** (M): remaining economic maturity of the exposure

- Another model parameter (set by the accord) is the **asset correlation**

Basel IRB Credit Capital Formula

- Basel II model: ASFM, heterogeneous portfolio, *default and migration risk*

$$Basel\ Capital = \sum_{j=1}^{N} LGD_j \cdot EAD_j \cdot \left[\Phi\left(\frac{\Phi^{-1}(PD_j) - \sqrt{\rho_j}\,\Phi^{-1}(0.001)}{\sqrt{1-\rho_j}} \right) - PD_j \right] \cdot MF(M_j, PD_j)$$

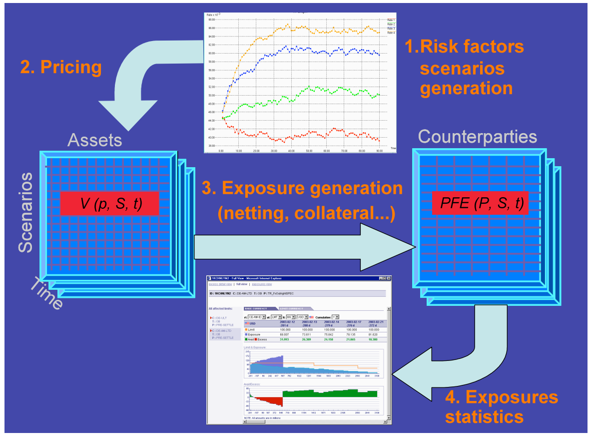
Capital at 99.9% over one year
- Default credit losses
- Single-factor Merton-type model
- Systematic risk (asymptotically fine-grained portfolio)

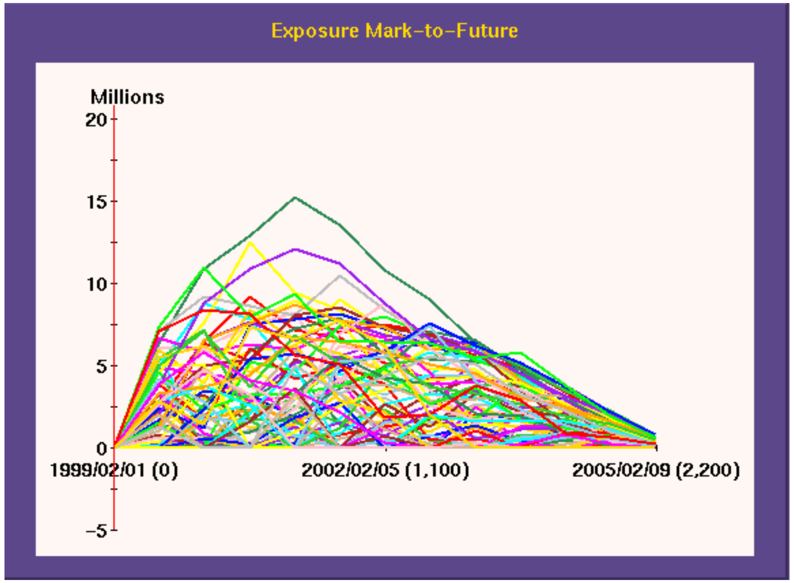
MF = maturity factor
Captures "incremental" credit risk capital due to credit migration

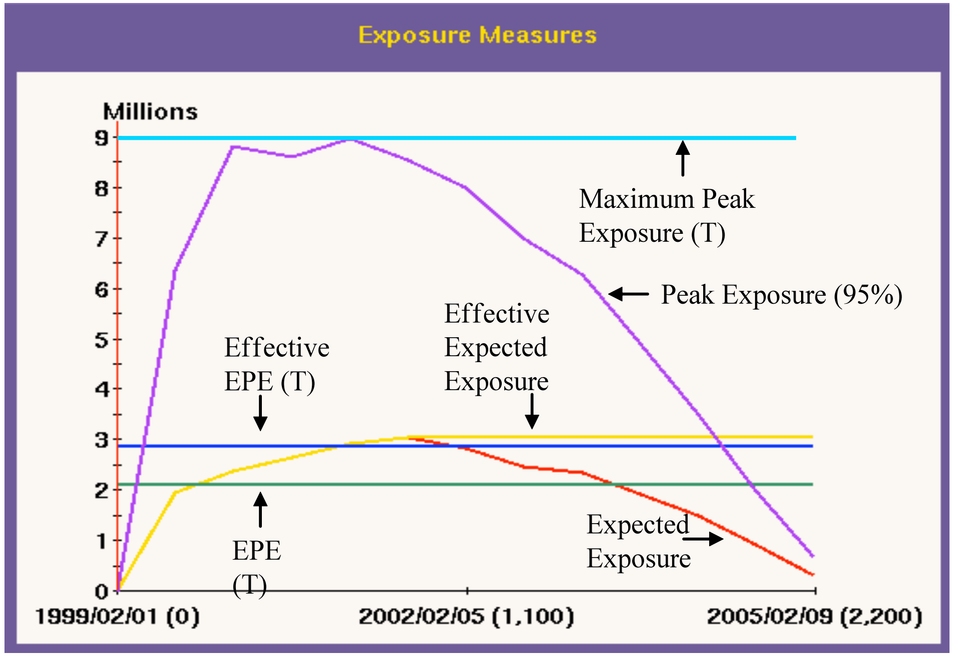
(MF function calibrated by the BCBS)

Basel and Potential Future Exposures (PFEs)

- Basel II – IRB credit capital
  - MtM + add-on → internal models for EAD



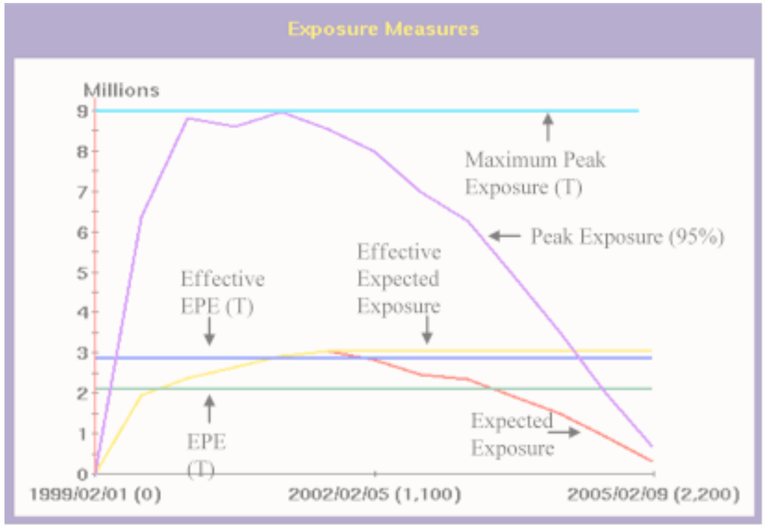




Source: de Prisco and Rosen (2005)

CCR and Basel – Internal Model

$$EAD = \alpha \cdot \text{Effective EPE}$$



Expected Exposure (over all scenarios) at t_k	$EE_j(t_k) = \sum_{s=1}^{S} PFE_j(\omega_s, t_k) p_s$
Time-Averaged Exposure (for scenario s, up to time t_k)	$\mu_j^{t_k}(\omega_s) = \frac{1}{t_k} \int_0^{t_k} PFE_j(\omega_s, t) dt$
Expected Positive Exposure (EPE)	$EPE_j(t_k) = \frac{1}{t_k} \int_0^{t_k} EE_j(t) dt = \sum_{s=1}^{S} \mu_j^{t_k}(\omega_s) p_s$
Effective Expected Exposure	$\mu_j^E(t_k) = \max_{0 \le i \le k} \left[EE_j(t_k) \right] = \max \left[\mu_j^E(t_{k-1}), EE_j(t_k) \right]$
Effective EPE	$\text{Effective } EPE_j(t_k) = \frac{1}{t_k} \int_0^{t_k} \mu_j^E(t) dt$

Internal Models for EAD in Basel

$$Basel\ Capital = \sum_{j=1}^{N} LGD_j \cdot \max\left(EAD_j - CVA_j, 0\right) \cdot \left[N\left(\frac{N^{-1}\left(PD_j\right) - \sqrt{\rho_j}\, N^{-1}(0.001)}{\sqrt{1-\rho_j}} \right) - PD_j \right] \cdot MF\left(M_j, PD_j\right)$$

$$EAD_j = Eff\ EPE_j \cdot \alpha \qquad \alpha = \frac{EC\left(L^T\right)}{EC\left(L^{EPE}\right)}$$

Credit losses – random exposures

Credit losses – deterministic exposures (EPE)

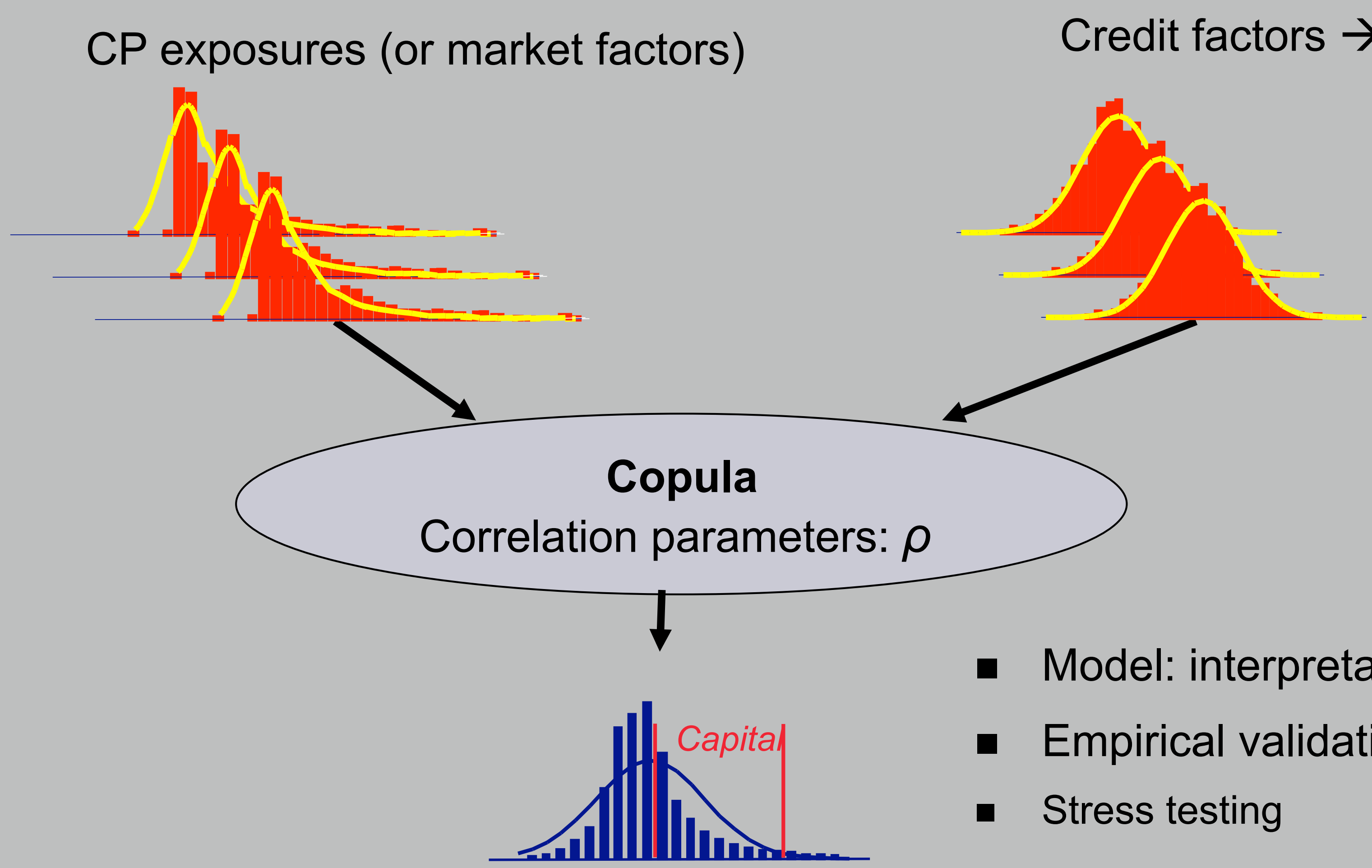
- *Eff EPE* = effective *EPE* (expected positive exposure)
- *Alpha*: measures the effect of using deterministic exposures (*EPE*) instead of stochastic exposures – ratio of
  - EC from a joint simulation of market and credit risk factors
  - EC when CP exposures are constant and equal to *EPE*
- *Eff EPE* and *M* (maturity) based on bank's internal model
- Supervisory alpha = *1.4* – allow for own estimate, subject to floor = *1.2*

Summary – CCR Methodology (Garcia et al. 2010)

- Computationally efficient approach CCR capital, alpha and CVA
  - Leverages CP exposure simulations and preserves their joint distribution
  - Can be applied within general integrated market-credit risk models
  - Simplified model that correlates (pre-computed) exposures with credit factors leads to a parsimonious, computationally tractable approach
    - Consistent with the Basel model and also easy to implement
- Stress testing framework
  - Wrong-way risk and for risk management and regulatory applications
  - Numerical solution for inverse problem: finding the minimum level of market-credit correlation which results in a floor for alpha (e.g. 1.2)
  - Stress test: market factors, correlations, exposures, time-steps
- Implemented and tested methodology at several international banks

CCR Capital – Correlated Market-Credit

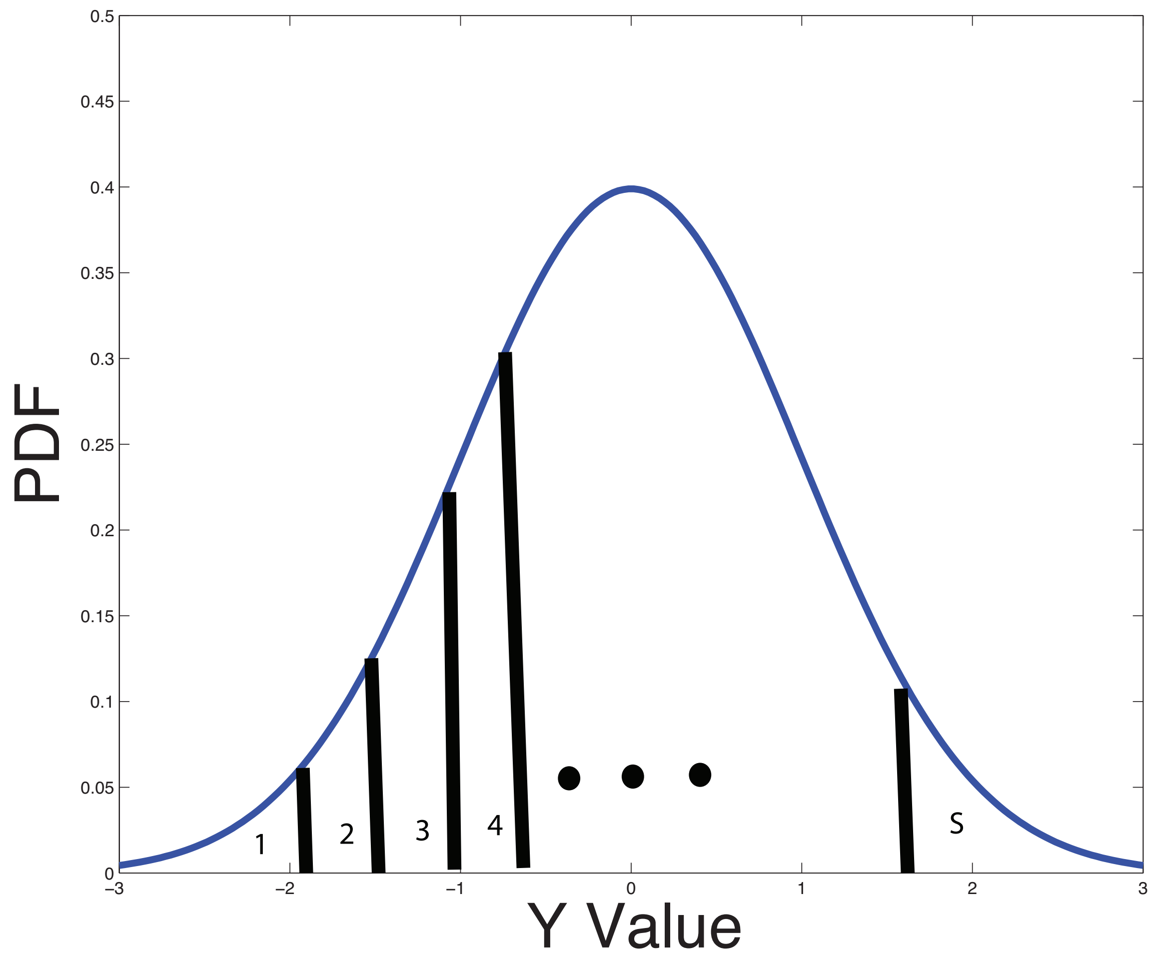
General market-credit codependence framework



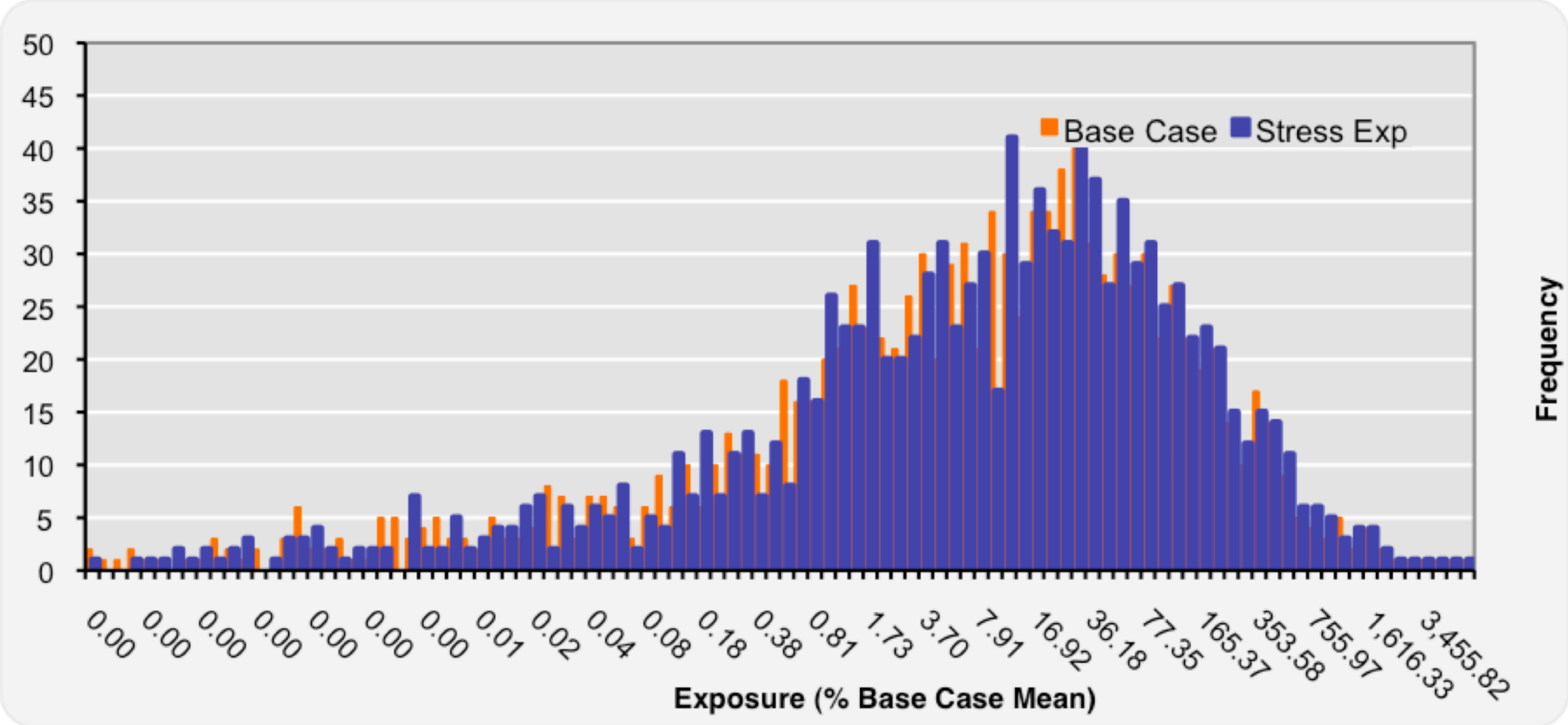
- Model: interpretable, auditable
- Empirical validation
- Stress testing

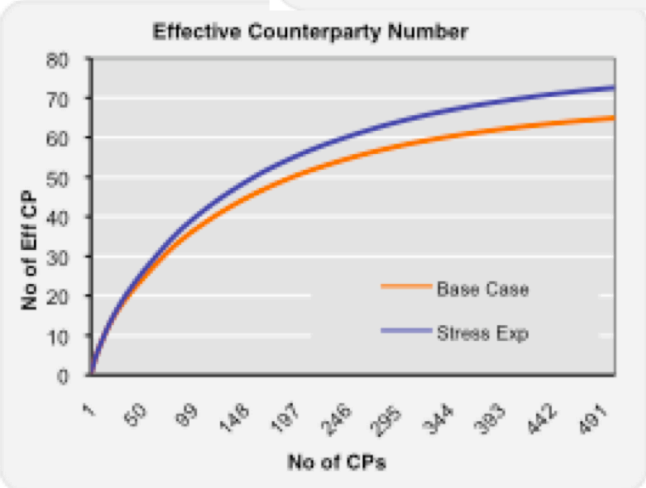
Exposure Simulation

- Exposure scenarios are sorted in order of increasing time-averaged exposure, mapped to a normal.
- Regions between bars have equal probability.



Exposures

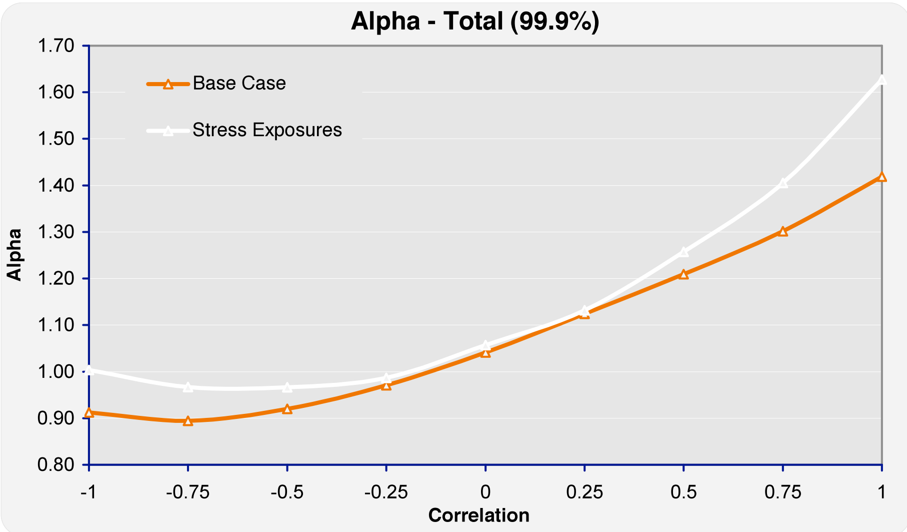


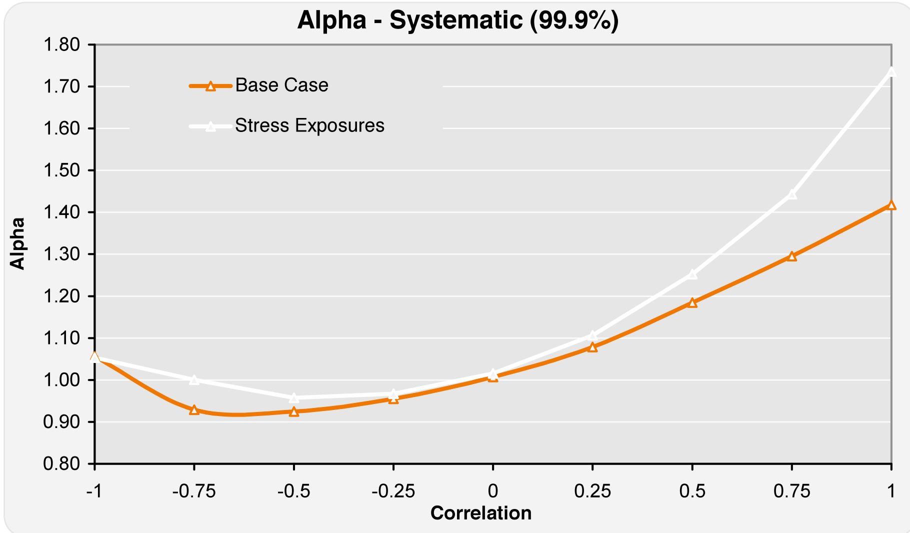


Sample portfolio: Approx. 2500 counterparties, 2000 market scenarios.

Stressed Exposures

Alpha (99.9%): Base Case vs. Stressed Exposures





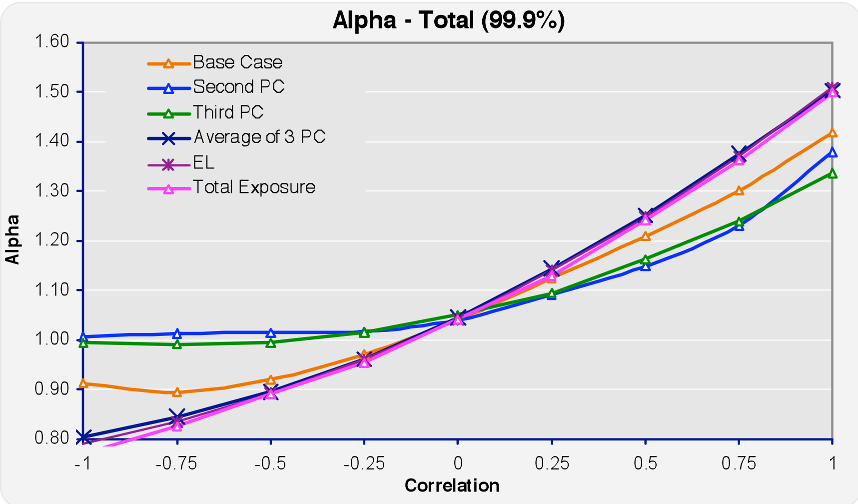
Alpha (99.9%) as a function of market-credit correlation

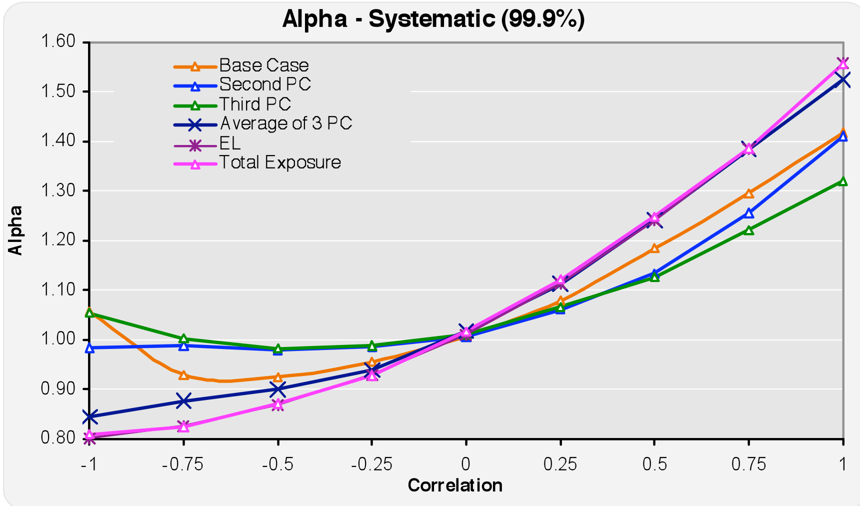
Correlation	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
Alpha Total (LTS/LTD)									
Base Case	0.91	0.89	0.92	0.97	1.04	1.12	1.21	1.30	1.42
Stressed Exposures	1.00	0.97	0.97	0.99	1.06	1.13	1.26	1.41	1.63
Alpha Systematic (LSS/LSD)									
Base Case	1.06	0.93	0.93	0.96	1.01	1.08	1.18	1.30	1.42
Stressed Exposures	1.05	1.00	0.96	0.97	1.02	1.11	1.25	1.44	1.74

Client Confidential

Sorting Scenarios in Different Ways

Alpha (99.9%): Exposure Factor Stress Tes





Alpha (99.9%) as a function of market-credit correlation

Correlation	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
Alpha Total (LTS/LTD)									
Base Case	0.91	0.89	0.92	0.97	1.04	1.12	1.21	1.30	1.42
Second PC	1.01	1.01	1.01	1.02	1.04	1.09	1.15	1.23	1.38
Third PC	0.99	0.99	0.99	1.01	1.05	1.09	1.16	1.24	1.34
Average of 3 PC	0.80	0.84	0.89	0.96	1.04	1.14	1.25	1.38	1.50
EL	0.79	0.83	0.89	0.96	1.04	1.14	1.25	1.37	1.51
Total Exposure	0.77	0.83	0.89	0.95	1.04	1.13	1.24	1.36	1.50
Alpha Systematic (LSS/LSD)									
Base Case	1.06	0.93	0.93	0.96	1.01	1.08	1.18	1.30	1.42
Second PC	0.98	0.99	0.98	0.99	1.01	1.06	1.13	1.26	1.41
Third PC	1.05	1.00	0.98	0.99	1.01	1.07	1.13	1.22	1.32
Average of 3 PC	0.85	0.88	0.90	0.94	1.02	1.11	1.24	1.38	1.52
EL	0.80	0.83	0.87	0.93	1.01	1.11	1.24	1.39	1.56
Total Exposure	0.81	0.82	0.87	0.93	1.02	1.12	1.25	1.39	1.56

Client Confidential

Optimal Transportation Problem

- Is the method described above conservative?
- What is the "worst case copula"?
- Given:
  - X_M : Market risk factors (exposures), with distribution P_M
  - X_C : Credit risk factors, with distribution P_C
  - Nonlinear loss function $L(X_M, X_C)$.
  - Risk Measure: ρ
- Solve:

$$\max_{\Pi(P_M, P_C)} \rho(L(X_C, X_M))$$

Related Work

- Bounds on Option Prices under Partial Information:
  - Bieglböck, Henry-Labodère and Penkner (2011), Galichon, Henry-Labodère and Touzi (2010), Haase, Ilg, and Werner, 2010, Avellaneda, Levy, and Parás (1995), Tankov (2011).
- Risk Measures under Model Uncertainty: Cont (2006), Nutz and Soner (2010), Bion-Nadal and Kervarec (2010), Talay and Zheng (2002).
- Bounds on Distribution Functions and VaR: Embrechts and Puccetti (2006a,b), Puccetti and Rüschendorf (2011), Wang and Wang (2011).

Default and Systematic Loss

- Counterparty *j* defaults if: $CWI_j \le \Phi^{-1}(PD_j)$

$$CWI_j = \sqrt{\rho_j} \cdot Z + \sqrt{1-\rho_j} \cdot \varepsilon_j$$

- Portfolio Losses:

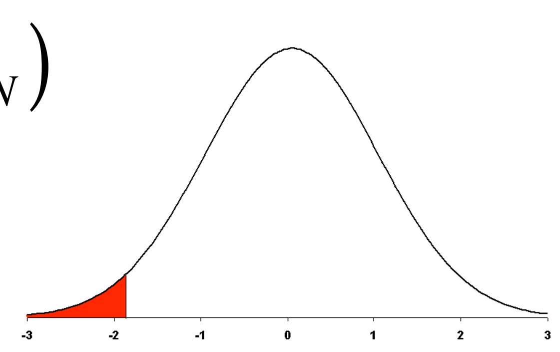
$$L = \sum_{j=1}^{N} LGD_j \cdot EAD_j \cdot 1\left\{CWI_j \le \Phi^{-1}\left(PD_j\right)\right\} = \sum_{j=1}^{N} L_j$$

- Market Factors:

$$X_M = \left(LGD_1, EAD_1, \ldots, LGD_N, EAD_N\right)$$

- Credit Factors:

$$X_C = \left(Z, \varepsilon_1, \ldots, \varepsilon_N\right)$$



Simplifications and Assumptions

- Use the existing exposure scenarios (from limits calculation) as the distribution of the market factors.
  - Resample for risk measurement calculations
- Only correlate *systematic credit factors* with market factors.
  - Use systematic losses in the optimal transportation problem.

$$L_S = E[L \mid Z] = \sum_{j=1}^{N} LGD_j \cdot EAD_j \cdot \Phi\left(\frac{\Phi^{-1}(PD_j) - \sqrt{\rho_j} \cdot Z}{\sqrt{1-\rho_j}}\right)$$

- Discretize systematic credit factor distribution.
- Use CVaR as the risk measure (instead of VaR).

Optimization Problem

- LGD adjusted exposure of CP *j* under market scenario *m*: y_{jm}
- Marginal probabilities of market scenarios: π_m
- Marginal probabilities of credit scenarios: $P(Z = Z_s) = q_s, \quad s = 1,\ldots,S$
- (Systematic) losses under a given market-credit scenario:

$$L_{ms} = \sum_{j=1}^{N} y_{jm} \cdot \Phi\left(\frac{\Phi^{-1}(PD_j) - \sqrt{\rho_j} \cdot Z_s}{\sqrt{1-\rho_j}} \right)$$

- Optimal transportation problem:

$$\max_{p \geq 0} \ \mathrm{Risk}_p(L)$$

$$\sum_{m=1}^{M} p_{ms} = q_s, \quad s = 1,\ldots,S$$

$$\sum_{s=1}^{S} p_{ms} = \pi_m, \quad m = 1,\ldots,M$$

Optimization Problem

- Using CVaR, Rockafellar and Uryasev (2002), and a minimax theorem, the problem becomes:

$$\max_{p\geq 0}\min_{C}\left\{C+(1-\alpha)^{-1}\sum_{m,s}p_{ms}\left(L_{ms}-C\right)_{+}\right\}$$
$$\sum_{m=1}^{M}p_{ms}=q_s,\quad s=1,\ldots,S$$
$$\sum_{s=1}^{S}p_{ms}=\pi_m,\quad m=1,\ldots,M$$

$\Longleftrightarrow$

$$\min_{C}\max_{p\geq 0}\left\{C+(1-\alpha)^{-1}\sum_{m,s}p_{ms}\left(L_{ms}-C\right)_{+}\right\}$$
$$\sum_{m=1}^{M}p_{ms}=q_s,\quad s=1,\ldots,S$$
$$\sum_{s=1}^{S}p_{ms}=\pi_m,\quad m=1,\ldots,M$$

- The inner (maximization) problem can be formulated as a linear program.
- The outer (minimization) problem is one-dimensional.

Optimal Transportation Problems

- Let $P = (p_1, p_{2,}, \ldots, p_m)$ and $Q = (q_1, q_{2,}, \ldots, q_n)$ be prob. mass vectors
- $C = (C_{ij}, 1 \le i \le m, 1 \le j \le n)$ be nonnegative matrix

$$\max \sum_i \sum_j H_{ij} C_{ij}$$

$$\text{s.t.} \quad \sum_j H_{ij} = p_i \qquad i = 1, \ldots, m$$

$$\sum_i H_{ij} = q_j \qquad j = 1, \ldots, n$$

- If $C = (C_{ij})$ is supermodular, i.e.,

$$C_{i_1 j_1} + C_{i_2 j_2} \ge C_{i_1 j_2} + C_{i_2 j_1} \qquad \text{for } i_1 \le i_2 \text{ and } j_1 \le j_2$$

- We can find $H = (H_{ij})$ by the following greedy algorithm

for i := 1 to m do
    for j := 1 to n do
        $H_{ij} = \min(p_i, q_j)$;
        $p_i := p_i - H_{ij}$ and $q_i := q_j - H_{ij}$

Optimal Transportation Problems

- For a fixed C,

$$\max_{p \in \Pi} C + (1-\beta)^{-1} \sum_{m,s} p_{ms} \left(L_{ms} - C \right)_+$$

When exposures are monotonic, the supermodularity condition holds and we can use the previous greedy algo. to find the optimal joint distribution.

- Now if we solve this LP for the following value of parameters, we have:

Counterparties	10	10	10	10
Market Scenarios	1000	1000	1000	2000
Credit Scenarios	100	100	200	200
Confidence Level	0.95	0.95	0.95	0.95
GSS tolerance	0.05	0.025	0.025	0.05
CVaR	37.8286	38.2714	39.0033	40.4004

Algorithm

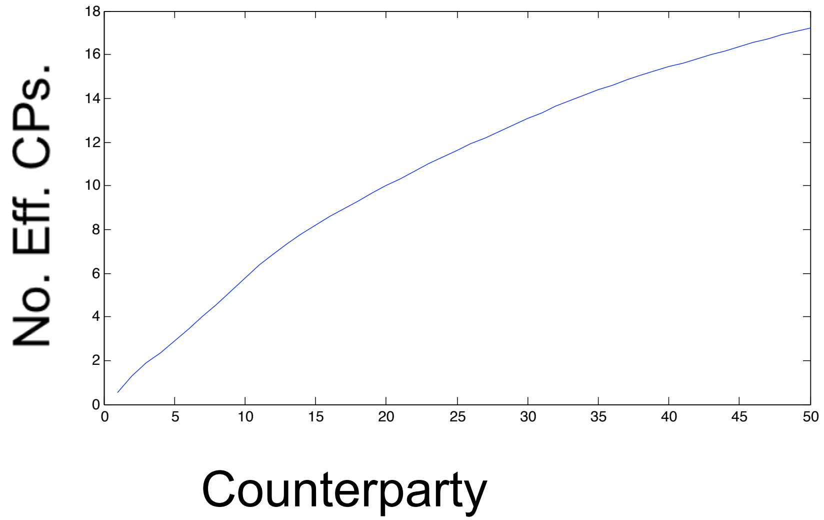
- Simulate from joint distribution of market factors and compute counterparty exposures.
  - Already carried out in practice for limits management
- Discretize systematic credit factors *Z*:

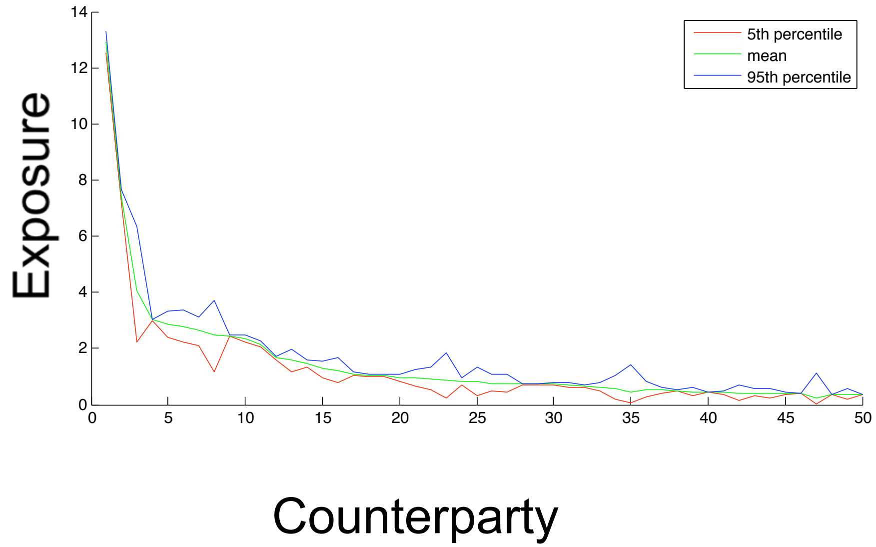
$$Z_1 \leq Z_2 \leq \cdots \leq Z_S$$

- Compute systematic losses under each combined market/credit scenario.
- Solve the worst-case copula optimal transportation problem.
- Sample from the resulting distribution to compute losses, risk measures, etc.

Example

- Portfolio of 50 counterparties, 2000 market scenarios.

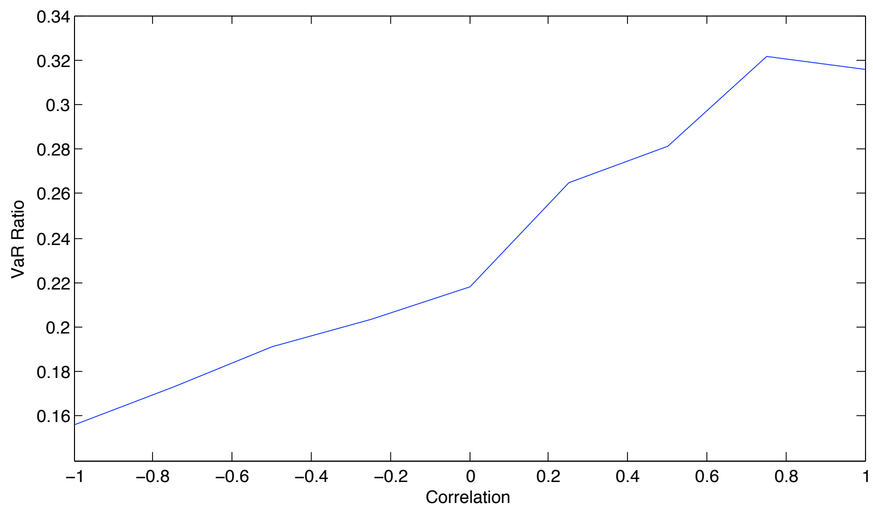




Example

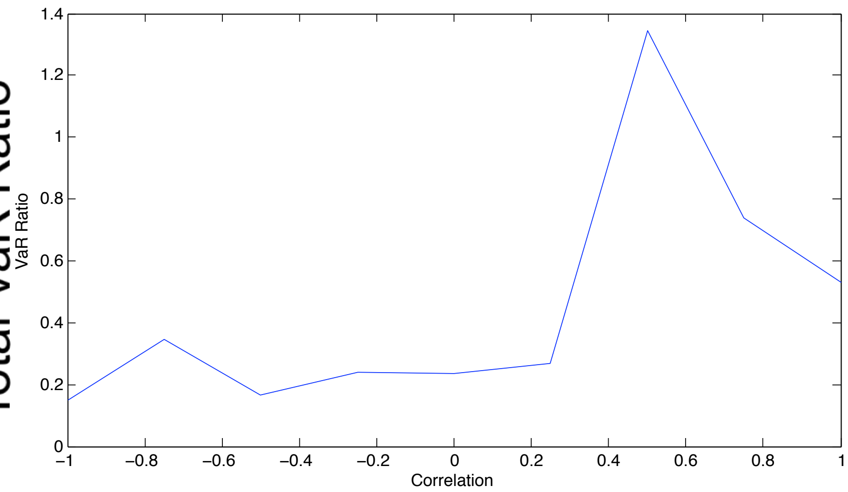
Comparing "worst-case" VaRs to VaRs calculated with a normal copula and sorting scenarios by total portfolio exposure.
Confidence level = 99.9%, No. scenarios = 100,000.
VaR Ratio = Sorting VaR / Worst Case VaR.

Systematic VaR Ratio



Correlation

Total VaR Ratio



Correlation

Future Work

- Importance sampling for the systematic credit factors.
- Accelerating computations through
  - Large scale optimization techniques
  - Exploitation of the structure of the LPs
- Convergence analysis of the discretized problems to the true optimal solution.
  - Error bounds
- Extensions to other problems in credit risk and risk management:
  - CVA, structured credit products,…

References

- Avellaneda, M., Levy, A., and Paràs, A., 1995, “Pricing and Hedging Derivatives Securities in Markets with Uncertain Volatilities”, Applied Mathematical Finance, 2, 73-88.
- Basel Committee on Banking Supervision, "International Convergence of Capital Measurements and Capital Standards: A Revised Framework, Comprehensive Version", 2006.
- Bieglböck, M., Henry-Labodère, P., and Penkner, F., 2011, “Model-Independent Bounds for Option Prices: A Mass Transportation Approach”, Working Paper.
- Bion-Nadal, J., and Kervarec, M., 2010, “Risk Measuring under Model Uncertainty”, Forthcoming in Annals of Applied Probability.
- E. Canabarro, E. Picoult and T. Wilde, "Analysing Counterparty Risk", Risk, 16(9), September, 2004.
- Cont, R., 2006, “Model Uncertainty and its Impact on the Pricing of Derivative Instruments”, Mathematical Finance, 16(3), 519-547.
- Counterparty Risk Management Policy Group, "Toward Greater Financial Stability: A Private Sector Perspective", 2005.
- B. de Prisco and D. Rosen, "Modelling Stochastic Counterparty Credit Exposures for Derivatives Portfolios", in "Counterparty Credit Risk Modelling", M. Pykhtin ed., Risk Books, 2005.
- Embrechts, P., and Puccetti, G., 2006a, “Bounds for Functions of Multivariate Risks”, Journal of Multivariate Analysis, 97, 526-547.

References

- Embrechts, P., and Puccetti., G, 2006b, Bounds for Functions of Dependent Risks”, Finance and Stochastics, 10, 341-352.
- Fleck, M. and Schmidt, A., 2005, “Analysis of Basel II Treatment of Counterparty Credit Risk”, Counterparty Credit Risk Modelling, M. Pykhtin Editor, Risk Books, London.
- Galichon A., Henry-Labodère, P., and Touzi, N., 2010, “A Stochastic Control Approach to No-Abitrage Bounds with Given Marginals, with an Application to Lookback Options”, Working Paper.
- Garcia Cespedes J. C., de Juan Herrero J. A., Rosen D., Saunders D. 2010, Effective modelling of Wrong-Way Risk, CCR Capital, and Alpha in Basel II”, Journal of Risk Model Validation, 4(1), pages 71-98.
- Glasserman, P., and Yao, D.D., 2004, “Optimal Couplings are Totally Positive and More”, Journal of Applied Probability, 41A, 321-332.
- Haase, J, Ilg, M., and Werner, R., 2010, “Model-Free Bounds on Bilateral Counterparty Valuation”, Working Paper.
- Nutz, M., and Soner, H.M., 2010, “Superhedging and Dynamic Risk Measures under Model Uncertainty”, Working Paper.
- E. Picoult, “Calculating and Hedging Exposure, Credit Value Adjustment and Economic Capital for Counterparty Credit Risk”, in "Counterparty Credit Risk Modelling", M. Pykhtin ed., Risk Books, 2005.

References

- Puccetti, G., and Rüschendorf, 2011, “Bounds for Joint Portfolios of Dependent Risks”, Working Paper.
- Rockafellar, R.T., and Uryasev, S., 2002, “Conditional value-at-risk for general loss distributions”, Journal of Banking & Finance, 26, 1443-1471.
- Rosen D. and Saunders D. 2010, Computing and Stress Testing Counterparty Credit Risk Capital, in Counterparty Credit Risk Modelling, (ed. E. Canabarro), Risk Books.
- Talay, D., and Zheng, Z., 2002, “Worst Case Model Risk Management”, Finance and Stochastics, 6, 517-537.
- Tankov, P., 2011, “Improved Frechet Bounds and Model-Free Pricing of Multi-Asset Options”, Journal of Applied Probability, 43, 389-403.
- Wang, B., and Wang, R., 2011, “The Complete Mixability and Convex Minimization Problems with Monotone Marginal Densities”, Forthcoming in Journal of Multivariate Analysis.