Valuing GWBs with Stochastic Interest Rates and Stochastic Volaility

2nd Québec - Ontario Workshop on Insurance Mathematics

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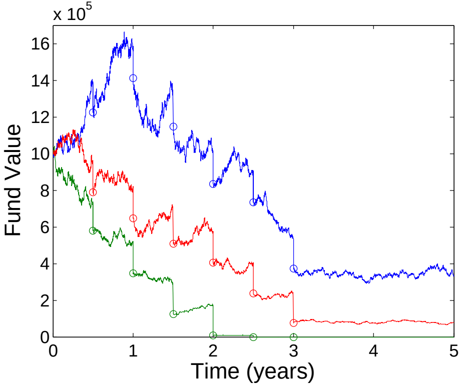
Feb 3, 2012

Background

Determine the behaviour of **fair management fees** for a class of **Guaranteed Withdrawal Benefit** insurance contracts

e.g.,

- 1 million paid back over 3 years
- semi-annual payments of $166,666.67
- Funds invested in equity / bond portfolio
- Funds (if any) returned to investor at year 5

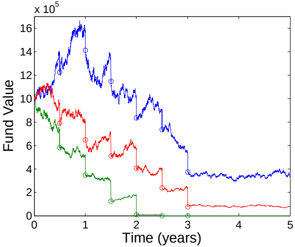


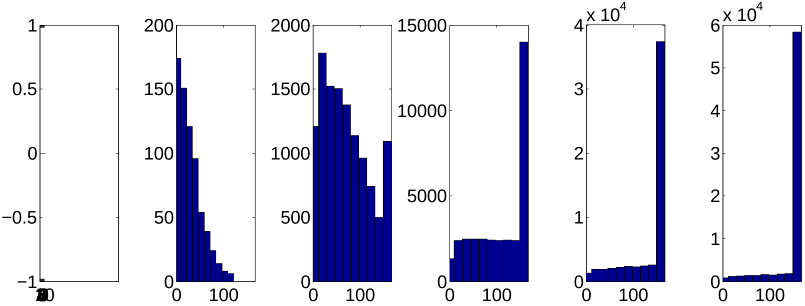
Background

- Attraction for Investors:
  - Provides investor with **equity participation**
  - Provides **guaranteed income** stream
  - **Drawndown protection**
  - Allows investor to **tune portfolio** through time
    - be aggressive now; be conservative later
  - Excess **funds returned** to investor

Background

- Insurer's embedded risks
  - Income draws the **fund below zero**
  - **Interest rates**
  - **Volatility**
  - **Mortality**





Background

- Several authors studied similar contracts. Limited list:

  Milevsky & Salisbury (2006); Dai, Kwok, & Zong (2008); Chen, Vetzal, & Forsyth (2008); Shah & Bertsimas (2008); Kling, Ruez & Ruß (2010); Forsyth (2011)

- What distinguishes this work
  - Using a time varying **mixed-fund** to back the sub-account
  - Allow for both **stochastic interest rates and volatility**
  - Using **dimensional reduction** techniques to simplify the PDEs
  - Applying **operator splitting** for numerical solutions
  - Derive **analytical approximation** for deterministic volatility and interest rates

Main Findings

- Stylized results
  - Stochastic Vol
    - Increasing vol-vol does not always increase mgt. fees
    - Reducing leverage effect tends to decrease mgt. fees
  - Stochastic Interest Rates
    - Increasing IR vol decreases mgt. fee
    - Increasing IR mean-reversion rate has little effect

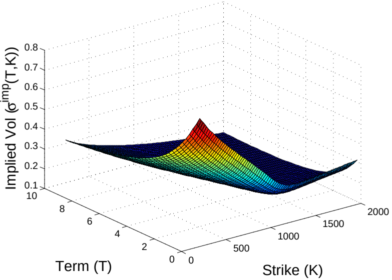
Underlying Assumptions

- The backing assets:
  - **Equity index** value S_t satisfies:

$$\frac{dS_t}{S_t} = r_t\, dt + \sqrt{v_t}\, dW_t^1\,, \qquad \textbf{Equity Value}, \qquad (1a)$$

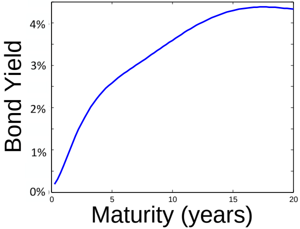
$$dv_t = \xi_t\, dt + \beta_t\, dW_t^2\,, \qquad \textbf{Stochastic Variance}, \qquad (1b)$$

$$dr_t = \theta_t\, dt + \sigma_t\, dW_t^3 \qquad \textbf{Stochastic Interest Rates}, \qquad (1c)$$



Underlying Assumptions

- The backing assets:
  - **Default-free bond** prices then satisfy the SDE:
$$\frac{dP_t(T)}{P_t(T)} = r_t\,dt + \sigma_t^P(T)\,dW_t^3, \tag{2}$$
where, $\sigma_t^P(T) = \sigma(t, r_t)\partial_r \ln P(t, r_t; T)$
  - **Fixed-Income** index P_t satisfies:
$$\frac{dP_t}{P_t} = r_t\,dt + \varsigma_t\,dW_t^3, \tag{3}$$
where $\varsigma_t = \left(\sum_{i=1}^m \psi_i\,\sigma_t(T_i)\,P(t, r_t; T_i)\right) / \left(\sum_{i=1}^m \psi_i\,P(t, r_t; T_i)\right)$

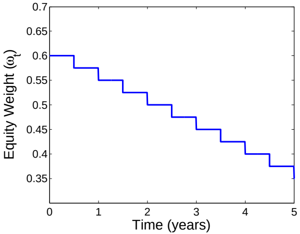


Underlying Assumptions

- The backing assets:
  - **Tracking index** value I_t satisfies:

$$\begin{aligned}\frac{dI_t}{I_t} &= \omega_t \frac{dS_t}{S_t} + (1-\omega_t)\frac{dP_t}{P_t} \\ &= r_t\,dt + \omega_t\sqrt{v_t}\,dW_t^1 + (1-\omega_t)\,\varsigma_t\,dW_t^3\,. \end{aligned} \tag{4}$$

where ω_t are deterministic weights:



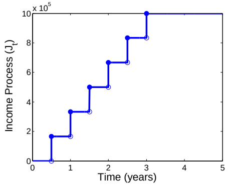
Allows investor to be aggressive early on and conservative later on.

Underlying Assumptions

- The **sub-account** or **fund value** F_t then satisfies:

$$dF_t = \left(\frac{dI_t}{I_t}\right) F_t - \alpha\, F_t\, dt - dJ_t$$
$$= (r_t - \alpha)\, F_t\, dt - \; dJ_t + \omega_t \sqrt{v_t}\, F_t\, dW_t^1 + (1-\omega_t)\, \varsigma_t\, F_t\, dW_t^3\,. \quad (5)$$

Here, $J_t = \sum_k \gamma_k\, \mathbb{I}(T_k \le t)$:



- "Four" sources of risk:
  - Equity index returns through W_t^1
  - Bond index returns through W_t^3
  - Volatility through v_t
  - Interest rates through r_t

Underlying Assumptions

Proposition

Explicit Fund Value. *The unique solution to the SDE* (5) *is given by*

$$F_T = e^{\int_0^T (r_u - \alpha)\, du}\, \eta_T \left(F_0 - \int_0^T e^{-\int_0^s (r_u - \alpha) du} \left(\eta_s\right)^{-1} dJ_s \right) , \qquad (6)$$

where η_t *is the following Dolean-Dades exponential*

$$\eta_t = \mathcal{E}\left(\int_0^t \omega_u \sqrt{v_u}\, dW_u^1 + \int_0^t (1 - \omega_u)\, \varsigma_u\, dW_u^3 \right) . \qquad (7)$$

This simplifies when the equity index is a GBM and interest rates are constant/deterministic.

[Milevsky & Salisbury (2006) have the constant ir and vol case].

Valuation

- The cash-flows provided by the product have value

$$V_0 = \underbrace{\sum_{k=1}^{n} \gamma_k \, P_0(T_k)}_{\text{Fixed-income portion}} + \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} (F_T)_+ | \mathscr{F}_0 \right]}_{\text{Option portion -- denote by } \mathcal{O}} . \tag{8}$$

- Fixed-Income portion is easy... bonds calibrated to market
- Option portion is hard... need an efficient way to deal with path-dependency

Valuation

- Use **"replicating portfolio"** to **reduce dimension**. Note,

$$\mathcal{O}_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,ds} \left(\frac{F_0 - \int_0^T Y_s\,dJ_s}{Y_T} \right)_+ \,\middle|\, \mathscr{F}_0 \right].$$

where $Y_t = e^{-\int_0^t (r_s - \alpha)ds} (\eta_t)^{-1}$

- Introduce a process X_t such that

$$dX_t = q_t\,dY_t, \qquad X_0 = F_0 - J_T\,, \qquad \text{and} \qquad q_t = J_t - J_T\,.$$

By integration by parts, it is not difficult to see that

$$X_T = F_0 - \int_0^T Y_t\,dJ_t\,,$$

- X_T replicates the the numerator in the expectation

Valuation

- Next, let $Z_t = X_t / Y_t$, then

$$\mathcal{O}_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\, ds} \left(Z_T\right)_+ \middle| \mathscr{F}_0 \right]$$

  Moreover,

$$dZ_t = (Z_t - q_t)\,(r_t - \alpha)\,dt + (Z_t - q_t)\left[\omega_t\,\sqrt{v_t}\;dW_t^1 + (1-\omega_t)\,\varsigma_t\,dW_t^3\right],$$
$$Z_0 = F_0 - J_T\,,$$

- Looks like we've only changed F_T into Z_T! True, but...
  - Z_t as a process has **no jump integrators**
  - Z_t **contains** ALL of the "info" in **both** Y_t **and** $\int_0^t Y_s\, dJ_s$

Valuation

- Use **forward-neutral measure** $\mathbb{Q}^T$ to remove discount factor

$$\mathcal{O}_0 = P_0(T)\, \underbrace{\mathbb{E}^{\mathbb{Q}^T}\left[(Z_T)_+ | \mathscr{F}_0\right]}_{\text{Expectation of interest}} . \tag{9}$$

where

$$\begin{aligned} dZ_t &= (Z_t - q_t)\,\Big[(r_t - \alpha) + \rho_{13}\,\omega_t\,\sqrt{v_t}\,\sigma_t^P(T) + (1-\omega_t)\,\varsigma_t\,\sigma_t^P(T)\Big]\;dt \\ &\qquad + (Z_t - q_t)\Big[\omega_t\,\sqrt{v_t}\;d\overline{W}_t^1 + (1-\omega_t)\,\varsigma_t\,d\overline{W}_t^3\Big]\;, \end{aligned} \tag{10a}$$

$$dv_t = (\xi_t + \rho_{23}\,\beta_t\,\sigma_t^P(T))\,dt + \beta_t\,d\overline{W}_t^2\,, \tag{10b}$$

$$dr_t = (\theta_t + \sigma_t\,\sigma_t^P(T))\,dt + \sigma_t\,d\overline{W}_t^3\,. \tag{10c}$$

Valuation

Proposition

Valuation PDE. *The process* $g_t = \mathbb{E}^{\mathbb{Q}^T}\left[(Z_T)_+ | \mathscr{F}_t\right]$ *is a martingale and there exists a function* $G(t,z,v,r) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ *such that* $g_t = G(t, Z_t, v_t, r_t)$*. Moreover, the function* $G(\cdot)$ *satisfies the PDE*

$$\begin{cases} \partial_t G + (\mathcal{L}_{z,t} + \mathcal{L}_{v,t} + \mathcal{L}_{r,t} + \mathcal{L}_t)\, G &= 0\,, \\ G(T,z,v,r) &= \max(z,0)\,, \end{cases} \tag{11}$$

where the various pieces of the infinitesimal generators are defined as follows:

$$\begin{aligned} \mathcal{L}_{z,t} &= (z - q_t)\left[(r-\alpha) + \rho_{13}\,\omega_t\,\sqrt{v}\,\sigma^P(t,r;T) + (1-\omega_t)\,\varsigma(t,r)\,\sigma^P(t,r;T)\right]\,\partial_z \\ &\quad + \ \tfrac{1}{2}(z-q_t)^2\left[\omega_t^2\,v\ + (1-\omega_t)^2\,\varsigma^2(t,r) + \rho_{13}\,\omega_t(1-\omega_t)\,\sqrt{v}\,\varsigma(t,r)\right]\,\partial_{zz}\,, \end{aligned} \tag{12a}$$

$$\mathcal{L}_{r,t} = \left(\theta(t,r) + \sigma(t,r)\sigma^P(t,r;T)\right)\,\partial_r + \tfrac{1}{2}\sigma^2(t,r)\partial_{rr}\,, \tag{12b}$$

$$\mathcal{L}_{v,t} = \left(\xi(t,v) + \rho_{23}\,\beta(t,v)\,\sigma^P(t,r;T)\right)\,\partial_v + \tfrac{1}{2}\beta^2(t,v)\partial_{vv}\,, \quad \textit{and} \tag{12c}$$

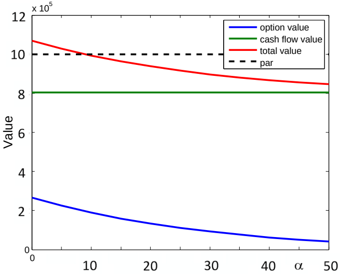
$$\begin{aligned} \mathcal{L}_t &= \rho_{23}\,\beta(t,v)\,\sigma(t,r)\,\partial_{rv} + \ (z-q_t)\left(\rho_{12}\,\omega_t\,\sqrt{v} + \rho_{23}\,(1-\omega_t)\,\varsigma(t,r)\right)\,\beta(t,v)\,\partial_{vz} \\ &\quad + \ (z-q_t)\left(\rho_{13}\,\omega_t\,\sqrt{v} + (1-\omega_t)\,\varsigma(t,r)\right)\,\sigma(t,r)\,\partial_{rz}\,. \end{aligned} \tag{12d}$$

Numerical Scheme

- With deterministic interest rates and volatility PDE reduces to

$$\begin{cases} \partial_t G + (z - q_t)(r(t) - \alpha)\,\partial_z G + \frac{1}{2}\,v(t)\,\omega_t^2\,(z - q_t)^2\,\partial_{zz} G & = 0\,, \\ G(T, z) & = (z)_+\,, \end{cases}$$

solve using standard implicit-explicit scheme.



	σ				
r	10%	20%	30%	40%	50%
1%	43.4	150.5	267.1	377.5	477.5
2%	12.3	72.4	148.2	227.5	301.0
3%	2.9	37.3	90.9	149.9	209.2
4%	0.5	19.8	57.4	103.3	150.9
5%	0.0	10.2	37.0	73.1	112.0

- $T = 20$ years, $F_0 = 10^6$, $\sigma = 30\%$, $r = 3\%$ and $\gamma = \frac{1}{9} \times 0.05 \times F_0$ paid monthly for the first 15 years.

Numerical Scheme

- For the general case, we use operator splitting
  - Treat cross-partial-derivative terms $\mathcal{L}_t$ explicitly
  - Treat partial-derivatives in a fixed direction ($\mathcal{L}_{z,t}$, $\mathcal{L}_{r,t}$, & $\mathcal{L}_{v,t}$) implicitly/explicitly

$$\begin{aligned}
V_0^n &= \left(1 - \delta t \left(L_z^n + L_v^n + L_r^n + L^n\right)\right) G^n \,, && \text{fully explicit} \\
\left(1 - \tfrac{1}{2}\,\delta t\, L_z^{n-1}\right) V_1^n &= V_0^n - \tfrac{1}{2}\,\delta t\, L_z^n\, G^n \,, && \text{implicit along } z \\
\left(1 - \tfrac{1}{2}\,\delta t\, L_v^{n-1}\right) V_2^n &= V_1^n - \tfrac{1}{2}\,\delta t\, L_v^n\, G^n \,, && \text{implicit along } v \\
\left(1 - \tfrac{1}{2}\,\delta t\, L_r^{n-1}\right) G^{n-1} &= V_2^n - \tfrac{1}{2}\,\delta t\, L_r^n G^n \,, && \text{implicit along } r
\end{aligned}$$

Numerical Experiments

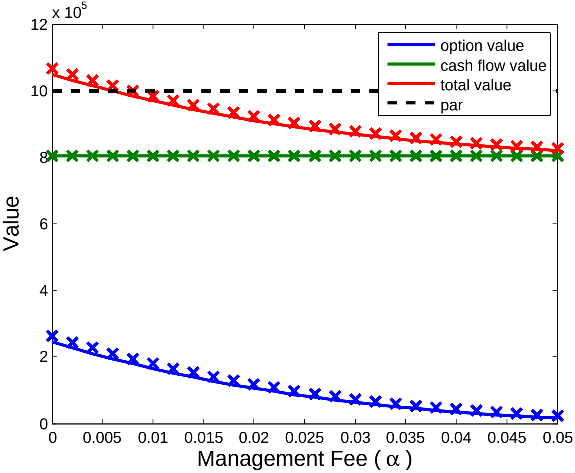


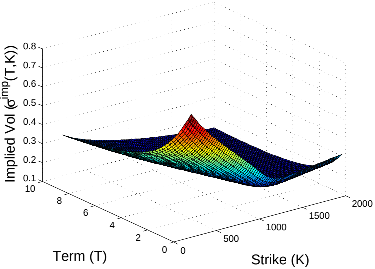
Figure: Comparison of Heston Model and Local Vol Model

Numerical Scheme

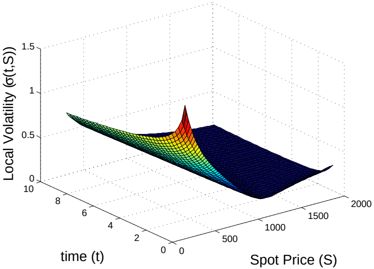
- Local volatility models often used in place of SV models

$$\frac{dS_t}{S_t} = r_t\,dt + \sqrt{v(t,S_t)}\,dW_t^1$$

Volatility/variance is an explicit function of time and equity level



(a) Heston Model



(b) Local Vol Model

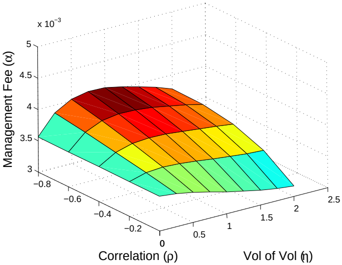
- Similar valuation equations can be derived in this case

Numerical Experiments

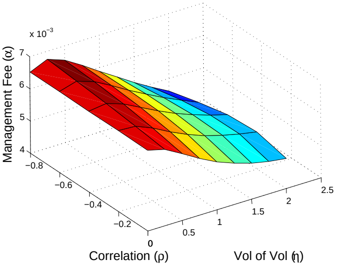
	Heston Model				Local Volatility Model			
	ρ_{12}				ρ_{12}			
η	-0.75	-0.5	-0.25	0	-0.75	-0.5	-0.25	0
0.001	64.6	64.6	64.6	64.6	81.3	81.3	81.3	81.3
0.5	65.0	64.1	62.5	60.2	88.8	86.9	81.0	78.6
1	58.4	58.4	57.2	54.4	79.2	75.6	72.6	70.0
2	46.4	49.4	51.1	48.9	66.0	59.4	54.9	51.0

Table: Implied management fee (in *bps*) versus skewness and vol-vol. The remaining model parameters are $\theta = 0.2^2$, $v_0 = 0.4^2$, $\kappa = 1$ and $r = 3\%$.

Numerical Experiments



(c) $\kappa = 2$, $\theta = v_0 = 0.2^2$



(d) $\kappa = 1$, $\theta = 0.2^2$, $v_0 = 0.4^2$

Figure: Fair management fee for various values of vol-vol η and correlation ρ under two different Heston model parameters.

Numerical Experiments

IR curve	κ	σ^r						
		10^{-5}	1%	2%	3%	4%	5%	6%
	0.5	6.5	5.8	4.4	0.5	-6.2	-16.6	-31.3
(a)	1	6.8	6.3	5.9	5.0	3.4	0.7	-2.7
	2	6.9	6.7	6.4	6.2	5.8	5.3	4.5
	0.5	46.2	44.5	42.0	36.4	25.3	7.2	-16.8
(b)	1	46.2	45.1	44.0	42.5	40.0	36.2	30.8
	2	46.2	45.6	45.0	44.4	43.7	42.7	41.5
	0.5	53.0	51.0	48.7	43.1	31.4	11.8	-15.6
(c)	1	52.7	51.5	50.3	48.9	46.5	42.6	37.0
	2	52.6	51.9	51.2	50.5	49.8	48.9	47.6

Table: Fair management fee (in basis points) versus interest rate volatility, $\sigma^S = 0.25$, $S_0 = \$1000$, $\rho_{13} = -0.3$, for the three yield curves.

Analytical Approximation

- In practice, often deterministic vol which match the ATM implied vol is used

$$v_t = (\sigma^{imp}(t))^2 + 2\,t\,\sigma^{imp}(t)\,\partial_t\sigma^{imp}(t). \tag{14}$$

together with deterministic interest rates

- An accurate approximation can be applied in this case by introducing the measure $\widehat{\mathbb{Q}}$

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \eta_T\,. \quad \text{so that,} \quad \mathcal{O}_0 = e^{-\alpha T}\,\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(F_0 - \int_0^T Y_s\,dJ_s\right)_+ \middle| \mathscr{F}_0\right],$$

Moreover,

$$\frac{dY_t}{Y_t} = -(r_t - \alpha)\,dt - \omega_u\,\sqrt{v_u}\,d\widehat{W}_t^1 - (1-\omega_u)\,\varsigma_u\,d\widehat{W}_t^3\,.$$

- Then, approximate $\int_0^T Y_s\,dJ_s$ in distribution as log-normal:

$$\int_0^T Y_s\,dJ_s \overset{d}{\sim} \widetilde{\mathcal{I}}_T = \exp\{a + b\,Z\}$$

Analytical Approximation

- The constants are determined such that first two moments are matched

$$a = 2\ln M_1 - \frac{1}{2}\ln M_2 \quad \text{and} \quad b = \sqrt{\ln M_2 - 2\ln M_1}$$

where

$$M_1 = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\sum_{k=1}^{N} Y_{t_k}\gamma_k\right] = \sum_{k=1}^{N} e^{-\int_0^{t_k}(r_u-\alpha)du}\,\gamma_k, \qquad \text{and}$$

$$M_2 = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\sum_{k=1}^{N} Y_{t_k}\gamma\right)^2\right] = \sum_{k=1}^{n} e^{-\int_0^{t_k}\left(2(r_u-\alpha)-\omega_u^2\,v_u\right)\,du}\,\gamma_k$$

$$+\,2\sum_{k<j=1}^{n} e^{-\int_0^{t_k}(r_u-\alpha)\,du-\int_0^{t_j}(r_u-\alpha)\,du+\int_0^{t_k}\omega_u^2\,v_u\,du}\,\gamma_k\,\gamma_j$$

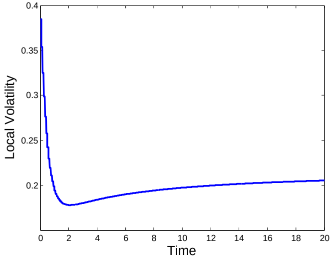
- Under this moment matching approximation we have

$$\begin{aligned}\mathcal{O}_0 \approx & e^{-\alpha T}\,\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(F_0 - \widetilde{\mathcal{I}}_T\right)_+ \middle| \mathscr{F}_0\right] \\ = & e^{-\alpha T}\,\left\{F_0\,\Phi(d) - e^{a+\frac{1}{2}b^2}\,\Phi(d-b)\right\}\end{aligned}$$

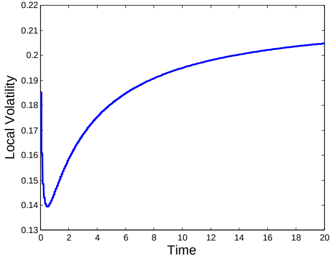
where $d = (\log(F_0) - a)\,/\,b$.

Analytical Approximation

	method	Mgt. Fee α (bp) 25	50	100	150	200
Model A	MM	208.3	189.2	155.2	126.5	102.3
	PDE (deterministic vol)	207.7	188.7	154.8	126.0	101.8
	PDE (local vol)	237.0	215.6	177.2	144.1	115.9
Model B	MM	192.7	173.4	139.4	110.9	87.2
	PDE (deterministic vol)	192.5	173.3	139.2	110.6	86.8
	PDE (local vol)	231.4	209.8	170.9	137.5	109.1



(a) Volatility term structure for Heston model with: $\kappa = 1$, $\theta = 0.2^2$, $v_0 = 0.4^2$, $\eta = 1$, $\rho_{12} = -0.7$.



(b) Volatility term structure for Heston model with: $\kappa = 1$, $\theta = v_0 = 0.2^2$, $\eta = 1$, $\rho_{12} = -0.7$.

Conclusions

- Demonstrated how to value a class of GWBs
  - Included stochastic interest rates and stochastic volatility
  - Accounted for path dependency can be neatly for through replicating portfolio
  - Solved PDE using operator splitting methods
- Stylized results
  - Stochastic Vol
    - Increasing vol-vol does not always increase mgt. fees
    - Reducing leverage effect tends to decrease mgt. fees
  - Stochastic Interest Rates
    - Increasing IR vol decreases mgt. fee
    - Increasing IR mean-reversion rate has little effect
- Analytical approximation is reasonably accurate

Thanks for your attention!

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