Valuing GWBs with Stochastic Interest Rates and Stochastic Volatility

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joint work with
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Background

Determine the behaviour of **fair management fees** for a class of **Guaranteed Withdrawal Benefit** insurance contracts

e.g.,

- 1 million paid back over 3 years
- semi-annual payments of $166,666.67
- Funds invested in equity / bond portfolio
- Funds (if any) returned to investor at year 5
Background

▶ Attraction for Investors:
  ▶ Provides investor with equity participation
  ▶ Provides guaranteed income stream
  ▶ Drawdown protection
  ▶ Allows investor to tune portfolio through time
    ▶ be aggressive now; be conservative later
  ▶ Excess funds returned to investor
Background

- Insurer’s embedded risks
  - Income draws the fund below zero
  - Interest rates
  - Volatility
  - Mortality
Background

Several authors studied similar contracts. Limited list:


What distinguishes this work

- Using a time varying mixed-fund to back the sub-account
- Allow for both stochastic interest rates and volatility
- Using dimensional reduction techniques to simplify the PDEs
- Applying operator splitting for numerical solutions
- Derive analytical approximation for deterministic volatility and interest rates
Main Findings

- Stylized results
  - Stochastic Vol
    - Increasing vol-vol does not always increase mgt. fees
    - Reducing leverage effect tends to decrease mgt. fees
  - Stochastic Interest Rates
    - Increasing IR vol decreases mgt. fee
    - Increasing IR mean-reversion rate has little effect
Underlying Assumptions

- The backing assets:
  - **Equity index** value $S_t$ satisfies:
    
    \[
    \frac{dS_t}{S_t} = r_t \, dt + \sqrt{v_t} \, dW_t^1, \quad \text{Equity Value,} \tag{1a}
    \]
    
    \[
    dv_t = \xi_t \, dt + \beta_t \, dW_t^2, \quad \text{Stochastic Variance,} \tag{1b}
    \]
    
    \[
    dr_t = \theta_t \, dt + \sigma_t \, dW_t^3 \quad \text{Stochastic Interest Rates,} \tag{1c}
    \]
Underlying Assumptions

- The backing assets:
  - **Default-free bond** prices then satisfy the SDE:
    \[
    \frac{dP_t(T)}{P_t(T)} = r_t \, dt + \sigma_t^P(T) \, dW_t^3,
    \] (2)
    where, \( \sigma_t^P(T) = \sigma(t, r_t) \partial_r \ln P(t, r_t; T) \)

- **Fixed-Income** index \( P_t \) satisfies:
    \[
    \frac{dP_t}{P_t} = r_t \, dt + \varsigma_t \, dW_t^3,
    \] (3)
    where \( \varsigma_t = (\sum_{i=1}^{m} \psi_i \sigma_t(T_i) P(t, r_t; T_i)) / (\sum_{i=1}^{m} \psi_i P(t, r_t; T_i)) \)

![Graph showing bond yield vs maturity (years)]
Underlying Assumptions

- The backing assets:
  - **Tracking index** value $I_t$ satisfies:

    \[
    \frac{dl_t}{I_t} = \omega_t \frac{dS_t}{S_t} + (1 - \omega_t) \frac{dP_t}{P_t} = r_t \, dt + \omega_t \sqrt{v_t} \, dW^1_t + (1 - \omega_t) \zeta_t \, dW^3_t. \tag{4}
    \]

    where $\omega_t$ are deterministic weights:

    Allows investor to be aggressive early on and conservative later on.
Underlying Assumptions

- The sub-account or fund value $F_t$ then satisfies:

$$dF_t = \left( \frac{dI_t}{I_t} \right) F_t - \alpha F_t \, dt - dJ_t$$

$$= (r_t - \alpha) F_t \, dt - dJ_t + \omega_t \sqrt{v_t} \, F_t \, dW_t^1 + (1 - \omega_t) \varsigma_t F_t \, dW_t^3. \quad (5)$$

Here, $J_t = \sum_k \gamma_k \mathbb{1}(T_k \leq t)$:

- “Four” sources of risk:
  - Equity index returns through $W_t^1$
  - Bond index returns through $W_t^3$
  - Volatility through $v_t$
  - Interest rates through $r_t$
Underlying Assumptions

Proposition

**Explicit Fund Value.** The unique solution to the SDE (5) is given by

\[ F_T = e^{\int_0^T (r_u - \alpha) \, du} \eta_T \left( F_0 - \int_0^T e^{-\int_0^s (r_u - \alpha) \, du} (\eta_s)^{-1} \, dJ_s \right), \tag{6} \]

where \( \eta_t \) is the following Dolean-Dades exponential

\[ \eta_t = \mathcal{E} \left( \int_0^t \omega_u \sqrt{v_u} \, dW_u^1 + \int_0^t (1 - \omega_u) \varsigma_u \, dW_u^3 \right). \tag{7} \]

This simplifies when the equity index is a GBM and interest rates are constant/deterministic.

[ Milevsky & Salisbury (2006) have the constant ir and vol case].
Valuation

- The cash-flows provided by the product have value

\[ V_0 = \sum_{k=1}^{n} \gamma_k P_0(T_k) + \mathbb{E}^{Q} \left[ e^{-\int_0^T r_s \, ds (F_T)_+} \bigg| \mathcal{F}_0 \right] . \]  

Fixed-income portion + Option portion – denote by \( \mathcal{O} \)

- Fixed-Income portion is easy... bonds calibrated to market
- Option portion is hard... need an efficient way to deal with path-dependency
Use “replicating portfolio” to reduce dimension. Note,

\[ O_0 = \mathbb{E}^Q \left[ e^{-\int_0^T r_s \, ds} \left( \frac{F_0 - \int_0^T Y_s \, dJ_s}{Y_T} \right) \bigg| \mathcal{F}_0 \right] . \]

where \( Y_t = e^{-\int_0^t (r_s - \alpha) \, ds} (\eta_t)^{-1} \)

Introduce a process \( X_t \) such that

\[ dX_t = q_t \, dY_t, \quad X_0 = F_0 - J_T, \quad \text{and} \quad q_t = J_t - J_T . \]

By integration by parts, it is not difficult to see that

\[ X_T = F_0 - \int_0^T Y_t \, dJ_t , \]

\( X_T \) replicates the the numerator in the expectation
Next, let $Z_t = X_t/Y_t$, then

$$
O_0 = \mathbb{E}^Q \left[ e^{-\int_0^T r_s \, ds} (Z_T)_+ \mid \mathcal{F}_0 \right]
$$

Moreover,

$$
dZ_t = (Z_t - q_t) (r_t - \alpha) \, dt + (Z_t - q_t) \left[ \omega_t \sqrt{v_t} \, dW^1_t + (1 - \omega_t) \varsigma_t \, dW^3_t \right],
$$

$Z_0 = F_0 - J_T$,

Looks like we’ve only changed $F_T$ into $Z_T$! True, but...

- $Z_t$ as a process has no jump integrators
- $Z_t$ contains ALL of the “info” in both $Y_t$ and $\int_0^t Y_s \, dJ_s$
Valuation

- Use **forward-neutral measure** \( \mathbb{Q}^T \) to remove discount factor

\[
O_0 = P_0(T) \mathbb{E}^{Q_T} \left[ (Z_T)^+ | \mathcal{F}_0 \right]. \tag{9}
\]

Expectation of interest

where

\[
dZ_t = (Z_t - q_t) \left[ (r_t - \alpha) + \rho_{13} \omega_t \sqrt{v_t} \sigma_t^P(T) + (1 - \omega_t) \varsigma_t \sigma_t^P(T) \right] dt \\
+ (Z_t - q_t) \left[ \omega_t \sqrt{v_t} \, dW_1^t + (1 - \omega_t) \varsigma_t \, dW_3^t \right], \tag{10a}
\]

\[
dv_t = (\xi_t + \rho_{23} \beta_t \sigma_t^P(T)) \, dt + \beta_t \, dW_2^t, \tag{10b}
\]

\[
dr_t = (\theta_t + \sigma_t^P(T)) \, dt + \sigma_t \, dW_3^t. \tag{10c}
\]
Valuation

Proposition

Valuation PDE. The process \( g_t = \mathbb{E}^{Q^T} [(Z_T)_{\mathcal{F}_t}] \) is a martingale and there exists a function \( G(t, z, v, r) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R} \) such that \( g_t = G(t, Z_t, v_t, r_t) \). Moreover, the function \( G(\cdot) \) satisfies the PDE

\[
\begin{array}{l}
\partial_t G + (L_{z,t} + L_{v,t} + L_{r,t} + L_t) G = 0, \\
G(T, z, v, r) = \max(z, 0),
\end{array}
\]

(11)

where the various pieces of the infinitesimal generators are defined as follows:

\[
L_{z,t} = (z - q_t) \left[ (r - \alpha) + \rho_{13} \omega_t \sqrt{v} \sigma^P(t, r; T) + (1 - \omega_t) \varsigma(t, r) \sigma^P(t, r; T) \right] \partial_z
\]

(12a)

\[
+ \frac{1}{2} (z - q_t)^2 \left[ \omega_t^2 v + (1 - \omega_t)^2 \varsigma^2(t, r) + \rho_{13} \omega_t (1 - \omega_t) \sqrt{v} \varsigma(t, r) \right] \partial_{zz},
\]

\[
L_{r,t} = \left( \theta(t, r) + \sigma(t, r) \sigma^P(t, r; T) \right) \partial_r + \frac{1}{2} \sigma^2(t, r) \partial_{rr},
\]

(12b)

\[
L_{v,t} = \left( \xi(t, v) + \rho_{23} \beta(t, v) \sigma^P(t, r; T) \right) \partial_v + \frac{1}{2} \beta^2(t, v) \partial_{vv}, \quad \text{and}
\]

(12c)

\[
L_t = \rho_{23} \beta(t, v) \sigma(t, r) \partial_{rv} + (z - q_t) \left( \rho_{12} \omega_t \sqrt{v} + \rho_{23} (1 - \omega_t) \varsigma(t, r) \right) \beta(t, v) \partial_{vz}
\]

(12d)

\[
+ (z - q_t) \left( \rho_{13} \omega_t \sqrt{v} + (1 - \omega_t) \varsigma(t, r) \right) \sigma(t, r) \partial_{rz}.
\]
Numerical Scheme

With deterministic interest rates and volatility PDE reduces to

\[
\begin{aligned}
\partial_t G + (z - q_t)(r(t) - \alpha) \partial_z G + \frac{1}{2} \nu(t) \omega^2 (z - q_t)^2 \partial_{zz} G &= 0, \\
G(T, z) &= (z)_+, 
\end{aligned}
\]

solve using standard implicit-explicit scheme.

\[
\begin{array}{cccccc}
\hline
\sigma & 10\% & 20\% & 30\% & 40\% & 50\% \\
\hline
r & 43.4 & 150.5 & 267.1 & 377.5 & 477.5 \\
1\% & 12.3 & 72.4 & 148.2 & 227.5 & 301.0 \\
2\% & 2.9 & 37.3 & 90.9 & 149.9 & 209.2 \\
3\% & 0.5 & 19.8 & 57.4 & 103.3 & 150.9 \\
4\% & 0.0 & 10.2 & 37.0 & 73.1 & 112.0 \\
5\% & & & & & \\
\hline
\end{array}
\]

\[
T = 20 \text{ years}, \quad F_0 = 10^6, \quad \sigma = 30\%, \quad r = 3\% \quad \text{and} \quad \gamma = \frac{1}{9} \times 0.05 \times F_0 \quad \text{paid monthly for the first 15 years.}
\]
Numerical Scheme

- For the general case, we use operator splitting
  - Treat cross-partial-derivative terms $\mathcal{L}_t$ explicitly
  - Treat partial-derivatives in a fixed direction ($\mathcal{L}_{z,t}$, $\mathcal{L}_{r,t}$, & $\mathcal{L}_{v,t}$) implicitly/explicitly

\[
\begin{align*}
V_0^n &= (1 - \delta t \, (L_z^n + L_v^n + L_r^n + L^n)) \, G^n, \quad \text{fully explicit} \\
\left(1 - \frac{1}{2} \delta t \, L_z^{n-1}\right) V_1^n &= V_0^n - \frac{1}{2} \delta t \, L_z^n \, G^n, \quad \text{implicit along } z \\
\left(1 - \frac{1}{2} \delta t \, L_v^{n-1}\right) V_2^n &= V_1^n - \frac{1}{2} \delta t \, L_v^n \, G^n, \quad \text{implicit along } v \\
\left(1 - \frac{1}{2} \delta t \, L_r^{n-1}\right) G^{n-1} &= V_2^n - \frac{1}{2} \delta t \, L_r^n \, G^n, \quad \text{implicit along } r
\end{align*}
\]
Numerical Experiments

Figure: Comparison of Heston Model and Local Vol Model
Numerical Scheme

- Local volatility models often used in place of SV models

\[
\frac{dS_t}{S_t} = r_t \, dt + \sqrt{\nu(t, S_t)} \, dW_t^1
\]

Volatility/variance is an explicit function of time and equity level

- Similar valuation equations can be derived in this case
Numerical Experiments

<table>
<thead>
<tr>
<th>η</th>
<th>Heston Model</th>
<th></th>
<th>Local Volatility Model</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ρ₁₂</td>
<td></td>
<td>ρ₁₂</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.75 -0.5 -0.25 0</td>
<td></td>
<td>-0.75 -0.5 -0.25 0</td>
<td></td>
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<tr>
<td>0.001</td>
<td>64.6 64.6 64.6 64.6</td>
<td></td>
<td>81.3 81.3 81.3 81.3</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>65.0 64.1 62.5 60.2</td>
<td></td>
<td>88.8 86.9 81.0 78.6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>58.4 58.4 57.2 54.4</td>
<td></td>
<td>79.2 75.6 72.6 70.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>46.4 49.4 51.1 48.9</td>
<td></td>
<td>66.0 59.4 54.9 51.0</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Implied management fee (in bps) versus skewness and vol-vol. The remaining model parameters are $\theta = 0.2^2$, $\nu_0 = 0.4^2$, $\kappa = 1$ and $r = 3\%$. 
Numerical Experiments

\[(c) \quad \kappa = 2, \quad \theta = v_0 = 0.2^2\]

\[(d) \quad \kappa = 1, \quad \theta = 0.2^2, \quad v_0 = 0.4^2\]

**Figure:** Fair management fee for various values of vol-vol $\eta$ and correlation $\rho$ under two different Heston model parameters.
### Numerical Experiments

Table: Fair management fee (in basis points) versus interest rate volatility, $\sigma^S = 0.25$, $S_0 = $1000, $\rho_{13} = -0.3$, for the three yield curves.

<table>
<thead>
<tr>
<th>IR curve</th>
<th>$\kappa$</th>
<th>$10^{-5}$</th>
<th>1%</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.5</td>
<td>6.5</td>
<td>5.8</td>
<td>4.4</td>
<td>0.5</td>
<td>-6.2</td>
<td>-16.6</td>
<td>-31.3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6.8</td>
<td>6.3</td>
<td>5.9</td>
<td>5.0</td>
<td>3.4</td>
<td>0.7</td>
<td>-2.7</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.9</td>
<td>6.7</td>
<td>6.4</td>
<td>6.2</td>
<td>5.8</td>
<td>5.3</td>
<td>4.5</td>
</tr>
<tr>
<td>(b)</td>
<td>0.5</td>
<td>46.2</td>
<td>44.5</td>
<td>42.0</td>
<td>36.4</td>
<td>25.3</td>
<td>7.2</td>
<td>-16.8</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>46.2</td>
<td>45.1</td>
<td>44.0</td>
<td>42.5</td>
<td>40.0</td>
<td>36.2</td>
<td>30.8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>46.2</td>
<td>45.6</td>
<td>45.0</td>
<td>44.4</td>
<td>43.7</td>
<td>42.7</td>
<td>41.5</td>
</tr>
<tr>
<td>(c)</td>
<td>0.5</td>
<td>53.0</td>
<td>51.0</td>
<td>48.7</td>
<td>43.1</td>
<td>31.4</td>
<td>11.8</td>
<td>-15.6</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>52.7</td>
<td>51.5</td>
<td>50.3</td>
<td>48.9</td>
<td>46.5</td>
<td>42.6</td>
<td>37.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>52.6</td>
<td>51.9</td>
<td>51.2</td>
<td>50.5</td>
<td>49.8</td>
<td>48.9</td>
<td>47.6</td>
</tr>
</tbody>
</table>
Analytical Approximation

- In practice, often deterministic vol which match the ATM implied vol is used

\[ \nu_t = (\sigma^{imp}(t))^2 + 2t \sigma^{imp}(t) \partial_t \sigma^{imp}(t). \]  \tag{14}

- An accurate approximation can be applied in this case by introducing the measure \( \hat{Q} \)

\[ \frac{d\hat{Q}}{dQ} = \eta_T. \]  
so that,

\[ \mathcal{O}_0 = e^{-\alpha T} \mathbb{E}^{\hat{Q}} \left[ \left( F_0 - \int_0^T Y_s \, dJ_s \right) \bigg| \mathcal{F}_0 \right]. \]

Moreover,

\[ \frac{dY_t}{Y_t} = -(r_t - \alpha) \, dt - \omega_u \sqrt{\nu_u} \, d\hat{W}_t^1 - (1 - \omega_u) \varsigma_u \, d\hat{W}_t^3. \]

- Then, approximate \( \int_0^T Y_s \, dJ_s \) in distribution as log-normal:

\[ \int_0^T Y_s \, dJ_s \sim \tilde{I}_T = \exp\{a + bZ\} \]
Analytical Approximation

- The constants are determined such that first two moments are matched

\[
a = 2 \ln M_1 - \frac{1}{2} \ln M_2 \quad \text{and} \quad b = \sqrt{\ln M_2 - 2 \ln M_1}
\]

where

\[
M_1 = \mathbb{E}^{\tilde{Q}} \left[ \sum_{k=1}^{N} Y_{tk} \gamma_k \right] = \sum_{k=1}^{N} e^{- \int_{0}^{tk} (r_u - \alpha) du} \gamma_k, \quad \text{and}
\]

\[
M_2 = \mathbb{E}^{\tilde{Q}} \left[ \left( \sum_{k=1}^{N} Y_{tk} \gamma \right)^2 \right] = \sum_{k=1}^{n} e^{- \int_{0}^{tk} (2(r_u - \alpha) - \omega_u^2 \nu_u) du} \gamma_k
\]

\[
+ 2 \sum_{k<j}^{n} e^{- \int_{0}^{tk} (r_u - \alpha) du - \int_{0}^{tj} (r_u - \alpha) du + \int_{0}^{tk} \omega_u^2 \nu_u du} \gamma_k \gamma_j
\]

- Under this moment matching approximation we have

\[
\mathcal{O}_0 \approx e^{-\alpha T} \mathbb{E}^{\tilde{Q}} \left[ \left( F_0 - \tilde{I}_T \right) \bigg| \mathcal{F}_0 \right] = e^{-\alpha T} \left\{ F_0 \Phi(d) - e^{a + \frac{1}{2} b^2} \Phi(d - b) \right\}
\]

where \( d = (\log(F_0) - a) / b \).
## Analytical Approximation

<table>
<thead>
<tr>
<th>method</th>
<th>Mgt. Fee $\alpha$ (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td><strong>Model A</strong></td>
<td></td>
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<tr>
<td>MM</td>
<td>208.3</td>
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<tr>
<td>PDE (deterministic vol)</td>
<td>207.7</td>
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<tr>
<td>PDE (local vol)</td>
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<tr>
<td><strong>Model B</strong></td>
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<tr>
<td>MM</td>
<td>192.7</td>
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<tr>
<td>PDE (deterministic vol)</td>
<td>192.5</td>
</tr>
<tr>
<td>PDE (local vol)</td>
<td>231.4</td>
</tr>
</tbody>
</table>

(a) Volatility term structure for Heston model with: $\kappa = 1$, $\theta = 0.2^2$, $\nu_0 = 0.4^2$, $\eta = 1$, $\rho_{12} = -0.7$.

(b) Volatility term structure for Heston model with: $\kappa = 1$, $\theta = \nu_0 = 0.2^2$, $\eta = 1$, $\rho_{12} = -0.7$. 
Conclusions

- Demonstrated how to value a class of GWBs
  - Included stochastic interest rates and stochastic volatility
  - Accounted for path dependency can be neatly for through replicating portfolio
  - Solved PDE using operator splitting methods
- Stylized results
  - Stochastic Vol
    - Increasing vol-vol does not always increase mgt. fees
    - Reducing leverage effect tends to decrease mgt. fees
  - Stochastic Interest Rates
    - Increasing IR vol decreases mgt. fee
    - Increasing IR mean-reversion rate has little effect
- Analytical approximation is reasonably accurate
Thanks for your attention!

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