Credit risk: a complex system seen from an actuarial perspective

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FROM THE PRESIDENT
RALPH BLANCHARD

Getting to the Right Answer

Question: "What is the difference between God and an actuary?"
Answer: "God knows that He's not an actuary."
(Joke overheard during my recent travels.)

Actuaries are typically involved in modeling or analyzing risks using faulty data—data that are both incomplete and error-prone. This data is from the past, but the present is different from the past and the future will be different from the present. We combine all this with subjective inputs from various sources, some of which are of uncertain reliability if not outright biased. If we then produce a single answer as a result, the only thing we know for certain is that the final outcome won't match our answer. In other words, our answer will be wrong.

If the only assurance we have is that our answer will be wrong, where does the value from the actuarial analysis come from? How do we get to the "right" answer?

Perhaps we can get there by acknowledging that the real value of our work is not from the single answer that we may produce. The value instead comes from the understanding that arises from the analysis—the understanding of the options and the possibilities, the opportunities and the risks regarding the business decision to be taken. It is when this understanding is shared with the various decision makers (such as those in claims, underwriting, and ceded reinsurance) that the real value to the operation is achieved.

Yet many actuaries insist on focusing on the single answer. Why is this? What creates this blind spot for many in the profession?

Perhaps it is our training.

Nearly all actuaries excelled in math, from the early elementary grades right up through college. These math courses were typically focused on having us come up with the one right answer, and in nearly all cases there was only one right answer. The classroom training (at least the training that I had) rewarded those that were first to produce the one right answer. The focus was on speed and having the result represent your work—and only your work. If you needed help it was a sign of reduced skill level. A math test was not a team event.

As a result, it should be no surprise that many actuaries focus exclusively on "the answer" when they communicate their work product to others. It should also be no surprise that they tend to defend this result against any and all doubters or naysayers, which no doubt provided fodder for the actuarial joke at the beginning of this column. This is what they've been trained to do (and selected for) since their early days in the classroom.

However, the business world is not the same as the classroom. In the world outside the classroom, success is usually measured on a group basis, not an individual basis. Actuaries need to be team players and recognize (and trumpet) the insights that were gleaned in their work towards an answer. The stronger actuaries are able to do this, adapting successfully from the classroom to the business environment. But how do we get more actuaries to successfully transition from the classroom to the business environment?

As is the case for most long-standing problems, there are no easy solutions. There is also unlikely to be a single solution. The following are some possible paths for various contributors to the current environment to muse upon.

Academia—Incorporate into math courses more team-oriented assignments and foster more of an environment where success for an individual is at least partly a function of the success as a group. Some colleges already incorporate this team-oriented mindset in certain areas, encouraging classmates to work together on some projects and to leverage each other's strengths so that all can excel.[1] Yes, this may be harder to grade, and yes, it risks giving too much credit to a freeloader, but the reward is hopefully worth the risk.

Supervisors and clients of actuaries—Refuse to accept an actuarial communication that stops at "the answer." Insist that the findings, risks, and possible alternatives be included in any such communication. This mindset can be found in the documentation requirements in Unpaid Claim Estimates ASOP.[2]

Employers of actuaries—Include teambuilding training for your staff.

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[1] I heard of such an approach a few years ago at Ringling College of Art & Design. They stressed to their students that they were all in it together, that the reputation of one depended at least partly on the reputation of the other—that alumni had a strong reputation if the current students had a strong reputation and vice versa.
[2] Actuarial Standard of Practice No. 43, as issued by the U.S. Actuarial Standards Board.

[Source: The (CAS) Actuarial Review, August 2011]

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Looking Backward

"The future ain't what it used to be."

—Yogi Berra

Several years ago, I was browsing through some old magazines at a relative's house and came across a 1976 article from a group of futurists. Each contributor was giving their view of what the world would look like in the future. By the time I read the article, that "future" was at least partly in the past.[1]

I obtained two valuable lessons from that retrospective look at the futurists' predictions. The first was that being a futurist is a difficult profession, as many (most?) of the predictions made in the article were very wrong.[2] There was a clear tendency to view the world through the filter of what the futurist had previously experienced. There was no recognizable ability to anticipate how the world would change. For example, one of the authors appeared to believe that by around the year 2000 "we will be living in the post-petroleum era."

The second lesson came from what some of the futurists got right. A few, not all, made note of major new developments that were happening in their time. The authors' noted correctly that these developments were going to have a major impact on the future—they weren't necessarily correct on what that impact might be, but they noted that it was probably going to be major and was worth paying attention to. For example, Isaac Asimov predicted that we will live in a "world or global village, tied together electronically, with every citizen able to communicate instantly with each other," although he didn't seem to anticipate the growth of the "virtual office."[3]

What does this have to do with the CAS? I believe it provides a useful lesson in our planning for the future. We need to identify and focus on what is changing in our world and be ready to adjust to those influences. We also need to acknowledge that the world is always changing and that we need to change or adapt along with it, even if we don't know exactly how it will change. This does not mean change for change's sake, "throwing out the baby with the bath water," but it does mean that standing still is not an option.

So what are some of the changes in our world that will affect our future in a significant but yet unknowable way? The following is a humble attempt to identify some.

1. **Tremendous increase in the volume of data captured.** When I first started my career, computer storage was expensive. There was an active attempt to reduce electronic file sizes and nearly all the data captured was captured manually. In contrast, memory is now cheap and more and more data is captured electronically. This expansion is opening up new avenues for analysis that were previously inconceivable. What kind of analyses might this lead to? For what uses?
2. **Tremendous increase in the accessibility of data.** This is a function of both the construction of databases that allow such access, and new hardware and software to bring that data to your fingertips. This includes the ability to access the data almost anywhere in the world. How will this change the profession? Will the impact be bigger for outsourcing or for the virtual office?
3. **GLMs, data mining, predictive modeling.** These are the new tools that continue to evolve and allow us to leverage the items above. Where will this lead us? What lessons are there from the recent financial crisis, with the heavy usage of data-intensive modeling by hedge funds and the like?

On a related note (linking the first three observations), given that much of the data of property/casualty companies is proprietary, will these models significantly split the market into two categories—those with the data and those without? Will the data needed to run these models become more proprietary or less proprietary in the future, and how will this affect the market?

4. **The rise of "analytics."** You don't have to be an actuary to apply the new tools mentioned above. In fact, the training of some folks may be more tailored toward the use of these tools than the actuarial profession (e.g., those with advanced degrees in statistics). My personal belief is that understanding the data and the environment where data are used still counts

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[1] For those interested, the article was "Five Noted Thinkers Explore the Future" from the July 1976 issue of *National Geographic*. A similar example of the perils of being a futurist is also evident from the book *Looking Backward* by Edward Bellamy in 1988, from which the title of this article was taken.

[2] Nearly all would be viewed as wrong, except for the open-endedness of some of the predictions. Those predictions wisely didn't give a firm date for their future, leaving open the possibility that they might eventually be right.

[3] To his credit, Isaac Asimov also said, "None of the conclusions we arrive at [in their attempt at predicting the future] may have any meaning whatsoever 25 years from now."

[Source: The (CAS) Actuarial Review, November 2011]

Baseball and the actuary

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Rating agencies, like Moody‘s or Standard and Poor‘s use econometrics models with several variables, some quite subjective, to produce their credit ratings. We propose to revisit the problem with a more classical actuarial approach.

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Rating agencies, like Moody's or Standard and Poor's use econometrics models with several variables, some quite subjective, to produce their credit ratings. We propose to revisit the problem with a more classical actuarial approach.

Classical arbitrage–free, consistent market assumptions rule out price differences that allow risk–free profits at zero cost. Yet, over– or under–estimation of the underlying risks generate such arbitrage opportunities, as with current credit ratings on European bonds. We introduce an alternate ranking based on risk measures.

Overview:

1. Credit risk models, a review
   1.1 Introduction
   1.2 Classical credit risk models
   1.3 Information based models
2. Credit risk under under jump processes
   2.1 Introduction
   2.2 Preliminaries and definitions
   2.3 Locally risk minimization hedging
3. Credit risk models with risk measures
   3.1 Introduction
   3.2 Inconsistencies and market integration
   3.3 Credit risk measures

1. Credit risk models, a review

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The drawback is that this is inconsistent with market observations; investors are aware of the probable default time in advance, leading to zero short–credit spreads.

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The bond payoff decomposes as

$$\min(X_T, D) = D - \max(0, D - X_T),$$

and can be considered a credit derivative, where $\max(0, X_T - D)$ represents the firm's value at time T.

Extension of Black and Cox (1976, JoF)

The payoff $D - \max(0, D - X_T)$ is that of a risk–free account and a vanilla option. Under the absence of arbitrage, if X_t is based on Brownian motion returns (GMB), then Black–Scholes formula can be used to price this credit derivative.

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The yield y_t at $t < T$ (if $t < \tau$) is given as the solution of

$$e^{-\int_t^T r_s ds}\mathbb{P}[\tau > T|\mathfrak{F}_t] = e^{-y_t(T-t)},$$

(for $D = 1$) and the credit spread is $y_t - \frac{\int_t^T r_s ds}{T-t} = -\frac{\ln \mathbb{P}[\tau > T|\mathfrak{F}_t]}{T-t}$.

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Here the default time is totally inaccessible (not predictable). Jarrow and Turnbull model the default time as the first jump time of a Poisson process. Investors cannot be aware of the default time, which yields to non–zero credit spreads and useful formulas for pricing credit derivatives.

Intensity process

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Difficult to use, except for simple stopped processes X, such as (non–homogeneous) Poisson.

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For a review of both, intensity and hazard based models, see Jeanblanc and LeCam (2007, preprint).

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Information based credit risk models are an attempt to link the structural and reduced form approaches. The key element here is the information flow available to investors.

Figure: Agual Azúl Falls, Chiapas, Mexico

Asymmetric information

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This investor's filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t \geq 0}$ is an expansion of $\mathfrak{G}$ that makes τ a stopping time.

Filtration expansions

Three 3 main methods of expanding $\mathfrak{G}$ in credit risk models:

- Progressive filtration expansion, Gieseke (2006, JoEDC):

$$\mathfrak{F}_t = \{B \in \mathfrak{F}_\infty; \text{for some } B_t \in \mathfrak{G}_t, B \cap \{t < \tau\} = B_t \cap \{t < \tau\}\}.$$

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- Guo and Zeng (2008, AAP), any filtration $\mathfrak{F}$ that satisfies:

$$\mathfrak{G}_t \cap \{t < \tau\} = \mathfrak{F}_t \cap \{t < \tau\}, \text{ for all } t \geq 0.$$

Further topics

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  - some Lévy processes.

2. Credit risk under under jumps processes

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We use local risk–minimization approach to analyze the risk of defaultable claims.

Previous work

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- The underlying process is a (local) martingale,
- The underlying process has continuous sample paths,
- The contingent claim is not path dependent,
- The Girsanov's theorem is applicable (MELMM Method).

We relax all these assumptions.

Defaultable security

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- The market is composed of
  - the risky asset $X = (X_t)_{0 \le t < \infty}$,
  - and a risk-free one.
- We study financial products with payoffs in the form of

$$F(X_T) 1_{\{\tau > T\}},$$

where $\tau = \inf\{t : X_t < 0\}$, $F : \mathbb{R} \to \mathbb{R}$ a convex function, and $T > 0$ is the maturity or expiration of the security.

- Assume that the process X is a finite variation Lévy process with non-negative drift

$$X_t = u + \mu t + \int_0^t \int_{\mathbb{R}-\{0\}} x J_X(ds \times dx), \quad t \geq 0.$$

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- There are no restrictions on the probability measure.
- The usual convenient assumptions are made:
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  - Finally, the market is frictionless, and there is no arbitrage.

Objectives

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- The second question is if it is possible to design a customized payoff $F(X_T)$ specifically, that makes the product $F(X_T)1_{\{\tau>T\}}$ completely risk free.

Itô's formula or Dynkin's formula

- By Dynkin's formula $f(t, X_t) - \int_0^t \mathcal{A}f(s, X_s)ds$ is a martingale for a $C^{1,2}$ function $f = f(t, x)$ where

$$\mathcal{A}f(s,x) = \frac{\partial f}{\partial s}(s,x) + \mu\frac{\partial f}{\partial x}(s,x) + \int_{\mathbb{R}} (f(s,x+y) - f(s,x))\, v(dy)$$

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- By the Optional Sampling Theorem

$$f(t \wedge \tau, X_{t \wedge \tau}) - \int_0^{t \wedge \tau} \mathcal{A}f(s, X_s)ds$$

is also a martingale.

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$$\mathcal{A}f(s,x) = \frac{\partial f}{\partial s}(s,x)+\mu\frac{\partial f}{\partial x}(s,x)+\int_{\mathbb{R}}(f(s,x+y)-f(s,x))\, \nu(dy)$$

- By the Optional Sampling Theorem

$$f(t \wedge \tau, X_{t\wedge\tau}) - \int_0^{t\wedge\tau} \mathcal{A}f(s, X_s)ds$$

is also a martingale.

- $f(t \wedge \tau, X_{t\wedge\tau}) = f(\tau, X_\tau)1_{\{\tau\leq t\}} + f(t, X_t)1_{\{\tau>t\}}$.

The compensator of $\left(f(t,X_t)1_{\{\tau>t\}}\right)_{(t\geq 0)}$

Theorem: Assume that X is a bounded variation Lévy process. Let $f:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a $C^{1,2}$ function. Then under some integrability conditions, the process

$$\left(f(t,X_t)1_{\{\tau>t\}}-\int_0^t \mathfrak{A}f(s,X_s)1_{\{\tau>s\}}\,ds\right)_{t\geq 0},$$

is an $\mathfrak{F}^X$- martingale, where

$$\begin{aligned}\mathfrak{A}f(s,x) &= \frac{\partial f}{\partial s}(s,x)+\mu\frac{\partial f}{\partial x}(s,x)-\int_{-\infty}^{-x} f(s,x+y)\,v(dy)\\ &\quad +\int_{-\infty}^{\infty}(f(s,x+y)-f(s,x))\,v(dy).\end{aligned}$$

Application in martingale theory

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- $f(t, X_t)1_{\{\tau > t\}} - \int_0^{t\wedge\tau} \mathfrak{A} f(s, X_s)ds$ is a martingale.

2.3 Locally risk minimization hedging

Definition: For an L^2-strategy θ, a small perturbation Δ, and a partition P of $[0, T]$, we set

$$r^P(\theta, \Delta) = \sum \frac{R_{t_i}(\theta + \Delta|_{(t_i, t_{i+1}]}) - R_{t_i}(\theta)}{\mathbb{E}[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} | \mathfrak{F}_{t_i}]} 1_{(t_i, t_{i+1}]}.$$

Then θ is called locally risk–minimizing if

$$\lim_{n \to \infty} r^{P_n}(\theta, \Delta) \geq 0, \quad (\mathbb{P} \times \langle M \rangle) - -a.e. \quad \text{on} \quad \Omega \times [0, T],$$

for every small perturbation Δ and every increasing sequence P_n of partitions tending to the identity; see Schweizer (1999, HUB).

Pseudo–local risk–minimization

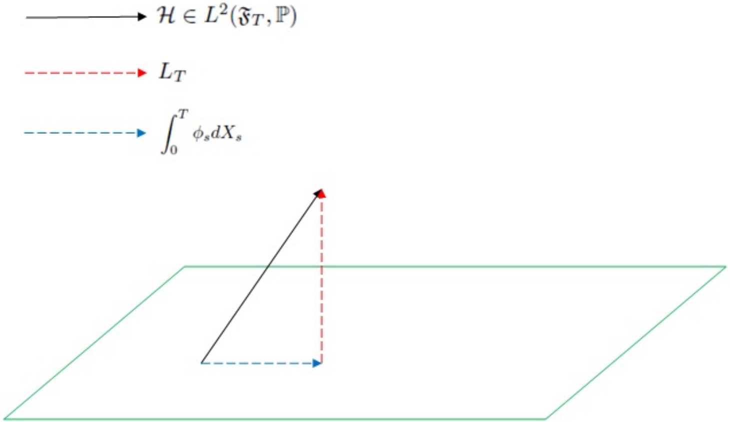
Definition: Let

$$V(\theta) = \phi X + \eta,$$

$$C_t(\theta) = V_t(\theta) - \int_0^t \phi_u dX_u,$$

and $\mathcal{H} \in L^2(\mathfrak{F}_T, \mathbb{P})$ is a contingent claim. An L^2-strategy $\theta = (\phi, \eta)$ with $V_T(\theta) = \mathcal{H}\ \mathbb{P} - a.s.$ is called pseudo–locally risk–minimizing or pseudo–optimal for $\mathcal{H}$ if $C(\theta)$ (the cost process) is a martingale, strongly orthogonal to the martingale part of X, see Schweizer, M.(1999, HUB).

Pseudo–local risk–minimization



PIDE

Remark: Given a convex function $F = F(x)$, it is assumed that there is a $C^{1,2}$ function $f = f(t,x)$ that is the solution of the following PIDE

$$\mathfrak{A}f(t,x) = \frac{(\mathfrak{A}K(t,x) - x\mathfrak{A}f(t,x) - \beta f(t,x))}{\int_{-\infty}^{\infty} y^2 \ v(dy)}\beta, \text{ for all } 0 \leq t \leq T,$$

and

$$f(T,x) = F(x), \text{ for all real numbers } x,$$

where $K(t,x) = xf(t,x)$ and $\beta = \mu + \int_{-\infty}^{\infty} y \ v(dy)$.

Theorem: Under the appropriate assumptions including the previous remark, there is an L^2–strategy $\phi = (\theta, \eta)$ as follows. The number of risky assets for the hedging process is given by the process $\theta = (\theta_t)_{0 \leq t \leq T}$, where

$$\theta_t = \frac{(\mathfrak{A}K(t, X_t) - X_t\mathfrak{A}f(t, X_t) - \beta f(t, X_t))}{\int_{-\infty}^{\infty} y^2 \; v(dy)} 1_{\{\tau > t\}}.$$

Hedging error

The hedging error L belongs to $\mathcal{M}_0^2$. It is strongly orthogonal to M and given by

$$L_t = f(t, X_t)1_{\{\tau > t\}} - f(0, X_0) - \int_0^t \theta_s \, dX_s, \; 0 \leq t \leq T.$$

The value process of the portfolio is equal to

$$V_t(\theta) = f(0, X_0) + \int_0^t \theta_s \, dX_s + L_t, \; 0 \leq t \leq T,$$

the number of the risk-free assets is

$$\eta_t = V_t(\theta) - \theta_t X_t, \; 0 \leq t \leq T,$$

and finally the cost process is provided by

$$C_t = f(0, X_0) + L_t, \; 0 \leq t \leq T.$$

Example 2.1: $F(x) = 1$, exponential jump sizes

- Assume that $X_t = u + \mu t + \sum_{j=1}^{N_t} Y_i$, for $\mu > 0$, $-Y_1 \sim \mathit{exponential}(\delta)$, and the process X a martingale.

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- Consider a defaultable zero–coupon bond that pays 1 unit if there is no default, i.e. $F(x) = 1$ for all x.

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- The number of risky assets of the hedging strategy is:

$$\theta_s = \frac{\left(\int_{-X_s}^{\infty} y f(s, X_s + y) F_Y(dy) - \mathbb{E}[Y_1] f(s, X_s)\right) 1_{\{\tau > s\}}}{\mathbb{E}[Y_1^2]},$$

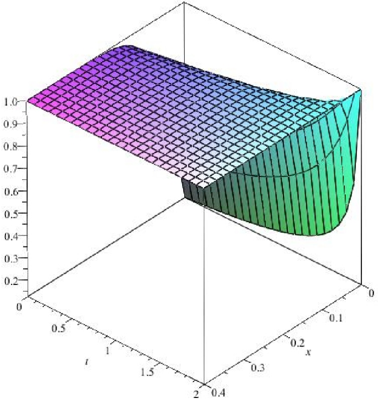
where $f = f(t, x)$ satisfies the following PIDE

$$\mathfrak{A} f(t, x) = 0, \text{ for all } 0 \leq t \leq T \text{ and all } x \in \mathbb{R},$$

$$f(T, x) = 1, \text{ for all } x \in \mathbb{R}.$$

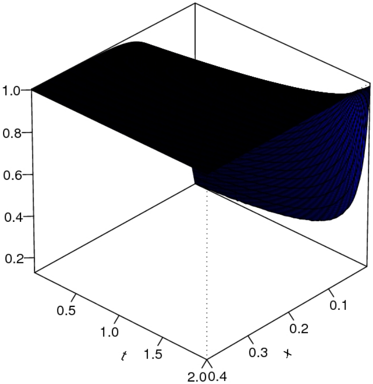
Example 2.1: $F(x) = 1$, exponential jump sizes

The graph of $f = f(t, x)$ on $[0, 2] \times [0, 0.4]$ is given for $\mu = 0.1$, $\delta = 100$, $\lambda = 10$, and $T = 2$.

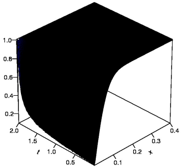


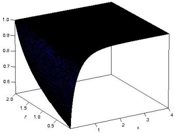
Example 2.1: $F(x) = 1$, exponential jump sizes

The function $f = f(t; x)$ can also be estimated numerically by simulation (for the same parameters as above).



Example 2.2: $F(x) = 1$, gamma jump sizes





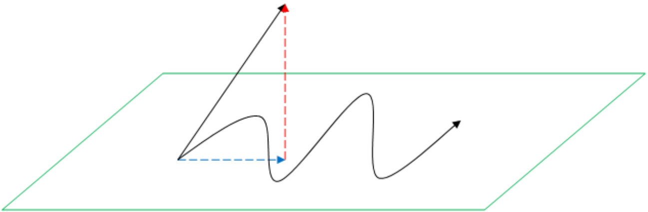
$\alpha = 5$ and $\beta = 0.002$ $\alpha = 0.5$ and $\beta = 0.02$

Perfect defaultable security

$\longrightarrow \mathcal{H} \in L^2(\mathfrak{F}_T, \mathbb{P})$

$\dashrightarrow L_T$

$\dashrightarrow \int_0^T \phi_s dX_s$



Here the curvy line is representing a risk-free product of the type of $G(X_t)1_{\{\tau>t\}}$.

Perfect defaultable security

Lemma: Assume that (f, F) satisfies the following conditions

$$\mathfrak{A}f(t,x) = 0,$$

$$\mathcal{L}f(t,x) = 0,$$

$$f(T,x) = F(x),$$

then the product $F(X_T)1_{\{\tau > T\}}$ is risk–free,

Perfect defaultable security

where

$$\begin{aligned}\mathfrak{A}f(t,x) &= \mu\frac{\partial f}{\partial x}(t,x)+\frac{\partial f}{\partial t}(t,x)\\ &\quad +\lambda\int_{-x}^{\infty} f(t,x+u)\mathbb{P}[Y_1\in du]-\lambda f(t,x),\end{aligned}$$

$$\begin{aligned}\mathcal{L}f(t,x) &= \mathfrak{A}f(t,x)^2\\ &\quad -\lambda\frac{\left(\int_{-x}^{\infty} yf(s,x+y)\mathbb{P}[Y_1\in dy]-\mathbb{E}[Y_1]f(s,x)\right)^2}{\mathbb{E}[Y_1^2]}.\end{aligned}$$

Use of risk measures

Assume that $F = F(X_T)1_{\{\tau > T\}}$ is a risk free defaultable security. Then the riskiness of the defaultable security $\mathcal{H}1_{\{\tau > T\}}$ can be measured by

$$CD_\rho(\mathcal{H}1_{\{\tau > T\}}) = \rho(G - \mathcal{H}),$$

where ρ is a risk measure.

Application to finite time ruin probabilities

Proposition: Under appropriate conditions we have the following decompositions:

$$1_{\{\tau > T\}} = f(0, X_0) + \int_0^T \theta_{s^-} \, dX_s + L_T,$$

Application to finite time ruin probabilities

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$$F(X_T)1_{\{\tau > T\}} = f(0, X_0) + \int_0^T \theta_{s^-} \; dM_s + L_T.$$

Finite time ruin probabilities

Example 2.3: Assume that $X_t = u + \mu t + \sum_{j=1}^{N_t} Y_i$ and $F(x) = 1 - \frac{\lambda}{\mu\delta} e^{(\frac{\lambda}{\mu} - \delta)x}$, then

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$$\mathbb{P}(\tau > t) - \frac{\lambda}{\mu\delta} \mathbb{E}[e^{(\frac{\lambda}{\mu} - \delta)X_t} 1_{\{\tau > t\}}] = F(u).$$

Examples 2.3-2.4: finite time ruin probabilities

- Let $-Y_1 \sim \textit{exponential}(\delta)$, for $\lambda = 0.2$, $\delta = 0.02$, $u = 10$, and $\mu = 1$. Then $\mathbb{P}(\tau \leq 1) \approx 0.1499320558$, compared to the exact value 0.1499305134.

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- If $-Y_1 \sim \textit{gamma}(5, 10)$, for $\lambda = 0.2$, $u = 10$, and $\mu = 1$. Then $\mathbb{P}(\tau \leq 1) \approx 0.181257660$.

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- If $-Y_1 \sim \textit{gamma}(5, 10)$, for $\lambda = 0.2$, $u = 10$, and $\mu = 1$. Then $\mathbb{P}(\tau \leq 1) \approx 0.181257660$.
- In both cases above the upper bound for the error is 1.8%

In summary

- The compensator of the process $(f(t, X_t)1_{\{\tau > t\}})_{(t \geq 0)}$ is obtained,

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- The hedging strategies and pricing rules were calculated under a physical measure.

In summary

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- The locally risk–minimization approach is carried out for defaultable claims under finite variation Lévy processes,
- The hedging strategies and pricing rules were calculated under a physical measure.
- The analysis does not use the MELMM method to obtain the strategies. However, the final solution is based on the solution of a PIDE.

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3. Credit risk models with risk measures

3.1 Introduction

3.1.1 Arbitrage opportunities:

- Arbitrage opportunities can arise from over– or under–estimation of the financial derivatives underlying risk.

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- We investigate the type of the arbitrage that comes from the (intuitive) properties of risk measures.

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3.1 Introduction

3.1.1 Arbitrage opportunities:

- Arbitrage opportunities can arise from over– or under–estimation of the financial derivatives underlying risk.
- We investigate the type of the arbitrage that comes from the (intuitive) properties of risk measures. In other words, under a specific risk measure, if the risk of a portfolio is less than or equal to 0, then a possible positive income of the portfolio is considered as an arbitrage income.

Portfolio 1

Is this portfolio consistent?

	Bond 1	Bond 2
Price of the bonds	$1,010$	908.9
Cash flow 1	10	9
Cash flow 2	$101,000$	909

Portfolios 2 and 3

Now, which of the following portfolios seems consistent?

	Bond 1	Bond 2
Price of the bonds	$1,010$	908.9
Cash flow 1	10	1
Cash flow 2	$1,010$	909

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A five–bond portfolio

Dates	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
7-Dec-2010	98.77	99.96	99.00	112.70	114.35
1-Jan-2011	0	0	0	5	5.4
1-May-2011	1.375	0	0	0	0
15-May-2011	0	0	1.5	0	0
1-July-2011	0	0	0	5	5.4
15-July-2011	0	2.6	0	0	0
1-Nov-2011	1.375	0	0	0	0
15-Nov-2011	0	0	1.5	0	0
1-Jan-2012	0	0	0	5	5.4
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15-July-2013	0	2.6	0	0	0
1-Nov-2013	1.375	0	0	0	0
15-Nov-2013	0	0	1.5	0	0
1-Jan-2014	0	0	0	5+100	5.4+100
15-Jan-2014	0	2.6	0	0	0
1-May-2014	1.375	0	0	0	0
15-May-2014	0	0	1.5	0	0
15-July-2014	0	2.6	0	0	0
1-Nov-2014	1.375	0	0	0	0
15-Nov-2014	0	0	1.5	0	0
15-Jan-2015	0	2.6	0	0	0
1-May-2015	1.375	0	0	0	0
15-May-2015	0	0	1.5	0	0
15-July-2015	0	2.6	0	0	0
1-Nov-2015	100+1.375	0	0	0	0
15-Nov-2015	0	0	1.5+100	0	0
15-Jan-2016	0	2.6+100	0	0	0
1-May-2016	0	2.6+100	0	0	0

Corporate bond 1

COLGATE PALMOLIVE CO MTNS BE
As of 7-Dec-2010
OVERVIEW

Price:	98.77
Coupon(%):	1.375
Maturity Date:	1-Nov-2015
Yield to Maturity(%):	1.639
Current Yield(%):	1.392
Fitch Rating:	AA
Coupon Payment Frequency:	Semi-Annual
First Coupon Date:	1-May-2011
Type:	Corporate
Callable:	No

OFFERING INFORMATION

Quantity Available:	1240
Minimum Trade Qty:	1
Dated Date:	3-Nov-2010
Settlement Date:	13-Dec-2010

(Source: Yahoo Finance: http://screen.yahoo.com/bonds.html)

Corporate bonds 2 and 3

JPMORGAN CHASE & CO
As of 7-Dec-2010
OVERVIEW

Price:	99.96
Coupon(%):	2.600
Maturity Date:	15-Jan-2016
Yield to Maturity(%):	2.607
Current Yield(%):	2.601
Fitch Rating:	AA
Coupon Payment Frequency:	Semi-Annual
First Coupon Date:	15-Jul-2011
Type:	Corporate
Callable:	No

OFFERING INFORMATION

Quantity Available:	300
Minimum Trade Qty:	1
Dated Date:	18-Nov-2010
Settlement Date:	13-Dec-2010

COCA COLA CO
As of 7-Dec-2010
OVERVIEW

Price:	99.00
Coupon(%):	1.500
Maturity Date:	15-Nov-2015
Yield to Maturity(%):	1.714
Current Yield(%):	1.515
Fitch Rating:	AA
Coupon Payment Frequency:	Semi-Annual
First Coupon Date:	15-May-2011
Type:	Corporate
Callable:	No

OFFERING INFORMATION

Quantity Available:	175
Minimum Trade Qty:	1
Dated Date:	15-Nov-2010
Settlement Date:	13-Dec-2010

(Source: Yahoo Finance: http://screen.yahoo.com/bonds.html)

Municipal bonds 4 and 5

CHICAGO ILL GO BDS
As of 7-Dec-2010
OVERVIEW

State:	Illinois
Price:	112.70
Coupon(%):	5.000
Maturity Date:	1-Jan-2014
Yield to Maturity(%):	0.780
Current Yield(%):	4.437
Fitch Rating:	AA
Coupon Payment Frequency:	Semi-Annual
First Coupon Date:	1-Jan-2008
Callable:	No

BOND PROFILE

Type:	Municipal
Insured:	Yes
Alternative Minimum Tax:	No

CHICAGO ILL TAXABLE GO BONDS
As of 7-Dec-2010
OVERVIEW

State:	Illinois
Price:	114.35
Coupon(%):	5.400
Maturity Date:	1-Jan-2014
Yield to Maturity(%):	0.644
Current Yield(%):	4.722
Fitch Rating:	AA
Coupon Payment Frequency:	Semi-Annual
First Coupon Date:	1-Jan-2005
Callable:	No

BOND PROFILE

Type:	Municipal
Insured:	Yes
Alternative Minimum Tax:	No

(Source: Yahoo Finance: http://screen.yahoo.com/bonds.html)

Risk Measures

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$\rho(X)$ measures the risk associated to the random variable X.

Coherent risk measures

Assume that random variables X and Y represent the gains on two different investments.

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- Translation invariance: $\rho(X+a) = \rho(X) - a$, for all $X \in \mathfrak{R}$ and all $a \in \mathbb{R}$,
- Monotonicity: if $X \le Y$ then $\rho(Y) \le \rho(X)$.

Other types of risk measures

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- Expectation bounded risk measures: Rockafellar et al. (2006, F&S).

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- Expectation bounded risk measures: Rockafellar et al. (2006, F&S).
- Deviation risk measures: Rockafellar et al. (2006, J&S).
- Distortion risk measures: Wang (2000, JRI).

Examples of risk measures

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$$VaR_\alpha(X) = -\inf\{z : F_X(z) > \alpha\}.$$

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- The conditional value–at–risk is given by

$$CVaR_\alpha(X) = -\mathbb{E}[X | X \leq -VaR_\alpha(X)],$$

when F_X is continuous at $-VaR_\alpha(X)$.

$A = [a_{ij}]$ is an $m \times n$ matrix representing n bonds with possible future cash flows at times $t_1, t_2, \ldots, t_m$,

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \\ \\ \downarrow & \downarrow & \cdots & \downarrow \end{array} \\ \begin{array}{c} t_1 \longrightarrow \\ t_2 \longrightarrow \\ \cdot \\ \cdot \\ \cdot \\ t_m \longrightarrow \end{array} & \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{21} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \end{array} .$$

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\left(\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{21} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
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\end{array}
$$

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$T = t_m$ represents the final date of the cash flow.

Each vector $\overline{x} = (x_1, x_2, ..., x_n)$ represents a portfolio consisting x_1 units of B_1, x_2 units of B_2,..., x_n units of B_n.

Accumulated Wealth Function

We consider the accumulated wealth function $\Pi_T : \mathbb{R}^m \to \Pi$ where

$$\Pi = \{\Pi_T(\overline{x}) : \quad \overline{x} \in \mathbb{R}^m\},$$

and $\overline{x} = (x_1, x_2, \ldots, x_m)$ is a vector in $\mathbb{R}^m$.

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Π_T is a linear function.

Assume that $\rho : \Pi \to \mathbb{R}$ is any risk measure that satisfies the conditions of sub–additivity, $\rho(x+y) \leq \rho(x) + \rho(y)$ and positive homogeneity, $\rho(tx) = t\rho(x)$ for every $t \geq 0$ and $x, y \in \Pi$.

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The composition of ρ and Π_T, denoted $\overline{\rho} = \rho \circ \Pi_T$, defines a risk measure on $\mathbb{R}^m$ into $\mathbb{R}$ that is sub–additive and positively homogeneous.

Representation theorem

Assume that $\overline{\rho} : \mathbb{R}^m \longrightarrow \mathbb{R}$ is as above, then

$$\overline{\rho}(\overline{y}) = \max\{-\overline{y} \cdot \overline{z} : \overline{z} \in \Delta_{\overline{\rho}}\},$$

where

$$\Delta_{\overline{\rho}} = \{\overline{z} \in \mathbb{R}^m; \quad \overline{\rho}(\overline{y}) \geq -\overline{y} \cdot \overline{z} \quad ,\forall \quad \overline{y} \in \mathbb{R}^m\},$$

and $\overline{y} \cdot \overline{z}$ is the usual inner product on $\mathbb{R}^m$.

Definition of ρ-arbitrage

The portfolio $\overline{x}$ is said to be a ρ-arbitrage portfolio if:

- $\overline{p} \cdot \overline{x} < 0$

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The portfolio $\overline{x}$ is said to be a ρ-arbitrage portfolio if:

- $\overline{p} \cdot \overline{x} < 0$
- $\overline{\rho}(\sum_{j=1}^{n} x_j a_{1j}, \ldots, \sum_{j=1}^{n} x_j a_{mj}) \leq 0$ or, equivalently, $\overline{\rho}(\overline{x} \cdot \overline{b}_1, \ldots, \overline{x} \cdot \overline{b}_m) \leq 0$ where $\overline{b}_i = (a_{i1}, ..., a_{in}) \in \mathbb{R}^n$ is the row i of matrix A for $i = 1, 2, ..., m$.

3.2 Inconsistencies and market integration

Definition: A (bond) market is said to be free of inconsistencies, if it does not provide any ρ-arbitrage opportunities to investors.

Measuring ρ–arbitrage

To measure inconsistencies the following optimization problem is proposed:

$$\begin{aligned} &\mathit{Maximize} \ -\overline{p}\cdot\overline{x}, \\ &\mathit{such\ that}\ \overline{\rho}(\overline{x}\cdot\overline{a}_1,\ldots,\overline{x}\cdot\overline{a}_m)\leq 0, \\ &\qquad\qquad \overline{x}+\overline{h}\geq 0, \\ &\qquad\qquad \overline{h}\cdot\overline{p}\leq 1, \\ &\qquad\qquad \overline{h}\geq 0, \end{aligned} \tag{1}$$

where $(\overline{x},\overline{h})\in\mathbb{R}^n\times(\mathbb{R}^+\cup\{0\})^n$ are the decision variables.

Existence of ρ–arbitrage

Lemma: Assume that θ^* is the optimal value of the primal problem above. Then the market is ρ–arbitrage free if and only if $\theta^* = 0$.

The dual problem

Theorem: Assume that the above primal problem is always finite, it reaches its optimal value, and so the optimal solutions always exist.

The dual problem

Theorem: Assume that the above primal problem is always finite, it reaches its optimal value, and so the optimal solutions always exist. Then the equivalent dual form of the primal problem is

$$\begin{aligned} &\textit{Minimize } \theta, \\ &\textit{such that } p_j = \lambda_j + \alpha \overline{b}_j \cdot \overline{z}, \\ &\qquad\qquad \overline{\lambda} \leq \theta \overline{p}, \\ &\qquad\qquad \theta \geq 0,\ \overline{\lambda} \geq 0,\ \alpha \geq 0,\ \overline{z} \in \Delta_{\overline{\rho}}. \end{aligned} \tag{2}$$

Karush–Kuhn–Tucker conditions

Theorem: $(\overline{x}^*, \overline{h}^*)$ and $(\theta^*, \overline{\lambda}^*, \alpha^*, \overline{z}^*)$ solve problems (1), and (2) respectively, if and only if they satisfy the following K–K–T conditions:

$$
\begin{aligned}
& p_j = \lambda_j^* + \alpha^* \overline{b}_j \cdot \overline{z}^*, \quad j = 1, 2, \ldots, n, \\
& \overline{\lambda}^* \leq \theta^* \overline{p}, \\
& \overline{x}^* \cdot (\overline{p} - \overline{\lambda}^*) = 0, \\
& \overline{\lambda}^* \cdot (\overline{x}^* + \overline{h}^*) = 0, \\
& \theta^* (\overline{h}^* \cdot \overline{p} - 1) = 0, \\
& \overline{h}^* \geq 0,\ \overline{h}^* \cdot \overline{p} \leq 1,\ \overline{x}^* + \overline{h}^* \geq 0, \\
& \theta^* \geq 0,\ \overline{\lambda}^* \geq 0,\ \overline{z}^* \in \Delta_{\overline{\rho}},\ \alpha^* \geq 0.
\end{aligned}
$$

Existence of ρ–arbitrage

Theorem: There is no ρ–arbitrage in a portfolio if and only if there exists $(\overline{z}^*, \alpha^*) \in \Delta_{\overline{\rho}} \times \mathbb{R}^+$ such that for every j,

$$p_j = \alpha^* \overline{z}^* \cdot \overline{b_j},\ j = 1, 2, ..., n,$$

where $\overline{b_j} = (a_{1j}, a_{2j}, ..., a_{mj})$ is the j's column of matrix A.

Classical arbitrage

Corollary: Suppose that the solution set of the following system of equations is empty:

$$p_j = \alpha^* \overline{z}_* \cdot \overline{b_j},\ j = 1, 2, ..., n,$$

where $\overline{b_j} = (a_{1j}, a_{2j}, ..., a_{mj})$ is the jth column of matrix A.

Classical arbitrage

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$$p_j = \alpha^* \overline{z}_* \cdot \overline{b_j},\ j = 1, 2, ..., n,$$

where $\overline{b_j} = (a_{1j}, a_{2j}, ..., a_{mj})$ is the jth column of matrix A. Then the existence of a classical arbitrage in the market is guaranteed.

Extension of the risk measurement

- So far, we have looked at maximizing the arbitrage income subject to some constraints.

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- Another perspective is to maximize the arbitrage income and minimize the risk simultaneously. In other words, trying to maximize a vector objective function $(-\overline{p} \cdot \overline{x}, -\overline{\rho}(\overline{x} \cdot \overline{a}_1, \ldots, \overline{x} \cdot \overline{a}_m))$.

Simple numerical example

Assume that changes in interest rates are modeled by

$$r = \begin{cases} r_1, & \text{with probability } 1 - q, \\ r_2, & \text{with probability } q, \end{cases}$$

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that is $\Omega = \{\omega_1, \omega_2\}$, $\mathbb{P}(\omega_1) = 1-q$, $\mathbb{P}(\omega_2) = q$, and assume that $\rho = CVaR_\alpha$, with $\alpha \in (0,1)$.

Numerical example: Portfolio 2

- Assume that $q = 0.9$, $r_1 = 0.05$, $r_2 = 0.03$, and $\alpha = 0.05$.

Numerical example: Portfolio 2

- Assume that $q = 0.9$, $r_1 = 0.05$, $r_2 = 0.03$, and $\alpha = 0.05$.
- The portfolio consists of the 2 following bonds:

	Bond 1	Bond 2
Price of the bonds	1010	908.9
Cash flow 1	10	1
Cash flow 2	1010	909

Numerical example: Portfolio 2

For this portfolio the optimal value is approximately equal to $\theta^* = 0.00869$.

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- The solution of the dual problem is equal to $\theta^* \approx 0.00869$, $\lambda_1^* \approx 0$, $\lambda_2^* \approx 7.8999$, $\alpha^* \approx 0.9901$, $\overline{z}_1^* \approx 1$, $\overline{z}_2^* \approx 0.9901$.

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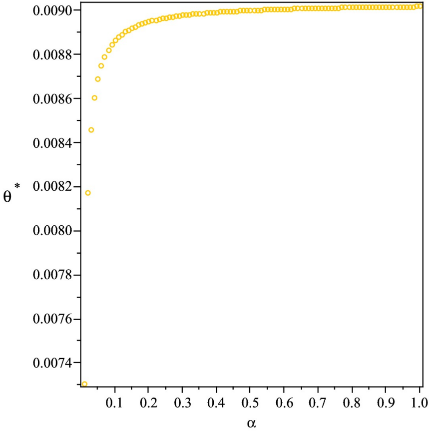
- The optimal solution is sensitive to changes in α, q, r_1, and r_2.

Sensitivity with respect to α

The following graph gives the optimal value θ^* as a function of α.



Empirical data

- When applying the model to real market data, the problem of determining the sub–gradient set $\Delta_{\overline{\rho}}$ can be difficult.

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- When applying the model to real market data, the problem of determining the sub–gradient set $\Delta_{\overline{\rho}}$ can be difficult.
- A possible avenue is to use risk statistics, defined in the next section.

Risk statistics

- Natural risk statistics,
- Coherent risk statistics,
- Law–invariant coherent risk statistics.

Numerical example: Portfolio 2 (revisited)

	Bond 1	Bond 2
Price of the bonds	1010	908.9
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The result of the optimization problem here is 0 for all natural risk statistics, coherent risk statistics, and law–invariant coherent risk statistics under the maximal set Δ.

Numerical example: Portfolio 3

	Bond 1	Bond 2
Price of the bonds	1010	908.9
Cash flow 1	10	9
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- The optimal value under the above risk statistics is still zero.

Numerical example: Portfolio 3

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Price of the bonds	1010	908.9
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- Under the assumptions of the tree model, the optimal value is ≈ 0.99899874.
- The optimal value under the above risk statistics is still zero.
- Can we conclude that natural risk measures cannot detect inconsistencies?

Using risk statistics

- The main problem with natural risk statistics and law–invariant coherent risk statistics is their permutation invariance. For example $\overline{\rho}(1,0) = \overline{\rho}(0,1)$.

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- In Heyde et al. (2007), they take data statically (i.e. at a fixed time), while we use data at different times.

3.3 Credit risk measures

Thank you for your attention

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