Asset-liability management for pension fund using an international investment model

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- We present an international economic model
- We introduce an Asset-Liability optimal model for a DC pension fund
- We solve the optimal problem by a stochastic programming technique

We consider the case of two economies : domestic economy (Canada) vs foreign economy (U.S). Investors are allowed to hold :

- Domestic and foreign T-bill.
- Stock shares in both domestic and foreign markets.
- A rolling bond portfolios : short-term, mid-term and long-term.

We assume that there is no default risk

The idea : We extend the affine term structure model for single-country presented by Dai and Singleton (2000) to the case of two economies.
The term structures of interest for domestic and foreign market are given by the same vector of three state variables $\underline{F} = \{F(t) = (F_1(t), F_2(t)F_3(t)), t \geq 0\}$, where F(t)follows an affine dynamic under the physical measure P :

$$dF(t) = K(\Theta - F(t))dt + \Sigma\sqrt{S(t)}dW^r(t) \quad (1)$$

S(t)is a diagonal matrix, where :

$$S_{ii}(t) = \alpha_i + \beta_i' F(t) \quad (2)$$

Result : An affine model assume that the short rates are an affine function of the three state variables, i.e.

$$r^j(t) = \delta_0^j + \delta_F^{j'} F(t), \qquad j \epsilon \{d, f\} \tag{3}$$

We show that the yields of zero-coupon bonds with maturity τ are affine function of factors,

$$Y^j(t,\tau) = -\frac{1}{\tau} A^j(\tau) + \frac{1}{\tau} B^j(\tau)' F(t) \tag{4}$$

A^j and B^j are functions of model parameters and satisfy a Riccati's ODE :

$$\begin{aligned} \frac{dA^j(\tau)}{d\tau} &= -\Theta^{q'} K^{q'} B^j(\tau) + \frac{1}{2} \sum_{i=1}^{N} [\Sigma' B^j(\tau)]_i^2 \alpha_i - \delta_0^j, & (5) \\ \frac{dB^j(\tau)}{d\tau} &= -K^{q'} B^j(\tau) - \frac{1}{2} \sum_{i=1}^{N} [\Sigma' B^j(\tau)]_i^2 \beta_i + \delta_F^j, & (6) \end{aligned}$$

We consider two stock indexes $X(t) = [X^i(t)]$ and we assume that the price process follows the stochastic differential equations :

$$(diag(X(t)))^{-1} dX(t) = \mu^X_{y(t)} dt + \Sigma^X_{y(t)} dW^x(t), \tag{7}$$

Where diag(X(t))is a diagonal matrix whose elements are X(t).
W^x is a two-dimensional standard Brownian motion.
We model the stock markets by continuous time Markov regime switching with two states, the transition probabilities is given by :

$$P_{ij}(t) = P(y(t+dt) = e_j | y(t) = e_i) = g_{ij} dt + o(dt) \tag{8}$$

Where g is the transition rate matrix.

The joint CPI (Consumer Price Index) $\Psi = (CPI^d, CPI^f)$ has the following dynamic :

$$(diag(\Psi_t))^{-1} d\Psi_t = \Pi_t dt + \Sigma_\Psi dW^\Psi \tag{9}$$

Where $diag(\Psi_t)$ is a diagonal matrix whose elements are ψ_t for each economy. Σ_Ψ introduce correlation between the CPI of each economy. The joint expected inflation Π has a Vascicek dynamic :

$$d\Pi_t = \beta_{\Pi_t}(\bar{\Pi} - \Pi_t)dt + \Sigma_\Pi dW^\Pi \tag{10}$$

Σ_Π introduce correlation between the expected inflation of each economy.

Under no-arbitrage and complete markets assumptions (bond markets), one could derive the dynamic of exchange rate $S(t)$ from the joint ATSM.
We specify the kernel pricing dynamics as follow :

$$\frac{dM^j(t)}{M^j(t)} = -r^j(t)dt - \Lambda^{j'}(t)dW(t) \tag{11}$$

Λ^j is the market price of risk.
We find the dynamic of the process s(t),$s(t) = log(S(t))$:

$$\begin{aligned} ds(t) &= (r^d(t) - r^f(t) + \frac{1}{2}(\|\Lambda^d(t)\|^2 - \|\Lambda^f(t)\|^2))dt \\ &\quad +(\Lambda^d(t) - \Lambda^f(t))' dW(t) \end{aligned} \tag{12}$$

- Previous works showed that ASTM led to a poor fit of the exchange rate and the model can't outperform the GARCH model or the random walk model
- We introduce an other economic model to explain the exchange rate dynamic

- We assume that the exchange rate dynamic is explained by the differential of short rates, the stocks indices and inflation rates in both economies :

$$ds(t) = a_1(r^d(t) - r^f(t))dt + a_2(X^d(t) - X^f(t))dt + ...$$

$$a_3(CPI^d(t) - CPI^f(t))dt + \sigma_s dW_s(t) \quad (13)$$

The Canadian labor income L follows the dynamic

$$dL(t) = \mu_L L(t)dt + \sigma_{L,r} L(t)dr^d(t) + \sigma_{L,X} L(t)dW^{X_d}(t) + \sigma_{L,\psi} L(t)dW^{\psi_d}(t) \tag{14}$$

Where $L(0) = L_0$ and we suppose the pension fund contribution to be a constant proportion γ_0

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be the probability space , where $\mathbf{P}$ represents the physical measure. $\mathfrak{F} = \{\mathfrak{F}(t), t \geq 0\}$ is the information which we suppose generated by the following Brownian motion vector $\underline{W} = \{\underline{W}^r, \underline{W}^s, \underline{W}^\psi, \underline{W}^\pi, \underline{W}^x\}$, where :

- $\underline{W}^r = \{W^r(t), t \geq 0\}$ for the joint interest rates market ;
- $\underline{W}^s = \{W^s(t), t \geq 0\}$ introduces an additional uncertainty specific to the exchange market ;
- $\underline{W}^\psi = \{W^\psi(t), t \geq 0\}$ corresponds to the CPI's dynamic and $\underline{W}^\pi = \{W^\pi(t), t \geq 0\}$for the long-term inflation rates ;
- $\underline{W}^x = \{W^x(t), t \geq 0\}$ for the joint stock model.

An additional information is generated by the continuous-time Markov chain

- We calibrate the model to the Canadian date(domestic economy) and the U.S data (the foreign economy)
- We use a sample of 7 yields in each economy with maturities

$$\tau = [0.5; 1; 2; 5; 7; 10; 15]$$

- Two stock indexes are considered : S&P500 and TSX.
- The period of calibration : January 1987- January 2010.

- We consider a state-space formulation to estimate all parameters.
- An extension of state-space formulation in the case of regime switching as proposed by Kim and Nelson(1999)
- Gaussian assumption : instead of MLE $\Rightarrow$ we get a QML estimates.
- The gaussian assumption does not hold which introduces some bias $\rightsquigarrow$ 'Solution' : bootstrap estimation.
- The measurement and transition equation are obtained by a discrete time formulation of stochastic process using 'Euler Scheme'
- Due to the nonlinear constrained likelihood function, we use the interior point optimization algorithm with logarithm penalty function

We define $DC(t)$ is the fund accumulation value up to time t by holding the portfolio $\{x_t\}$

$$\frac{dDC(t)}{DC(t)} = x_t' \frac{dA(t)}{A(t)} + \gamma_0 \frac{dL(t)}{L(t)} \tag{15}$$

$A(t)$ is a vector of assets prices at time t.

We consider a DC pension fund with the following optimization problem with an exponential utility function :

$$(P) : \begin{cases} maxE\left(U\left(\frac{DC(T)}{DB(T)}\right)/\mathfrak{F}\right) \\ Subject\ to\ 1^{tr}x_t = 1, t = 0, ..., T-1 \end{cases}$$

Where :

- U is the exponential utility function with an absolute risk aversion γ $U(y) = -\exp(-\gamma y)$
- $\frac{DC(t)}{DB(t)}$ is the pension ratio at time end of the year t.
- $DB(t) =$ is the value a DB pension fund which is equal to $\frac{2}{3}L(t)$ 'a benchmark model'

- We generate a scenario paths using our economic model
- Let Ξ be the set of all scenarios paths, i.e. $\Xi = \{\xi_i^t; t = 1, ..., T, i = 1, ..., S_n\}$ where ξ_i^t is the i^{th} scenario at period t. We have S_n scenarios.
- The stochastic programming approximate the problem (P) by considering a discrete probability space which is generated by our scenarios ξ_i^t. The approximated problem $(P1)$ is given by :

$$(P1) : \begin{cases} max \sum_{i=1}^{S_n} Pr(\xi_i)(U\left(\frac{DC^i(T)}{DB^i(T)}\right) \\ Subject\ to\ 1^{tr} x_t^i = 1; t = 0, ..., T-1; i = 1, ..., S_n \end{cases}$$

An equivalent deterministic problem. and $\xi_i = \{\xi_i^t; t = 1, ..., T\}$ is the i^{th} path.

It is very useful to work with scenario trees and to transform scenarios path to scenario trees.
Scenario tree should give a representative information about risk factors : we use a forward clustering method

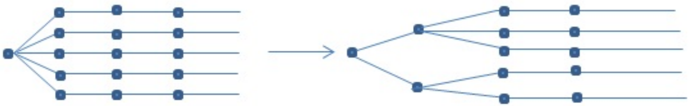

FIGURE: Tree construction

Clustering algorithm for N stages and b branching scheme at each node :

1. Set the tree root and t=0.
2. Put t=t+1. Assign cluster for each scenario according to the squared Euclidean distance and using b clustering center
3. Modify scenarios by replacing ξ_i^t by the mean of its cluster
4. For next stages : repeat steps 2 and 3.

A Numerical example

- We generate 2 periods tree with 5 branches at each nodes, i.e. $T = 4$ and $S_n = 81$
- We solve our optimal problem for $\gamma = -1$' the absolute risk aversion coefficient'
- We compare our result with the asset allocation method Buy-and-hold

A Numerical example

Strategy	Stock(can)	Stock(US)	Cash	Can20y	U.S.20y
Buy-and-Hold	24.9%	12.4%	29.2%	18.4%	15.1 %
SP	35.9%	27.1%	6.5%	16.3 %	14.2 %

Objective function for Buy-and-Hold : -0.3012

Objective function for SP : -0.2466

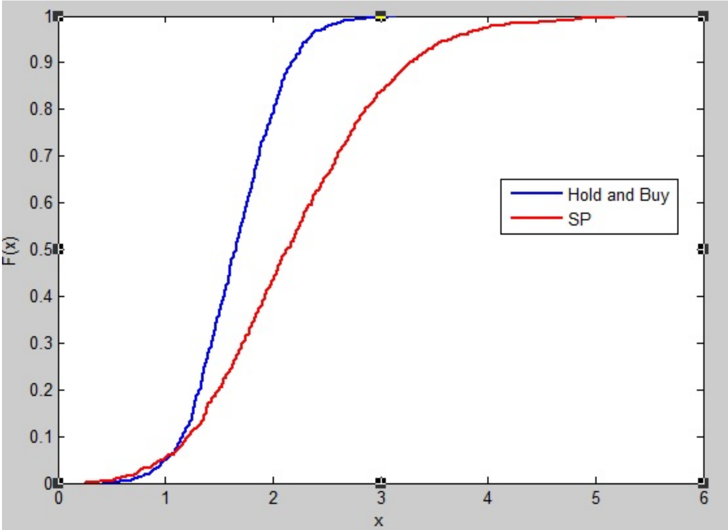

FIGURE: The accumulated value

Conclusion :

- We introduce a stochastic framework that tries to fit the asset dynamic.
- The model can be extended to more than two economies.
- We set a stochastic programming optimization

Current and future works :

- Introduce liabilities management
- Set a robust stochastic framework for more than two economies
- Use international liabilities : life insurance

- Thank you !
- All comments and remarks welcomed